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On the A_2 Andrews–Schilling–Warnaar identities

Shashank Kanade*[1]

[1]*Department of Mathematics, University of Denver, USA*

Abstract. In a groundbreaking work, Andrews–Schilling–Warnaar invented an A_2 generalization of the A_1 Bailey machinery and discovered many identities related to the principal characters of standard modules for $\widehat{\mathfrak{sl}}_3$, or equivalently, for the vertex operator algebra $\mathcal{W}_3(3,p')$. Jointly with Russell, we have given conjectures for completing this set of identities and proved these conjectures for small values of p'. In another direction, character of $\mathcal{W}_r(p,p')$ has been related to an appropriate limit of certain $\mathfrak{sl}_r$ coloured Jones polynomials of torus knots $\mathrm{T}(p,p')$ under some restrictions on r,p,p'. This note summarizes these developments.

Keywords: Rogers–Ramanujan-type identities, Affine Lie algebras, Vertex operator algebras, Jones polynomials

1 Introduction

1.1 Rogers–Ramanujan identities and their origins

Let us begin by recalling the celebrated pair of Rogers–Ramanujan identities:

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q)_n}=\frac{1}{\theta(q;q^5)},\quad \sum_{n\geq 0}\frac{q^{n^2+n}}{(q)_n}=\frac{1}{\theta(q^2;q^5)}. \tag{1.1}$$

where we use the standard notations for the Pochhammer symbols $(q)_n$ (and more generally $(a;q)_n$) for $n\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$ as found in [2] and we use the notation:

$$\theta(a_1,a_2,\cdots,a_k;q)=\prod_{1\leq i\leq k}(a_i;q)_\infty(q/a_i;q)_\infty. \tag{1.2}$$

Rogers–Ramanujan identities, their various generalizations, and analogues appear in many branches of mathematics. Most importantly for us, they appear in the study of affine Lie algebras and vertex operator algebras (henceforth, VOAs). Secondly, they also appear in relation to the coloured Jones polynomials of knots and links.

*shashank.kanade@du.edu. S. Kanade is supported by Collaboration Grant for Mathematicians #636937 awarded by the Simons Foundation.

The identities (1.1) were first proved completely representation-theoretically by Lepowsky and Wilson [18, 19, 20, 21] by considering standard (i.e., integrable, highest-weight) level 3 modules for the affine Lie algebra $\widehat{\mathfrak{sl}}_2$. Identities (1.1) also arise by considering the representation theory of Virasoro VOA $\mathsf{Vir}(2,5)$. In general, representation theory of VOAs is an enormously rich source of Rogers–Ramanujan-type identities having the "sum=product" shape of (1.1).

On the knot-theoretical side, the second Rogers–Ramanujan identity in (1.1) seems to have been proved first by Armond and Dasbach [6] by considering tails of Jones polynomials for torus knots $\mathrm{T}(2,5)$ coloured with finite dimensional irreducible $\mathfrak{sl}_2$ modules. Here, the product side is a result of using the Rosso–Jones formula [24] as explained by Morton, [23]. The sum side is calculated using a combinatorial walk model [7]. More generally, Armond and Dasbach proved one Andrews–Gordon identity at each modulus $2k+1$ by considering the torus knot $\mathrm{T}(2,2k+1)$.

1.2 Bailey machinery

In the world of q-series, Bailey machinery provides a highly productive mechanism to prove Rogers–Ramanujan-type identities. By Bailey machinery, we mean the framework that takes as input a Bailey pair (as defined in [1], for instance) and modifies this pair iteratively (using the Bailey lemma, the Bailey lattice, or any other suitable transformations). The modified pairs then lead to requisite identities by taking appropriate limits.

From the point of view of representation theory and knot theory, this mechanism is especially well suited to handle cases that are closely related to the Lie algebra $\mathfrak{sl}_2$. In particular, it produces identities related to principal characters (see (2.3) below) for standard modules for affine Lie algebras $\widehat{\mathfrak{sl}_2} = \mathrm{A}_1^{(1)}$ and $\mathrm{A}_2^{(2)}$, rational Virasoro algebras $\mathsf{Vir}(p,q)$, etc. This machinery has also been successfully used (see [15], for instance) in understanding Jones polynomials of links coloured with irreducible, finite-dimensional $\mathfrak{sl}_2$ modules.

