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A Murnaghan–Nakayama rule for quantum cohomology of the flag manifold

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Abstract. We give a rule for the multiplication of a Schubert class by a tautological class in the (small) quantum cohomology ring of the full flag manifold. As an intermediary step, we establish a formula for the multiplication of a Schubert class by a quantum Schur polynomial indexed by a hook partition.

Résumé. On décrit une règle pour la multiplication d'une classe de Schubert par une classe tautologique dans le (petit) anneau de cohomologie quantique de la variété de drapeaux. En cours de route, on obtient une formule pour la multiplication d'une classe de Schubert par un polynôme de Schur quantique correspondant à une équerre.

Keywords: Murnaghan–Nakayama Formula, Schubert Polynomials.

The *Murnaghan–Nakayama rule* is a method for computing values of the irreducible characters of symmetric groups. Under the identification of the character ring with the ring of symmetric functions, it translates into a rule for the multiplication of a Schur symmetric function s_λ by a Newton power sum p_r. As explained in [15, Sect. 2], in the cohomology ring of the full flag manifold, the Newton power sums correspond to tautological classes that are components of the Chern character. Accordingly, a formula describing multiplication by a tautological class is also called a *Murnaghan–Nakayama rule* since it corresponds to multiplication by a Newton power sum.

Here, in Corollary 2.3 we describe a rule for multiplication by a tautological class in the small quantum cohomology ring of the full flag manifold. As an intermediary step,

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we show in Theorem 2.2 a formula for multiplication by a hook quantum Schur polynomial. This entails a detailed analysis of chains and intervals in the quantum Bruhat order. Making use of results by N. C. Leung and C. Li [10] and A. Postnikov [17], it turns out that quantum products by hook Schur polynomials can be reduced to the classical product from [18].

1 A Murnaghan–Nakayama rule for (classic) cohomology

We begin by reviewing combinatorial models for the cohomology ring of the flag manifold. (We omit the precise definitions of the flag manifold and its cohomology as the combinatorial models will suffice for our purposes; the interested reader can consult [7].) Then we will derive a formula for the product of a Schubert polynomial and a Newton power sum, which will be instructive in deriving our quantum analogue.

Let $\mathbb{F}\ell_n$ denote the manifold of complete flags in $\mathbb{C}^n$. Its cohomology ring $H^*\mathbb{F}\ell_n$ with coefficients in $\mathbb{Z}$ is a $\mathbb{Z}$-module with a distinguished basis $\{\mathfrak{S}_w\}_{w\in S_n}$ whose elements $\mathfrak{S}_w$ are called *Schubert classes* and are indexed by elements of the symmetric group S_n.

Borel [4] gave a presentation of $H^*\mathbb{F}\ell_n$ as a quotient of a polynomial ring: specifically, $H^*\mathbb{F}\ell_n \simeq \mathbb{Z}[x_1,\ldots,x_n]/I_n$, where $I_n = \langle e_l(x_1,\ldots,x_n) \mid l \in [n]\rangle$ where e_l is the lth elementary symmetric polynomial and $[n] = \{1,\ldots,n\}$.

Lascoux and Schützenberger [9] defined a set of polynomials in $\mathbb{Z}[x]$ called *Schubert polynomials* that correspond to the Schubert classes $\mathfrak{S}_w$ under Borel's isomorphism. We write $\mathfrak{S}_w(x)$ for the Schubert polynomial associated with $\mathfrak{S}_w$ on the set of variables $x := \{x_1,\ldots,x_n\}$. Every Schur polynomial $s_\lambda(x_1,\ldots,x_k)$ is a Schubert polynomial, as follows. Let $w \in S_n$ be a permutation with a unique descent at k. Then the sequence $(w(k)-k,\ldots,w(2)-2,w(1)-1)$ is a partition contained in the rectangular partition $R_{k,n-k}$ of $k(n-k)$ into k parts with each part of size $n-k$. This is a bijection and for $\lambda \subseteq R_{k,n-k}$ we write $v(\lambda,k) \in S_n$ for the corresponding permutation. Then $\mathfrak{S}_{v(\lambda,k)}(x) = s_\lambda(x_1,\ldots,x_k)$.

Open Problem 1.1. *Find a combinatorial rule expressing $\mathfrak{S}_u \cdot \mathfrak{S}_v$ in terms of the basis $\{\mathfrak{S}_w\}_{w\in S_n}$ of $H^*\mathbb{F}\ell_n$. That is, find a combinatorial rule for the coefficients $c^w_{u,v} \in \mathbb{Z}$ in the product*

$$\mathfrak{S}_u(x)\cdot\mathfrak{S}_v(x) = \sum_w c^w_{u,v}\mathfrak{S}_w(x). \tag{1.1}$$

A remarkable property of the Schubert polynomials is that the computation in Open Problem 1.1 is performed in the polynomial ring, not in the quotient. The resulting coefficients $c^w_{u,v}$ in the quotient will be the same.

Although Problem 1.1 remains open in general, several special cases are known. *Monk's Formula* [13] treats the case where v is the simple transposition $(k,k+1)$: for any $u \in S_n$, we have $\mathfrak{S}_u \cdot \mathfrak{S}_{(k,k+1)} = \sum \mathfrak{S}_{u(i,j)}$ with the sum ranging over all transpositions (i,j) satisfying $i \le k < j \le n$ and $\ell(u)+1 = \ell(u(i,j))$, where $\ell(u) := \#\{i<j \mid u(i) > u(j)\}$.