For these and some more reasons explained in [3], we refer to the classical Bailey machinery by the designation $\mathfrak{sl}_2$ or A_1.

1.3 Andrews–Schilling–Warnaar's A_2 Bailey machinery

In [3], Andrews–Schilling–Warnaar invented an A_2 generalization of the A_1 Bailey machinery. Specifically, they defined a notion of the A_2 Bailey pair, found the unit A_2 Bailey pair, and gave a transformation law for A_2 Bailey pairs. Note that this transformation law is not yet as general as the one for A_1 Bailey pairs.

In [3], the authors then used this mechanism to discover "sum=product" identities related to principal characters of certain standard modules at every level $\ell \in \mathbb{Z}_{>0}$ for the affine Lie algebra $\widehat{\mathfrak{sl}_3}$. At levels $\ell \in \mathbb{Z}_{>0}$ coprime to 3, these principal characters also arise by considering characters of modules for the rational VOA $\mathcal{W}_3(3, \ell+3)$ (see [3, 12]).

The actual number of possible identities at level $\ell \in \mathbb{Z}_{>0}$ grows as a quadratic in ℓ, but the number of identities found in [3] grows linearly in ℓ. Consequently, a major subset of identities were yet to be found.

Our recent work [17] gives precise conjectures for all these missing identities. We are further able to prove our conjectures at levels 3 and 7. A further family of results valid for all levels divisible by 3 is detailed in Section 3 below.

1.4 Certain $\mathfrak{sl}_r$ coloured Jones polynomials of torus knots

Morton's work [23] effectively relates characters of $\mathsf{Vir}(p,q)$ VOA to $\mathfrak{sl}_2$ coloured Jones polynomials of torus knots $\mathrm{T}(p,q)$ where $2 \le p < q$ are coprime. Now, $\mathsf{Vir}(p,q)$ VOA is the $r=2$ member in the family of principal $\mathcal{W}_r$ algebras of type A. It is thus natural to investigate whether characters of $\mathcal{W}_r(p,q)$ algebras are analogously related to torus knots. In [16] we show that limits of certain $\mathfrak{sl}_r$ coloured invariants of torus knots $\mathrm{T}(p,q)$ similarly give rise to characters of $\mathcal{W}_r(p,q)$ VOAs. For precise conditions and statements, see in Section 4. Importantly, our hope is that generalizing the work of Armond and Dasbach [6] to $\mathfrak{sl}_3$ coloured invariants of torus knots $\mathrm{T}(3, p+3)$ will put the Andrews–Schilling–Warnaar identities in a very different light.

1.5 Recent and related developments

Recently, Andrews–Schilling–Warnaar's A_2 identities have gained much attention. Especially, the interested reader may look at [10], [29], [26], [27] and [30].

2 Foundations

In this section, we begin by collecting the basic setup required for this note.

2.1 Notation regarding the Lie algebras $\mathfrak{sl}_r$

As usual, we let the simple roots of $\mathfrak{sl}_r$ be $\alpha_1, \dots, \alpha_{r-1}$. The set of positive roots will be denoted by Φ_r^+ and the set of all roots will be Φ_r. The fundamental weights will be denoted by $\Lambda_1, \dots, \Lambda_{r-1}$. Q_r and P_r will denote the root and the weight lattices, respectively. The Weyl group will be denoted by $\mathfrak{S}_r$. The lattices $Q_r \subsetneq P_r$ come equipped with a symmetric, positive definite and $\mathfrak{S}_r$-invariant bilinear form $(\cdot,\cdot)$ such that $(\alpha_i, \alpha_i) = 2$ for all $0 \le i \le r-1$. We will let $\|\cdot\|$ be the norm corresponding to $(\cdot,\cdot)$. The Weyl vector will be denoted by δ. Recall that $w \mapsto (-1)^{\ell(w)}$ is the sign representation of $\mathfrak{S}_r$ where $\ell(\cdot)$ is the standard length function on $\mathfrak{S}_r$.

2.2 Affine Lie algebras $\widehat{\mathfrak{sl}_r}$

For $\widehat{\mathfrak{sl}_r}$, we will reuse some notation from $\mathfrak{sl}_r$ case. We shall provide adequate clarification in the cases of potential confusion.

For $0 \leq i \leq r-1$, let α_i be the simple roots and let Λ_i be the fundamental weights of the affine Lie algebra $\widehat{\mathfrak{sl}_r}$. Fix $\ell \in \mathbb{Z}_{\geq 0}$ (called the level) and let $\lambda = c_0\Lambda_0 + \cdots + c_{r-1}\Lambda_{r-1}$, where $c_i \in \mathbb{Z}_{\geq 0}$ and $c_0 + \cdots + c_{r-1} = \ell$. By $L(\lambda)$ we denote the irreducible highest-weight $\widehat{\mathfrak{sl}_r}$ module with highest-weight λ. The character of this module is denoted by $\mathrm{ch}(L(\lambda))$ and it belongs to:

$$\mathrm{ch}(L(\lambda)) \in e^{\lambda}\mathbb{Z}[[e^{-\alpha_0}, \cdots, e^{-\alpha_{r-1}}]]. \tag{2.1}$$

Here, e^{λ} and e^{α_i} are formal symbols. We define the *principally specialized character* and the *principal character*[1] of $L(\lambda)$ to be respectively –

$$\chi(L(\lambda)) = \left(e^{-\lambda}\,\mathrm{ch}(L(\lambda))\right)\big|_{e^{-\alpha_0}\mapsto q, \dots, e^{-\alpha_{r-1}}\mapsto q}, \tag{2.2}$$

$$\chi(\Omega(\lambda)) = \frac{\chi(L(\lambda))}{\chi(L(\Lambda_0))} = \frac{(q)_\infty}{(q^r;q^r)_\infty}\chi(L(\lambda)) \tag{2.3}$$

Due to the Dynkin diagram symmetries in the $\widehat{\mathfrak{sl}_r}$ case, we also have that:

$$\chi(\Omega(\lambda)) = \chi(\Omega(\sigma\lambda)), \tag{2.4}$$

where σ is any dihedral permutation of $c_0, c_1, \dots, c_{r-1}$. In the case of $\widehat{\mathfrak{sl}_3}$, we have the following for $c_0, c_1, c_2 \in \mathbb{Z}_{\geq 0}$, $c_0 + c_1 + c_2 = \ell$, $m = \ell + 3$ (see [12, 28], etc.):

$$\chi(\Omega(c_0\Lambda_0 + c_1\Lambda_1 + c_2\Lambda_2)) = \frac{(q^m;q^m)_\infty^2}{(q)_\infty^2}\theta(q^{c_0+1};q^m)\theta(q^{c_1+1};q^m)\theta(q^{c_2+1};q^m). \tag{2.5}$$

2.3 Principal $\mathcal{W}_r$ algebras

Let $r \leq p, p'$ be a pair of coprime integers. Now, principal $\mathcal{W}_r(p,p')$ algebras are certain simple vertex operator algebras (VOAs) that can be obtained as the quantum Hamiltonian reductions of affine VOAs based on $\widehat{\mathfrak{sl}_r}$ at levels $\frac{p}{p'} - r$. For deep and foundational properties of these VOAs, see [4, 5].

From [22] and [12], we recall characters of $\mathcal{W}_r(p,p')$ VOAs and their modules.

1. In the notation of [12], the character of the VOA $\mathcal{W}_r(p,p')$ is $\chi_{0,0}^{r,p,p'}$ and up to a pure power of q it equals the following normalized character:

$$\overline{\chi}_{0,0}^{r,p,p'} = (q)_\infty^{1-r}\sum_{\alpha\in Q_r}\sum_{\sigma\in\mathfrak{S}_r}(-1)^{\ell(\sigma)}q^{\frac{1}{2}pp'\|\alpha\|^2 - p'(\alpha,\delta) + p(\alpha,\sigma(\delta)) - (\delta,\sigma(\delta)-\delta)}. \tag{2.6}$$

[1]Note that the first equality in (2.3) as a definition of principal characters is only valid for the affine Lie algebras of type $X_r^{(t)}$ with $X = \mathrm{A}, \mathrm{D}, \mathrm{E}$ and $t \in \{1,2,3\}$.