The terms appearing in Monk's Formula define a partial order on S_n graded by $\ell(u)$. The *k-Bruhat order* $\leq_k$ is defined by the covering relation $u \lessdot_k u(i,j)$ if $i \leq k < j \leq n$ and $\ell(u)+1 = \ell(u(i,j))$; that is, $u \lessdot_k u(i,j)$ if $\mathfrak{S}_{u(i,j)}$ appears in the product $\mathfrak{S}_u(x) \cdot \mathfrak{S}_{(k,k+1)}$.

Write $[u,w]_k$ for the interval between u and w in the k-Bruhat order. Label each cover relation $u \lessdot_k u(i,j)$ by $u(i)$ and write $u \xrightarrow{u(i)} u(i,j)$. The maximal element of a (saturated) chain $\gamma = (u \xrightarrow{a_1} u_1 \xrightarrow{a_2} \cdots \xrightarrow{a_r} u_r)$ is denoted $\mathrm{end}(\gamma) = u_r$ and this chain has *length r*. If $a_1 > \cdots > a_l < a_{l+1} < \cdots < a_r$, then γ is a *peakless chain of height l and length r*.

The following case of Open Problem 1.1 was proved in [18].

Proposition 1.2. *Let $u \in S_n$, $l \leq k$, and $m \leq n-k$. Then*

$$\mathfrak{S}_u(x) \cdot s_{(m,1^{l-1})}(x_1,\ldots,x_k) \;=\; \sum_{\gamma} \mathfrak{S}_{\mathrm{end}(\gamma)}(x) \tag{1.2}$$

summing over all peakless chains of height l and length $m+l-1$ in the labeled k-Bruhat order.

To describe the multiplicities in (1.2), we define the *Grassmannian–Bruhat order* on S_n by $\eta \preceq \zeta$ if there are a permutation $u \in S_n$ and an integer k with $u \leq_k \eta u \leq_k \zeta u$. This is a graded partial order on S_n with minimal element the identity permutation e. Its rank function is $\mathcal{L}(\zeta) := \ell(\zeta u) - \ell(u)$ for any $u \in S_n$ and $k \in \mathbb{N}$ with $u \leq_k \zeta u$. The cover relations also inherit the labelling from the k-Bruhat orders. As shown in [1], neither $\preceq$, $\mathcal{L}$ nor the labelling depend on the choice of k or u.

We say that a permutation ζ is *minimal* if $\mathcal{L}(\zeta) = \#\mathrm{supp}(\zeta) - s(\zeta)$, where $\mathrm{supp}(\zeta) = \{i \in [n] \mid \zeta(i) \neq i\}$ and $s(\zeta)$ is the number of nontrivial cycles in the factorization of ζ into disjoint cycles. The *height* of a minimal permutation ζ is $\mathrm{ht}(\zeta) := \#\{i \in [n] \mid i < \zeta(i)\}$. Using results from [3], the following characterizes minimal permutations in terms of the Grassmannian–Bruhat order.

Lemma 1.3. *A permutation ζ is minimal if and only if the interval $[e,\zeta]_\preceq$ has a peakless chain. The number of peakless chains of height l in $[e,\zeta]_\preceq$ is the binomial coefficient $\binom{s(\zeta)-1}{\mathrm{ht}(\zeta)-l}$.*

Proposition 1.2 now reads as follows.

Proposition 1.4. *Let $u \in S_n$, $l \leq k$ and $m \leq n-k$. Then*

$$\mathfrak{S}_u(x) \cdot s_{(m,1^{l-1})}(x_1,\ldots,x_k) \;=\; \sum \binom{s(\zeta)-1}{\mathrm{ht}(\zeta)-l} \mathfrak{S}_{\zeta u}(x)\,, \tag{1.3}$$

summing over all minimal permutations $\zeta \in S_n$ such that $u \leq_k \zeta u$ and $\mathcal{L}(\zeta) = m+l-1$.

We use this to describe the product of a Schubert polynomial and a power sum symmetric polynomial. Let $p_r(x_1,\ldots,x_k) := x_1^r + \cdots + x_k^r$ denote the rth power sum polynomial. Expanded in terms of Schur polynomials [11], we have

$$p_r(x_1,\ldots,x_k) = \\ s_{(r)}(x_1,\ldots,x_k) - s_{(r-1,1)}(x_1,\ldots,x_k) + \cdots + (-1)^{r+1} s_{(1^r)}(x_1,\ldots,x_k). \tag{1.4}$$

Corollary 1.5. *Let $u \in S_n$. Then*

$$\mathfrak{S}_u(x) \cdot p_r(x_1, \ldots, x_k) \;=\; \sum (-1)^{\mathrm{ht}(\zeta)+1} \mathfrak{S}_{\zeta u}(x)\,,$$

summing over all minimal cycles $\zeta \in S_n$ such that $u \leq_k \zeta u$ and $\mathcal{L}(\zeta) = r$.