2. When $p = r$, inequivalent irreducible modules of $\mathcal{W}_r(r, p')$ are enumerated by $\mathfrak{sl}_r$ weights $\zeta = c_1\Lambda_1 + \cdots + c_{r-1}\Lambda_{r-1} \in P_r$ such that $c_1 + \cdots + c_{r-1} \le p' - r$. The character of the corresponding irreducible $\mathcal{W}_r(r, p')$ module is denoted by $\chi^{r,r,p'}_{0,\zeta}$ [12] and up to a pure power of q it equals the normalized character:

$$\overline{\chi}^{r,r,p'}_{0,\zeta} = (q)^{1-r}_\infty \sum_{\alpha\in Q_r} \sum_{\sigma\in\mathfrak{S}_r} (-1)^{\ell(\sigma)} q^{\frac{1}{2}rp'\|\alpha\|^2 - p'(\alpha,\delta) + r(\alpha,\sigma(\zeta+\delta)) - (\delta,\sigma(\zeta+\delta)-\zeta-\delta)}. \tag{2.7}$$

2.4 Cylindric partitions

We let $r \ge 1$ and $\ell \ge 1$.

A *composition* of ℓ of length r is an (ordered) sequence of non-negative integers $c = (c_0, c_1, \ldots, c_{r-1})$ that adds up to ℓ. Note crucially that ℓ stands for the level and r for rank as in Section 2.2.

A *cylindric partition* of profile c is defined to be an (ordered) sequence of partitions $\Lambda = (\lambda^{(0)}, \ldots, \lambda^{(r-1)})$ such that:

1. Each $\lambda^{(j)} = \lambda^{(j)}_1 + \lambda^{(j)}_2 + \cdots$ is written in a non-increasing order and is assumed to continue indefinitely by appending an infinite string of zeroes.

2. For all $0 \le i \le r-2$ and all j, $\lambda^{(i)}_j \ge \lambda^{(i+1)}_{j+c_{i+1}}$ and $\lambda^{(r-1)}_j \ge \lambda^{(0)}_{j+c_0}$.

Given a composition $c = (c_0, \ldots, c_{r-1})$ of ℓ, define its set of non-negative indices by

$$I_c = \{0 \le i \le r-1 \mid c_i > 0\}.$$

For $\emptyset \subsetneq J \subseteq I_c$, define $c(J) = (c_0(J), \ldots, c_{r-1}(J))$ as

$$c_i(J) = \begin{cases} c_i - 1 & i \in J, i-1 \notin J \\ c_i + 1 & i \notin J, i-1 \in J \\ c_i & \text{otherwise,} \end{cases} \tag{2.8}$$

where we think of the indices $0, \ldots, r-1$ (and consequently elements of the sets I_c and J) as elements of the cyclic group $\mathbb{Z}_r$. Note that if c is a composition of ℓ, then so is $c(J)$.

Let $\mathcal{C}_c$ be the collection of all cylindric partitions of profile c and define:

$$H_c(z, q) = \frac{(zq)_\infty}{(q)_\infty} \sum_{\Lambda\in\mathcal{C}_c} z^{\max(\Lambda)} q^{\mathrm{wt}(\Lambda)}. \tag{2.9}$$

Now, the following properties of cylindric partitions and the functions H_c are crucial.

1. The functions H_c are governed by the Corteel–Welsh recurrence [11]. For a fixed r and ℓ, the functions H_c as c varies over length r compositions of ℓ are the unique solutions in $\mathbb{Z}[[z,q]]$ to the following finite system of recurrences and initial conditions [2]:
$$H_c(z,q) = \sum_{\emptyset \subsetneq J \subseteq I_c} (-1)^{|J|-1}(zq)_{|J|-1} H_{c(J)}(zq^{|J|}, q), \tag{2.10}$$
$$H_c(0,q) = (q)_\infty^{-1}, \quad H_c(z,0) = 1. \tag{2.11}$$

2. H_c appear as characters as follows ([8, 25, 14, 12]):
$$(q)_\infty H_{(c_0,\ldots,c_{r-1})}(1,q) = \chi(\Omega(c_0\Lambda_0 + \cdots + c_{r-1}\Lambda_{r-1})) = \overline{\chi}^{r,r,\ell+r}_{0,\,c_1\Lambda_1+\cdots+c_{r-1}\Lambda_{r-1}}. \tag{2.12}$$

3. H_c functions have certain symmetries. The first follows from the definition of cylindric partitions. The second follows from the previous relation to principal characters, but can also be deduced from the properties of cylindric partitions.
$$H_{(c_0,\cdots,c_{r-1})}(z,q) = H_{(c_1,c_2,\ldots,c_{r-1},c_0)}(z,q), \tag{2.13}$$
$$H_{(c_0,\cdots,c_{r-1})}(1,q) = H_{(c_{r-1},c_{r-2},\ldots,c_1,c_0)}(1,q). \tag{2.14}$$
Due to (2.13), we cyclically permute c and assume that c_0 is the largest part.