Proof. Expanding the product $\mathfrak{S}_u(x) \cdot p_r(x_1, \ldots, x_k)$ using Equations (1.4) and (1.3) shows that the coefficient of $\mathfrak{S}_{\zeta u}$ for $u \leq_k \zeta u$ is zero unless $\mathcal{L}(\zeta) = r$ and ζ is minimal. When ζ is the product of $s(\zeta)$ minimal cycles whose supports are disjoint, this coefficient is

$$(-1)^{\mathrm{ht}(\zeta)+1} \sum_{l=1}^{r} (-1)^{l-\mathrm{ht}(\zeta)} \binom{s(\zeta)-1}{\mathrm{ht}(\zeta)-l} \;=\; \begin{cases} 0, & \text{if } s(\zeta) \neq 1, \\ (-1)^{\mathrm{ht}(\zeta)+1}, & \text{if } s(\zeta) = 1. \end{cases} \qquad \square$$

This proof differs from that given in [14, 15].

2 A Murnaghan–Nakayama rule for quantum cohomology

The (small) quantum cohomology ring *$qH^*\mathbb{F}\ell_n$* of the flag manifold is a deformation the cohomology ring $H^*\mathbb{F}\ell_n$ whose product encodes the three-point Gromov–Witten invariants of $\mathbb{F}\ell_n$. To simplify notation, write $\mathbb{Z}[q]$ for $\mathbb{Z}[q_1, \ldots, q_{n-1}]$, where $q_1, \ldots, q_{n-1}$ are indeterminates. As a $\mathbb{Z}$-module, we have $qH^*\mathbb{F}\ell_n = \mathbb{Z}[q] \otimes_{\mathbb{Z}} H^*\mathbb{F}\ell_n$. It follows that the Schubert classes $\{\mathfrak{S}_w\}_{w \in S_n}$ in $H^*\mathbb{F}\ell_n$ form a $\mathbb{Z}[q]$-basis of $qH^*\mathbb{F}\ell_n$. We write $*$ to distinguish this *"quantum product"* from the *"classical product"* in $H^*\mathbb{F}\ell_n$.

Open Problem 2.1. *Find a combinatorial rule for the coefficients $C^w_{u,v} \in \mathbb{Z}[q]$ in the product*

$$\mathfrak{S}_u * \mathfrak{S}_v = \sum_{w \in S_n} C^w_{u,v} \mathfrak{S}_w. \tag{2.1}$$

Givental and Kim [8] proved an analogue of Borel's presentation, expressing $qH^*\mathbb{F}\ell_n$ as a quotient of $\mathbb{Z}[q][x]$. We write $\psi : \mathbb{Z}[q][x] \to qH^*\mathbb{F}\ell_n$ for the corresponding surjection. Fomin, Gelfand, and Postnikov [5] defined *quantum Schubert polynomials $\mathfrak{S}^q_w(x)$* $\in \mathbb{Z}[q][x]$ so that the quantum product of Schubert classes satisfies $\mathfrak{S}_u * \mathfrak{S}_v = \psi(\mathfrak{S}^q_u(x) \cdot \mathfrak{S}^q_v(x))$. To simplify notation, for $f(x) \in \mathbb{Z}[q][x]$, we will write $\mathfrak{S}_u * f(x)$ for $\psi(\mathfrak{S}^q_u(x) \cdot f(x))$.

As we already mentioned, if $\lambda \subseteq R_{k,n-k}$, then $s_\lambda(x_1, \ldots, x_k) = \mathfrak{S}_{v(\lambda,k)}(x)$. Analogously, for $\lambda \subseteq R_{k,n-k}$, define the *quantum Schur polynomials* as

$$s^q_\lambda(x_1, \ldots, x_k) := \mathfrak{S}^q_{v(\lambda,k)}(x),$$

and following Equation (1.4), define the *quantum power sum* as:

$$p^q_r(x_1, \ldots, x_k) := \\ s^q_{(r)}(x_1, \ldots, x_k) - s^q_{(r-1,1)}(x_1, \ldots, x_k) + \cdots + (-1)^{r+1} s^q_{(1^r)}(x_1, \ldots, x_k). \tag{2.2}$$

We derive a formula for $\mathfrak{S}_u * p^q_r(x)$ via intermediary formulas for $\mathfrak{S}_u * s^q_{(b,1^{a-1})}(x)$, as in the proof of Corollary 1.5.

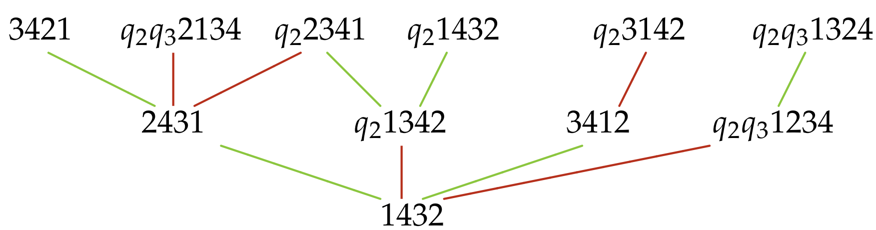
We begin with the *quantum Monk Formula* of Fomin, Gelfand, and Postnikov [5]: for $u \in S_n$ and $1 \le k < n$, $\mathfrak{S}_u * \mathfrak{S}_{(k,k+1)} = \sum \mathfrak{S}_{u(i,j)} + \sum q_{i,j}\mathfrak{S}_{u(i,j)}$, where the first sum is given by the classical Monk Formula and the second sum is over transpositions (i,j) with $i \le k < j$ and $\ell(u)+1 = \ell(u(i,j)) + 2(j-i)$, and $q_{i,j} := q_i q_{i+1} \cdots q_{j-1}$. For example,

$$\mathfrak{S}_{1432} * \mathfrak{S}_{(2,3)} = \mathfrak{S}_{2431} + \mathfrak{S}_{3412} + q_2\mathfrak{S}_{1342} + q_{2,4}\mathfrak{S}_{1234}.$$