3 Completing the A_2 identities from [3]

In this section, we will now solely focus on the $r = 3$ case. That is, we will only consider principal characters of standard $\widehat{\mathfrak{sl}_3}$ modules of level $\ell \in \mathbb{Z}_{>0}$. Equivalently, these are the characters $\overline{\chi}^{3,3,3+\ell}_{0,\zeta}$ for irreducible modules for the VOA $\mathcal{W}_3(3, 3+\ell)$ whenever ℓ is coprime to 3, however, we will stick with the former description.

In [3], using an appropriate generalization of the usual Bailey machinery to the A_2 root system, "sum=product" identities involving certain principal characters were found. For identities found in [3], the triple (c_0, c_1, c_2) appearing in (2.5) satisfies:
$$\begin{aligned}&(c_{\sigma(0)} + 1, c_{\sigma(1)} + 1, c_{\sigma(2)} + 1)\\ &\in \{(i, i, 3k - 2i + s), (1, k - t, 2k + t - 1 \pm 1) \mid k \geq 2;\ 1 \leq i \leq k;\ t = 0, 1;\ s = -1, 0, 1\}.\end{aligned} \tag{3.1}$$
where σ is some permutation of the indices $0, 1, 2$ (recall (2.4)). This implies that a majority of identities at every level were yet to be found. In [17], jointly with Russell, we were able to conjecture all of the missing identities and prove our conjectures at levels 3 and 7. In fact, our conjectures pertain to the two-variable generating functions $H_c(z,q)$. Upon setting $z = 1$, one gets conjectures for the principal characters of standard modules (see (2.12)).

[2] The referee has kindly pointed out that $H_c(z,0) = 1$ from (2.11) is redundant.

3.1 The arrangement of modules

Fix $\ell > 0$ and let $m = \ell + 3$ and let k be such that $m = \ell + 3 \in 3k + \{-1, 0, 1\}$.

We now arrange length 3 compositions of ℓ in a triangular array. Suppose that $c = (c_0, c_1, c_2)$ is a composition of ℓ. Due to the cyclic symmetry (2.13), we assume that c_0 is the largest part of c. Let $(\lambda_0, \lambda_1, \lambda_2)$ be the sequence obtained by permuting (c_0, c_1, c_2) in a weakly decreasing order. Now place the composition c in row λ_1 and column λ_2. This gives rise to a triangular arrangement of compositions. Lastly, we draw a horizontal line after row $k-1$. For example, here is the arrangement for $\ell = 11, m = 14, k = 5$:

$$\begin{array}{lllll}
\text{row } 0: & 11,0,0 & & & \\
\text{row } 1: & 10,1,0/10,0,1 & 9,1,1 & & \\
\text{row } 2: & 9,2,0/9,0,2 & 8,2,1/8,1,2 & 7,2,2 & \\
\text{row } 3: & 8,3,0/8,0,3 & 7,3,1/7,1,3 & 6,3,2/6,2,3 & 5,3,3 \\
\text{row } 4: & 7,4,0/7,0,4 & 6,4,1/6,1,4 & 5,4,2/5,2,4 & 4,4,3 \\
\hline
\text{row } 5: & 6,5,0/6,0,5 & 5,5,1 & &
\end{array} \tag{3.2}$$

Theorem 1. *[17, Thm 5.1] Fix $\ell > 0$. Suppose that expressions for all $H_c(z, q)$ are known where c runs over length 3 compositions of ℓ that lie above the line, i.e., $\lambda_1 \leq k - 1$. Then, the expressions for $H_c(z, q)$ where c is a length 3 composition of ℓ that lies below the line, i.e., $\lambda_1 \geq k$ are determined by the Corteel–Welsh recursions (2.10).*

This theorem implies that it is enough to produce conjectures for the compositions that lie above the line.