The terms appearing in the quantum Monk Formula define a ranked partial order on $S_n[q] := \{q^\alpha u : u \in S_n \text{ and } q^\alpha \text{ is a monomial in } q_1, \ldots, q_{n-1}\}$ as follows. The *quantum k-Bruhat order $\le^q_k$* on $S_n[q]$ is defined by the following cover relations:

1. $u \lessdot^q_k u(i,j)$ if $i \le k < j$ and $\ell(u)+1 = \ell(u(i,j))$, so that $u \lessdot_k u(i,j)$;
2. $u \lessdot^q_k q_{i,j}u(i,j)$ if $i \le k < j$ and $\ell(u)+1 = \ell(u(i,j)) + 2(j-i)$; and
3. extend q-multiplicatively: $u \le^q_k w$ if and only if $q^\alpha u \le^q_k q^\alpha w$ for $u, w \in S_n[q]$.

We display two levels of the quantum 2-Bruhat order on $S_4[q]$ above 1432.



The rank function ℓ on the quantum k-Bruhat order is given by $\ell(q^\alpha u) := 2\deg q^\alpha + \ell(u)$, where $\deg q^\alpha$ is the usual degree of the monomial q^α in $\mathbb{Z}[q]$. As far as we know, there is no analogue of the Grassmannian–Bruhat order related to the quantum k-Bruhat order. Consequently, rather than speak of minimal permutations (as in the classical case), we will define minimal intervals in the quantum k-Bruhat order. Suppose that $u, w \in S_n$ are permutations and $u \le^q_k q^\alpha w$. We say that the interval $[u, q^\alpha w]^q_k$ is *minimal* if its rank $\ell(q^\alpha w) - \ell(u)$ equals $\#\operatorname{supp}(wu^{-1}) - s(wu^{-1})$. When $q^\alpha = 1$, the interval is minimal if and only if wu^{-1} is a minimal permutation.

We state our two main results.

Theorem 2.2. *Let $u \in S_n$, $l \le k$ and $m \le n-k$. Then*

$$\mathfrak{S}_u * s^q_{(m,1^{l-1})}(x_1, \ldots, x_k) = \sum \binom{s(wu^{-1})-1}{\mathrm{ht}(wu^{-1})-l} q^\alpha \mathfrak{S}_w\,,$$

summing over all minimal intervals $[u, q^\alpha w]^q_k$ such that $\ell(q^\alpha w) - \ell(u) = m+l-1$.

The following quantum Murnaghan–Nakayama formula was conjectured in [14]. It follows from Theorem 2.2 in the same way that Corollary 1.5 follows from Proposition 1.4.

Corollary 2.3. *Let $u \in S_n$. Then*

$$\mathfrak{S}_u * p_r^q(x_1, \dots, x_k) = \sum (-1)^{\mathrm{ht}(w^{-1}u)+1} q^\alpha \mathfrak{S}_w,$$

summing over all minimal intervals $[u, q^\alpha w]_k^q$ of rank r such that $w^{-1}u$ is a single cycle.

For example, in $qH^*\mathbb{F}\ell_8$, if $u = 38254671$, then the product $\mathfrak{S}_u * p_4^q(x_1, \dots, x_4)$ is

$$\begin{gathered}\mathfrak{S}_{(2,5,6,3,4)u} + \mathfrak{S}_{(2,3,5,6,4)u} - \mathfrak{S}_{(2,5,7,6,4)u} - \mathfrak{S}_{(3,5,7,6,4)u} - q_4\mathfrak{S}_{(2,4,7,6,5)u} - q_4\mathfrak{S}_{(3,4,7,6,5)u} \\ + q_4\mathfrak{S}_{(2,3,4,6,5)u} + q_{2,5}\mathfrak{S}_{(2,4,6,5,8)u} + q_{2,5}\mathfrak{S}_{(2,5,6,4,8)u} - q_{2,5}\mathfrak{S}_{(2,3,4,5,8)u} \\ + q_{4,8}\mathfrak{S}_{(1,7,6,5,4)u} - q_{2,8}\mathfrak{S}_{(1,7,6,5,8)u} + q_{2,8}\mathfrak{S}_{(1,8,2,3,4)u} + q_4 q_{2,8}\mathfrak{S}_{(1,8,2,4,5)u}\,.\end{gathered}$$

3 Sketch of Proof and Remarks

We prove Theorem 2.2 in two parts. In the first part we show that if $[u, q^\alpha w]_k^q$ is a minimal interval, then there is a hook partition λ with $|\lambda| = \ell(q^\alpha w) - \ell(u)$ such that $C_{v(\lambda,k),u}^{q^\alpha w}$ is nonzero. This uses a study of chains in the interval $[u, q^\alpha w]_k^q$ in terms of certain left operators modeled on those for the ordinary k-Bruhat order as developed in [2]. This is sketched in Section 3.1.

Our second part of the proof uses a Corollary to the proof of Theorem 1.2 in [10], the result of Leung and Li that "quantum equals classical". We discuss this in Section 3.2. A consequence is that if $C_{v(\lambda,k),u}^{q^\alpha w} \neq 0$ for some partition λ, then there are $y, z \in S_n$ with $y \leq_k z$, $\ell(z) - \ell(y) = \ell(q^\alpha w) - \ell(u) = |\lambda|$, and $zy^{-1} = wu^{-1}$ such that for all partitions μ with $|\mu| = |\lambda|$, $C_{v(\mu,k),u}^{q^\alpha w} = q^\alpha c_{v(\mu,k),y}^z$. If λ is a hook partition, then Proposition 1.4 implies that the interval $[u, q^\alpha w]_k^q$ is minimal. Then Theorem 2.2 follows from these results and Proposition 1.4.

We point out Theorem 15 of [12] also claims that quantum equals classical for hook partitions, and from which one could in principle deduce our Theorem 2.2. Unfortunately, its proof invokes a lemma (Lemma 13 in [12]) whose hypotheses do not apply to the case (hook partitions) that is invoked. We sketch this in Section 3.3. This necessitates the alternative proof we provide.

3.1 Left Operators

We now sketch a proof of the first part in the following result.

Theorem 3.1. *Let $u, w \in S_n$ and $q^\alpha w \in S_n[q]$. Then $[u, q^\alpha w]_k^q$ is a minimal quantum interval if and only if there is a hook partition λ with $|\lambda| = \ell(q^\alpha w) - \ell(u)$ such that $C_{v(\lambda,k),u}^{q^\alpha w} \neq 0$.*

A cover $y \lessdot_k^q q^\alpha z$ in the quantum k-Bruhat order gives a transposition $(a,b) = zy^{-1}$. Therefore, a chain in $[u, q^\alpha w]_k^q$ provides a factorization of $\zeta := wu^{-1}$ into transpositions

$$\zeta \;=\; (a_r, b_r) \;\cdots\; (a_2, b_2)\,(a_1, b_1)\,, \tag{3.1}$$

which in turn defines a (multi) graph with vertex set $[n]$ whose edges are (a_i, b_i) for each transposition above.

Lemma 3.2. *Suppose that $[u, q^\alpha w]_k^q$ is minimal and (3.1) is a factorization of $\zeta = wu^{-1}$ corresponding to a chain in $[u, q^\alpha w]_k^q$. Let F be the corresponding multigraph. Then*

1. *F is a forest: it is acyclic without multiple edges and its connected components are trees.*
2. *The vertices in connected components of F form a noncrossing partition of $[n]$.*
3. *ζ is a product of disjoint cycles, one for each tree in F.*

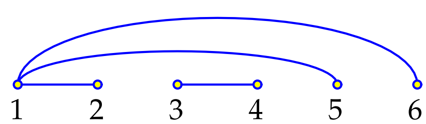
Example 3.3. We have the following chain in the quantum 3-Bruhat order:

$$\begin{aligned} 635214 = u \;\; &\lessdot_3^q \;\; q_{1,5}\,(1\,6)u \;\; \lessdot_3^q \;\; q_{1,5}\,(1\,2)(1\,6)u = q_{1,5}\,235164 \\ &\lessdot_3^q \;\; q_{1,5}\,(3\,4)(1\,2)(1\,6)u \;\; \lessdot_3^q \;\; q_{1,5}q_3\,(1\,5)(3\,4)(1\,2)(1\,6)u = q_1q_2q_3^2q_4\,241563. \end{aligned}$$

The interval $[635214, q_1q_2q_3^2q_4\,241563]_3^q$ has rank four, as

$$\ell(q_1q_2q_3^2q_4\,241563) = 10 + 5 = 15 \quad \text{and} \quad \ell(635214) = 11.$$

Set $\zeta = (1,5)(3,4)(1,2)(1,6) = (3,4)(1,6,2,5)$. Its support has six elements and ζ consists of two cycles, and thus this interval is minimal. The supports of the two cycles in ζ are non-crossing. We draw its corresponding forest.



Postnikov [17] observed that the quantum Bruhat order (union over k of all quantum k-Bruhat orders) has a cyclic symmetry. We describe this for the quantum k-Bruhat order. Write $\mathfrak{o}$ for the n-cycle $(1, 2, \ldots, n)$. If $i > j$, define $q_{i,j} := q_{j,i}^{-1}$, a Laurent monomial. For $u, w \in S_n$, define $q(u,w) := q_{w^{-1}(n), u^{-1}(n)}$.

Lemma 3.4. *Let $u, w \in S_n$. We have $u <_k^q q^\alpha w$ if and only if $\mathfrak{o}u <_k^q q(u,w) q^\alpha \mathfrak{o}w$.*

For $a < b$ in $[n]$, define an operator $\mathbf{v}_{a,b}$ on $S_n[q] \cup \{\mathbf{0}\}$ by $\mathbf{v}_{a,b} \bullet \mathbf{0} = \mathbf{0}$, and for $u \in S_n$,

$$\mathbf{v}_{a,b} \bullet u \;:=\; \begin{cases} (a,b)u & \text{if } u \lessdot_k (a,b)u, \\ q_{i,j}(b,a)u & \text{if } u \lessdot_k^q q_{i,j}(b,a)u \text{ with } u^{-1}(b) = i \leq k < u^{-1}(a) = j, \\ \mathbf{0} & \text{otherwise}\,, \end{cases}$$

and extend this action multiplicatively. We will say that $\mathbf{v}_{a,b}$ is a *classical operator* if the first case holds, and that it is a *quantum operator* when the second case holds.