3.2 Conjectures for compositions above the line

Let us continue to let ℓ, m, k as in the previous subsection. For $0 \leq j \leq k-1$, define:

$$\mathrm{e_j} = [\overbrace{0, \ldots, 0}^{j}, 1, \ldots, 1] \in \mathbb{Z}^{k-1}, \quad \mathrm{e}_{-1} = [2, 1, \ldots, 1] \in \mathbb{Z}^{k-1}. \tag{3.3}$$

For $\rho, \sigma \in \mathbb{Z}^{k-1}$, define (with the convention that $(q)_n^{-1} = 0$ if $n < 0$):

$$\begin{aligned}
S_{3k-1}(\rho \mid \sigma) &= \sum_{r,s \in \mathbb{Z}_{\geq 0}^{k-1}} \frac{z^{r_1} q^{\left(\sum_{1 \leq i \leq k-1} r_i^2 - r_i s_i + s_i^2\right) + \rho \cdot r + \sigma \cdot s}}{\prod_{1 \leq i \leq k-2} (q)_{r_i - r_{i+1}} (q)_{s_i - s_{i+1}}} \cdot \frac{q^{2 r_{k-1} s_{k-1}}}{(q)_{r_{k-1}} (q)_{s_{k-1}} (q)_{r_{k-1} + s_{k-1} + 1}} \\
S_{3k+1}(\rho \mid \sigma) &= \sum_{r,s \in \mathbb{Z}_{\geq 0}^{k-1}} \frac{z^{r_1} q^{\left(\sum_{1 \leq i \leq k-1} r_i^2 - r_i s_i + s_i^2\right) + \rho \cdot r + \sigma \cdot s}}{\prod_{1 \leq i \leq k-2} (q)_{r_i - r_{i+1}} (q)_{s_i - s_{i+1}}} \cdot \frac{1}{(q)_{r_{k-1}} (q)_{s_{k-1}} (q)_{r_{k-1} + s_{k-1} + 1}} \\
S_{3k}(\rho \mid \sigma) &= \sum_{r,s \in \mathbb{Z}_{\geq 0}^{k-1}} \frac{z^{r_1} q^{\left(\sum_{1 \leq i \leq k-1} r_i^2 - r_i s_i + s_i^2\right) + \rho \cdot r + \sigma \cdot s}}{\prod_{1 \leq i \leq k-2} (q)_{r_i - r_{i+1}} (q)_{s_i - s_{i+1}}} \cdot \frac{1}{(q)_{r_{k-1} + s_{k-1}} (q)_{r_{k-1} + s_{k-1} + 1}} \begin{bmatrix} r_{k-1} + s_{k-1} \\ r_{k-1} \end{bmatrix}_{q^3}
\end{aligned} \tag{3.4}$$

Conjecture 1. *[17, Conj. 5.1] Fix ℓ and let $c = (c_0, c_1, c_2)$ be a composition above the line (i.e., $\lambda_1 \le k-1$). We then have:*

$$H_{(c_0,c_1,c_2)}(z,q) = \begin{cases} S_{\ell+3}(\mathrm{e}_{c_1} \mid \mathrm{e}_{c_2}) - qS_{\ell+3}(\mathrm{e}_{c_1-1} \mid \mathrm{e}_{c_2-1}) & c_1, c_2 > 0 \\ S_{\ell+3}(\mathrm{e}_{c_1} \mid \mathrm{e}_0) & c_2 = 0 \\ S_{\ell+3}(\mathrm{e}_0 \mid \mathrm{e}_{c_2}) - q(1-z)S_{\ell+3}(\mathrm{e}_{-1} \mid \mathrm{e}_{c_2-1}) & c_1 = 0, c_2 \neq 0 \end{cases} \tag{3.5}$$

Theorem 2. *[17, Thm 5.3] The conjectures above satisfy the initial conditions (2.11), i.e., $H_c(0,q) = (q)_\infty^{-1}$ and $H_c(z,0) = 1$.*

3.3 Proving the conjectures at low levels and other results

We were able to prove our conjectures in some specific (small) levels:

Theorem 3. *[17] Conjecture 1 is true for $\ell \in \{2,3,4,5,7\}$ (the cases $\ell \in \{3,7\}$ are new).*

Idea of proof. Broadly, the proof has two steps. First we find certain fundamental (recurrence) relations satisfied by the S functions defined in (3.4). This step is straight-forward (see also [9]). The next step is to show that the Corteel–Welsh recursions (2.10) are a consequence of these fundamental relations. This step requires a non-trivial amount of computer assistance, especially in levels 5 and 7. □

Recently, Uncu has extended the theorem above to include two further cases.

Theorem 4. *[27] Conjecture 1 is true for $\ell \in \{8, 10\}$.*

In an important development, S. O. Warnaar has given a far reaching enhancement of the A_2 Bailey machinery of [3] and proved the following.

Theorem 5. *[30] Conjecture 1 is true with $z = 1$.*

In the case of $3 \mid \ell$ and with $z = 1$, we are in fact able to improve Theorem 1. In this case, we prove concrete formulas for $H_c(1,q)$ when c is below the line. This process crucially uses an identity of Weierstraß (see [17, Lem 6.1]). Note that the original formulation in [17] of the theorem below was conditional upon the validity of Conjecture 1 with $z = 1$, which in turn has been proved in [30].

Theorem 6. *[17, Thm. 6.4], [30] [3] Let $k \ge 3$. Let $c = (c_0, c_1, c_2)$ with $c_0 \ge c_1 \ge c_2 \ge 0$, $c_0 + c_1 + c_2 = \ell = 3k-3$, and $c_1 \ge k$. Then:*

$$H_c(1,q) = X(1,q) - q^{c_2+1}Y(1,q), \tag{3.6}$$

[3] Here we correct two minor errors in [17]: In equations (6.10) and (6.11) of [17], S_{3m} are supposed to be S_{3k}. Additionally, in equations (6.7) and (6.10) of [17], the second case never occurs.

where

$$X(z,q) = S_{3k}(\mathrm{e}_{2k-c_1-2} \mid \mathrm{e}_{2k-c_0-2}) - qS_{3k}(\mathrm{e}_{2k-c_1-3} \mid \mathrm{e}_{2k-c_0-3}), \tag{3.7}$$

$$Y(z,q) = \begin{cases} S_{3k}(\mathrm{e}_{c_0-k} \mid \mathrm{e}_{c_1-k}) - qS_{3k}(\mathrm{e}_{c_0-k-1} \mid \mathrm{e}_{c_1-k-1}), & c_1 > k \\ S_{3k}(\mathrm{e}_{c_0-k} \mid \mathrm{e}_0) & c_1 = k. \end{cases} \tag{3.8}$$

3.4 Examples

As examples, we single out two new identities that emerge from the theorems above. In both, $\sum$ stands for a sum over $r_1, r_2, s_1, s_2 \geq 0$.

$$\begin{aligned} &\sum \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 - r_2 s_2 + s_2^2 + r_2 + s_2}(1 - 2q^{1+r_1+s_1})}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2+s_2}(q)_{r_2+s_2+1}} \begin{bmatrix} r_2 + s_2 \\ r_2 \end{bmatrix}_{q^3} \\ &\quad = \frac{1}{(q)_\infty} \frac{1}{\theta(q, q^2, q^2, q^3, q^3; q^9)_\infty} = \frac{\chi(\Omega(3\Lambda_0 + 3\Lambda_1))}{(q)_\infty}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\sum \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 - r_2 s_2 + s_2^2}(q^{-r_1+s_1+s_2} - q^{r_2+s_2} + q^{1+r_1+r_2+s_1+s_2})}{(q)_{r_1-r_2}(q)_{s_1-s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2+s_2+1}} \\ &\quad = \frac{1}{(q)_\infty} \frac{1}{\theta(q, q^2, q^2, q^3, q^3, q^4; q^{10})} = \frac{\chi(\Omega(4\Lambda_0 + 3\Lambda_1))}{(q)_\infty}. \end{aligned} \tag{3.10}$$

Identity (3.9) arises from Theorem 6 with $(c_0, c_1, c2) = (3, 3, 0)$. (3.10) is one of the identities in Theorem 3 with $\ell = 7$.

4 Coloured $\mathfrak{sl}_r$ invariants of torus knots

Let us begin by collecting a few further facts about the representation theory of $\mathfrak{sl}_r$.

Recall that the Weyl denominator associated with $\mathfrak{sl}_r$ is:

$$\Delta_r = \sum_{w \in \mathfrak{S}_r} (-1)^{\ell(w)} e^{w(\delta)},$$

where $e^{w(\delta)}$ is a formal exponential, to be thought of as an element of the group algebra of the weight lattice P_r. We denote:

$$\mathrm{qdim}(\Delta_r) = \sum_{w \in \mathfrak{S}_r} (-1)^{\ell(w)} q^{(w(\delta), \delta)}.$$

Weyl denominator formula implies that:

$$\mathrm{qdim}(\Delta_r) = \prod_{\alpha \in \Phi_r^+} (q^{\frac{1}{2}(\delta, \alpha)} - q^{-\frac{1}{2}(\delta, \alpha)}).$$

The function qdim can be defined more generally by $\mathrm{qdim}(e^{\mu}) = q^{(\mu,\delta)}$ for $\mu \in P_r$. In particular, for a finite-dimensional irreducible module L, we may define $\mathrm{qdim}(L)$ as simply $\mathrm{qdim}(\mathrm{ch}(L))$ where the character $\mathrm{ch}(L)$ belongs to the group algebra of P_r.