A composition of operators $\mathbf{v} = \mathbf{v}_{a_r,b_r} \cdots \mathbf{v}_{a_1,b_1}$ is zero if for all $u \in S_n$, $\mathbf{v} \bullet u = \mathbf{0}$ and otherwise it is nonzero. If $\mathbf{v}' = \mathbf{v}_{c_r,d_r} \cdots \mathbf{v}_{c_1,d_1}$ is another composition of operators, then it is *equivalent to* $\mathbf{v}$ if for all $u \in S_n$, $\mathbf{v} \bullet u = \mathbf{v}' \bullet u$. When $\mathbf{v}$ is nonzero, there are $u, w \in S_n$ and $q^\alpha w \in S_n[q]$ such that $\mathbf{v} \bullet u = q^\alpha w$. In this case, $u \leq_k^q q^\alpha w$, $\mathbf{v}$ corresponds to a chain in $[u, q^\alpha w]_k^q$, and $\mathbf{v}$ corresponds to a factorization (3.1) of $\zeta = wu^{-1}$. Then the chains in $[u, q^\alpha w]_k^q$ correspond to compositions $\mathbf{v}'$ of operators equivalent to $\mathbf{v}$.

By Lemma 3.4, *cyclic shift* (multiplication by $\mathfrak{o}$) is an isomorphism of intervals $[u, q^\alpha w]_k^q$, and it acts (via conjugation) on chains in intervals and on the operators $\mathbf{v}_{a,b}$, preserving relations. If $b \neq n$, then $\mathfrak{o}\mathbf{v}_{a,b}\mathfrak{o}^{-1} = \mathbf{v}_{a+1,b+1}$ and this preserves classical/quantum operators. However, if $b = n$, then $\mathfrak{o}\mathbf{v}_{a,n}\mathfrak{o}^{-1} = \mathbf{v}_{1,a+1}$, and it interchanges classical operators with quantum operators. Key to our proof of the forward implication of Theorem 3.1 is showing that a chain in a minimal interval may be cyclically shifted (conjugated by some power of $\mathfrak{o}$) to become a chain of classical operators.

The relations among classical operators were determined in [2], and they are all of degree two or three. In particular, given two classical operators $\mathbf{v}_{a,b}$ and $\mathbf{v}_{c,d}$:

- they commute if $(a,b)(c,d)$ is non-crossing, in the sense of Lemma 3.2(2);
- their product is nonzero and they do not commute if $b = c$ or $a = d$; and
- their product is zero if the partition $\{a,b\} \sqcup \{c,d\}$ is crossing or if $a = c$ or $b = d$.

There is an interesting relation of degree three that we do not describe. If we also consider quantum operators, we do not have a complete set of relations, but do know those of degrees two and three, for they are induced from the classical relations by the cyclic symmetry of Lemma 3.4.

A composition $\mathbf{v} = \mathbf{v}_{a_r,b_r} \cdots \mathbf{v}_{a_1,b_1}$ is a *connected (classical) row* if $b_1 = a_2$, $b_2 = a_3$, …, $b_{r-1} = a_r$. It is a *connected (classical) column* if $a_1 = b_2, \ldots, a_{r-1} = b_r$. The graph with vertices $\{a_1, b_1, \ldots, a_r, b_r\}$ and edges $(a_i b_i)$ is a path in either case. A *classical row* is a composition of connected rows in which every pair has noncrossing support (a noncrossing composition), and the same for a *classical column*. A composition is a *row* (*column*) if may be cyclically shifted to a classical row (column).

Given these definitions, we may express Postnikov's quantum Pieri formula [16] as

$$\mathfrak{S}_u * s^q_{(m)}(x_1, \ldots, x_k) = \sum q^\alpha \mathfrak{S}_v,$$

summing over all $q^\alpha v = \mathbf{R} \bullet u$, where $\mathbf{R}$ is a row of length m. For quantum multiplication by $s^q_{1^m}(x_1, \ldots, x_k)$, replace row by column. A consequence of the Pieri formula and Lemma 3.4 is that if $u \leq_k^q q^\alpha w$ and λ is a partition with $|\lambda| = \ell(q^\alpha w) - \ell(u)$, then

$$q(u,w) C^{q^\alpha w}_{v(\lambda,k),u} = C^{q(u,w) q^\alpha \mathfrak{o} w}_{v(\lambda,k),\mathfrak{o} u}\,. \tag{3.2}$$

A composition $\mathbf{T} = \mathbf{v}_{a_t,b_t} \cdots \mathbf{v}_{a_1 b_1}$ is a *tree* if the graph with vertices $\{a_t, b_t, \ldots, a_1, b_1\}$ given by the corresponding factorization is a tree. By Lemma 3.2, if $[u, q^\alpha w]_k^q$ is a minimal quantum interval, then any chain has corresponding composition of operators that is a *noncrossing forest*: a composition of trees in which every pair has noncrossing support. We state our main technical result.