Let $L_r(n\Lambda_1)$ denote the (finite-dimensional) irreducible $\mathfrak{sl}_r$ module of highest weight $n\Lambda_1$ and let $\Pi_{r,n}$ be the set of weights of $L_r(n\Lambda_1)$.

Given a knot K with a fixed framing f, a finite-dimensional simple Lie algebra $\mathfrak{g}$ along with a finite-dimensional irreducible module L, let $J_K^f(L)$ denote the framing-dependent, un-normalized, L coloured Jones invariant of K. Here, un-normalized means that for the unknot with zero framing this invariant equals $\mathrm{qdim}(L)$.

Let p, p' be a pair of coprime positive integers, and let $\mathrm{T}(p,p')$ be the torus knot obtained by the (p,p') cabling of the unknot of zero framing, see [23] for details. In this case, $\mathrm{T}(p,p')$ has writhe pp'. We will fix (and then forget) this framing of $\mathrm{T}(p,p')$. Now, the Rosso–Jones formula [24] (as explained by [23]) in conjunction with some elementary properties of $\Pi_{r,n}$ leads to the following theorem.

Theorem 7. *Let p, p' be a pair of positive coprime integers.*

1. *[16, Thm. 5.1] The framing-dependent, un-normalized invariants of torus knots satisfy:*

$$J_{\mathrm{T}(p,p')}(L_r(n\Lambda_1)) = \frac{q^{-\frac{p'}{2p}\|\delta\|^2}}{\mathrm{qdim}(\Delta_r)} \sum_{\lambda\in\Pi_{r,n}, w\in\mathfrak{S}_r} (-1)^{\ell(w)} q^{(p\lambda+w\delta,\delta)+\frac{p'}{2p}\|p\lambda+w\delta\|^2}.$$

2. *[16, Thm. 6.1] We have:*

$$\lim_{n\to\infty} J_{\mathrm{T}(p,p')}(L_r(nr\Lambda_1)) = \frac{(q)_\infty^{r-1}}{\prod_{\alpha\in\Phi_r^+}(1-q^{(\alpha,\delta)})}\overline{\chi}_{0,0}^{r,p,p'}. \tag{4.1}$$

3. *[16, Prop. 3.4] If $p < p'$ and $1 \le p \le r-1$, then, $\overline{\chi}_{0,0}^{r,p,p'} = 0$ and the limit in (4.1) is 0.*

In part 3 above, the reason the limit equals 0 is that the lowest degree of the corresponding Jones invariant approaches ∞ as $n \to \infty$. Thus, it is interesting to understand limits of the shifted invariants $\widehat{J}_K^f(L)$ which equal $J_K^f(L)$ divided by its trailing monomial. We have the following conjecture:

Conjecture 2. *[16, Conj. 6.6] Let $2 \le p < p'$ be a pair of coprime positive integers such that $p < r$. Letting $\Phi_1^+ = \emptyset$ and denoting the Weyl vector of $\mathfrak{sl}_r$ by δ_r, we have:*

$$\lim_{n\to\infty} \widehat{J}_{\mathrm{T}(p,p')}(L_r(n\Lambda_1)) = \frac{\prod_{\alpha\in\Phi_{r-p}^+}(1-q^{(\alpha,\delta_{r-p})})}{\prod_{\alpha\in\Phi_r^+}(1-q^{(\alpha,\delta_r)})}(q)_\infty^{p-1}\,\overline{\chi}_{0,0}^{p,p,p'}.$$

Note that unlike (4.1) which involves $L_r(nr\Lambda_1)$, this conjecture involves $L_r(n\Lambda_1)$.

The $r = 3$ case of this conjecture holds due to the work [13] (see also [16]).

Our hope is that this connection of $\mathcal{W}_3(3, 3+\ell)$ algebra characters to knot theory will provide useful insights on the corresponding Andrews–Schilling–Warnaar identities.

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