Lemma 3.5. *Suppose that* $\mathbf{v}$ *is a nonzero noncrossing forest. Then* $\mathbf{v}$ *is equivalent to the product of a row* $\mathbf{R}$ *and a column* $\mathbf{C}$*, and the product* $\mathbf{R}\cdot\mathbf{C}$ *may be cyclically shifted to the product of a classical row and a classical column.*

Proof of the forward implication of Theorem 3.1. Suppose that $[u, q^\alpha w]^q_k$ is a minimal interval. By Lemma 3.2, there is a noncrossing forest $\mathbf{F}$ of left operators of length $\ell(q^\alpha w) - \ell(u)$ with $q^\alpha w = \mathbf{F} \bullet u$. By Lemma 3.5, there are a row $\mathbf{R}$ and a column $\mathbf{C}$ of lengths m and l, respectively, with $m + l = \ell(q^\alpha w) - \ell(u)$ and $q^\alpha w = (\mathbf{R}\cdot\mathbf{C}) \bullet u$. By Lemma 3.5 again, $\mathbf{R}\cdot\mathbf{C}$ may be cyclically shifted to the product of a classical row and a classical column. If r is that shift, then $\mathfrak{S}_{\mathfrak{o}^r w}$ appears in

$$s_{(m)}(x_1,\ldots,x_k)\cdot s_{1^l}(x_1,\ldots,x_k)\cdot\mathfrak{S}_{\mathfrak{o}^r u} = \\ \left(s_{(m,1^l)}(x_1,\ldots,x_k) + s_{(m+1,1^{l-1})}(x_1,\ldots,x_k)\right)\cdot\mathfrak{S}_{\mathfrak{o}^r u},$$

which is a product in $H^*\mathbb{F}\ell_n$. By (3.2), at least one of $C^{q^\alpha w}_{v((m,1^l),k),u}$ or $C^{q^\alpha w}_{v((m+1,1^{l-1}),k),u}$ is nonzero. □

3.2 Quantum equals Classical

Leung and Li [10] gave a different identity among the quantum coefficients than (3.2). In it, the exponent of q^α is reduced. For $i \in [n-1]$ let e_i be the exponent of q_i, so that $q^{e_i} = q_i$. We state the results of Leung and Li, in the form that we use.

Proposition 3.6. *Suppose that* $u, w \in S_n$*,* $q^\alpha w \in S_n[q]$*, and that* λ *is a partition such that* $C^{q^\alpha w}_{v(\lambda,k),u} \neq 0$*. Then there is an* $i \in [n-1]$ *such that one of the following two conditions hold:*

1. *We have* $i \neq k$ *and* $1 = 2\alpha_i - \alpha_{i-1} - \alpha_{i+1}$*,* $\ell(w(i,i+1)) = \ell(w) - 1$*,* $\ell(u(i,i+1)) = \ell(u) + 1$*, and* $q^{\alpha - d_i} w(i,i+1) \geq^q_k u(i,i+1)$*.*
2. *We have* $i = k$ *and* $2 = 2\alpha_k - \alpha_{k-1} - \alpha_{k+1}$*,* $\ell(w(k,k+1)) = \ell(w) - 1$*,* $\ell(u(k,k+1)) = \ell(u) + 1$*, and* $q^{\alpha - e_k} w(k,k+1) \geq^q_k u(k,k+1)$*.*

Proposition 3.7. *Suppose that* $u, w \in S_n$*,* $q^\alpha w \in S_n[q]$*, and* $i \in [n-1]$*, and either condition of Proposition 3.6 holds. Then for any partition* λ*,* $C^{q^\alpha w}_{v(\lambda,k),u} = q_i C^{q^{\alpha - e_i} w(i,i+1)}_{v(\lambda,k),u(i,i+1)}$*.*

In [10] these are formulated in a weaker form; the stronger result of Proposition 3.7 is not stated. The journal version of this abstract will contain a complete explanation.

Observe that in the statement of Proposition 3.7, $wu^{-1} = (w(i,i+1))(u(i,i+1))^{-1}$. An induction on $\deg(q^\alpha)$ using Propositions 3.6 and 3.7 shows that if $C^{q^\alpha w}_{v(\lambda,k),u} \neq 0$, then there exist $y,z \in S_n$ with $y \leq_k z$, $|\lambda| = \ell(z) - \ell(y) = \ell(q^\alpha w) - \ell(u)$, $zy^{-1} = wu^{-1}$, and $q^\alpha c^z_{v(\lambda,k),y} = C^{q^\alpha w}_{v(\lambda,k),u} \neq 0$. If λ is a hook partition, then this implies that zy^{-1} is a minimal permutation and thus $[u, q^\alpha w]^q_k$ is minimal, which is the reverse implication in Theorem 3.1.

Proof of Theorem 2.2. By Theorem 3.1, $[u, q^\alpha w]^q_k$ is a minimal interval if and only if there is a hook partition $\lambda = (m, 1^{l-1})$ with $C^{q^\alpha w}_{v(\lambda),u} \neq 0$.

It remains to compute the coefficient $C^{q^\alpha w}_{v(\lambda,k),u}$. By the induction sketched above, and using the same notation, this coefficient equals $q^\alpha c^z_{v(\lambda,k),y}$. Now by Proposition 1.4, this equals $q^\alpha \binom{s(wu^{-1})-1}{\mathrm{ht}(wu^{-1})-l}$, as $wu^{-1} = zy^{-1}$. □

3.3 Remarks

Proposition 1.4 may be deduced from a stronger result of Mészáros, Panova, and Postnikov [12, Thm. 8]. There, they establish a formula for a 'lift' of the Schur polynomial $s_{(m,1^{l-1})}(x_1, \dots, x_k)$ to the Fomin–Kirillov [6] algebra $\mathcal{E}_n$, which is a subalgebra (generated by certain Dunkl elements θ_i) of the quotient of a free associative algebra with generators $x_{i,j}$ for $1 \leq i < j \leq n$ by certain relations, which include $x^2_{i,j} = 0$. Their paper contains another result [12, Thm. 15] which is a quantum analog of [12, Thm. 8], giving a formula for a 'quantum lift' $s^q_{(m,1^{l-1})}(x_1, \dots, x_k)$ in the quantum Fomin–Kirillov algebra $\mathcal{E}^q_n$ [16]. This formula would imply Theorem 2.2. Unfortunately, as we now sketch, there is a gap in the proof of [12, Thm. 15] (we have communicated this to the authors).

In [12], the result [12, Thm. 15] is deduced from the proof of [12, Thm. 8] using a technical result [12, Lem. 13] which roughly asserts the following: If an identity involving positive terms holds in $\mathcal{E}_n$, and if no relations $x^2_{i,j} = 0$ were used to deduce the identity, then the same identity holds in $\mathcal{E}^q_n$. We are convinced this lemma is correct, but it unfortunately does not apply in the case invoked to prove [12, Thm. 15].

In addition to using relations (but not $x^2_{i,j} = 0$), the proof of [12, Thm. 8] uses in a fundamental way [12, Cor. 10], that $e_a(\theta_1, \dots, \theta_a) h_b(\theta_1, \dots, \theta_a) = 0$ in $\mathcal{E}_{a+b}$ (in their notation). However, this identity can only be proven directly in $\mathcal{E}_{a+b}$ using the relations $x^2_{i,j} = 0$. For example, when $a = b = 1$, as $\theta_1 = x_{1,2}$, it becomes $0 = e_1(x_{1,2}) h_1(x_{1,2}) = x^2_{1,2}$.

We can prove a stronger version of Theorem 3.1 which is nearly a quantum analog of Proposition 1.2. Its arguments involve a significantly more detailed study of the left operators and their relations than our proof of Theorem 3.1.

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References

[1] N. Bergeron and F. Sottile. "Schubert polynomials, the Bruhat order, and the geometry of flag manifolds". *Duke Math. J.* **95**.2 (1998), pp. 373–423.

[2] N. Bergeron and F. Sottile. "A monoid for the Grassmannian Bruhat order". *European J. Combin.* **20**.3 (1999), pp. 197–211.

[3] N. Bergeron and F. Sottile. "A Pieri-type formula for isotropic flag manifolds". *Trans. Amer. Math. Soc.* **354**.7 (2002), pp. 2659–2705.

[4] A. Borel. "Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts". *Ann. of Math. (2)* **57** (1953), pp. 115–207.

[5] S. Fomin, S. Gelfand, and A. Postnikov. "Quantum Schubert polynomials". *J. Amer. Math. Soc.* **10**.3 (1997), pp. 565–596.

[6] S. Fomin and A. N. Kirillov. "Quadratic algebras, Dunkl elements, and Schubert calculus". *Advances in geometry*. Vol. 172. Progr. Math. Birkhäuser, Boston, MA, 1999, pp. 147–182.

[7] W. Fulton. *Young tableaux*. Vol. 35. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 1997.

[8] A. Givental and B. Kim. "Quantum cohomology of flag manifolds and Toda lattices". *Comm. Math. Phys.* **168**.3 (1995), pp. 609–641.

[9] A. Lascoux and M.-P. Schützenberger. "Polynômes de Schubert". *C. R. Acad. Sci. Paris Sér. I Math.* **294**.13 (1982), pp. 447–450.

[10] N. C. Leung and C. Li. "Classical aspects of quantum cohomology of generalized flag varieties". *Int. Math. Res. Not. IMRN* 16 (2012), pp. 3706–3722.

[11] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. Second edition with contributions by A. Zelevinsky. The Clarendon Press, Oxford University Press, New York, 1995, pp. x+475.

[12] K. Mészáros, G. Panova, and A. Postnikov. "Schur times Schubert via the Fomin-Kirillov algebra". *Electron. J. Combin.* **21**.1 (2014), Paper 1.39, 22.

[13] D. Monk. "The Geometry of Flag Manifolds". *Proc. London Math. Soc.* **9** (1959), pp. 253–286.

[14] A. Morrison. "A Murgnahan-Nakayama rule for Schubert polynomials". *26th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2014)*. Assoc. Discrete Math. Theor. Comput. Sci., Nancy, 2014, pp. 525–536.

[15] A. Morrison and F. Sottile. "Two Murnaghan-Nakayama rules in Schubert Calculus". *Annals of Combinatorics* **22**.2 (2018), pp. 363–375.

[16] A. Postnikov. "On a quantum version of Pieri's formula". *Adv. in Geometry* **172** (1999), pp. 371–383.

[17] A. Postnikov. "Symmetries of Gromov-Witten invariants". *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*. Vol. 276. Contemp. Math. Amer. Math. Soc., Providence, RI, 2001, pp. 251–258.

[18] F. Sottile. "Pieri's formula for flag manifolds and Schubert polynomials". *Ann. Inst. Fourier (Grenoble)* **46**.1 (1996), pp. 89–110.