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Cyclic shuffle-compatibility and cyclic quasisymmetric functions

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Abstract. A permutation statistic st is said to be shuffle-compatible if the distribution of st over the set of shuffles of two disjoint permutations π and σ depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, and the lengths of π and σ. This notion is implicit in Stanley's work on P-partitions, and was first explicitly studied by Gessel and Zhuang, who developed an algebraic framework for shuffle-compatibility in which quasisymmetric functions play an important role. Later, Domagalski et al. defined a version of shuffle-compatibility for statistics on cyclic permutations. We develop an algebraic framework for cyclic shuffle-compatibility in which the role of quasisymmetric functions is replaced by the cyclic quasisymmetric functions recently introduced by Adin et al.

Keywords: permutation statistics, shuffle-compatibility, cyclic shuffles, quasisymmetric functions, cyclic descents, cyclic peaks

1 Introduction

We say that $\pi = \pi_1\pi_2\cdots\pi_n$ is a (*linear*) *permutation* of *length* n if it is a sequence of n distinct positive integers, not necessarily from 1 to n. (We refer to these as linear permutations to distinguish them from cyclic permutations, but we will often drop the descriptor "linear" if it is clear from context that we are referring to linear permutations.) For example, 826491 is a permutation of length 6. Let $|\pi|$ denote the length of a permutation π, and $\mathfrak{P}_n$ the set of permutations of length n.

Let $\pi \in \mathfrak{P}_m$ and $\sigma \in \mathfrak{P}_n$ be *disjoint* permutations, that is, permutations with no letters in common. We say that $\tau \in \mathfrak{P}_{m+n}$ is a *shuffle* of π and σ if both π and σ are subsequences of τ. The set of shuffles of π and σ is denoted $\pi \sqcup\!\sqcup \sigma$. For example, we have $71 \sqcup\!\sqcup 25 = \{7125, 7215, 7251, 2715, 2751, 2571\}$.

Following [4], a (*linear*) *permutation statistic* is a function st on permutations such that $\operatorname{st}\pi = \operatorname{st}\sigma$ whenever π and σ have the same relative order. Four well-studied permutation statistics are the descent set Des, descent number des, peak set Pk, and

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peak number pk. We say that $i \in [n-1]$ is a *descent* of $\pi \in \mathfrak{P}_n$ if $\pi_i > \pi_{i+1}$. Then the *descent set* and *descent number* of π are defined by

$$\operatorname{Des}\pi := \{\, i \in [n-1] : \pi_i > \pi_{i+1} \,\} \quad \text{and} \quad \operatorname{des}\pi := |\operatorname{Des}\pi|\,,$$

respectively. We say that $i \in \{2, 3, \ldots, n-1\}$ is a *peak* of $\pi \in \mathfrak{P}_n$ if $\pi_{i-1} < \pi_i > \pi_{i+1}$; similarly, the *peak set* and *peak number* of π are defined by

$$\operatorname{Pk}\pi := \{\, i \in \{2, 3, \ldots, n-1\} : \pi_{i-1} < \pi_i > \pi_{i+1} \,\} \quad \text{and} \quad \operatorname{pk}\pi := |\operatorname{Pk}\pi|\,,$$

respectively.

All four statistics defined above have a remarkable property related to shuffles, called "shuffle-compatibility". Given a set S of permutations and a permutation statistic st, the *distribution* of st over S is the multiset

$$\operatorname{st} S := \{\{\, \operatorname{st}\pi : \pi \in S \,\}\}$$

of all values of st among permutations in S, including multiplicity. Then st is called *shuffle-compatible* if the distribution of st over the shuffles of any two disjoint permutations π and σ depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, and the lengths of π and σ. In other words, st is shuffle-compatible if $\operatorname{st}(\pi \sqcup\!\sqcup \sigma) = \operatorname{st}(\pi' \sqcup\!\sqcup \sigma')$ whenever $\operatorname{st}\pi = \operatorname{st}\pi'$, $\operatorname{st}\sigma = \operatorname{st}\sigma'$, $|\pi| = |\pi'|$, and $|\sigma| = |\sigma'|$.

The shuffle-compatibility of the descent set and descent number are implicit consequences of Stanley's theory of P-partitions [7]. Likewise, Stembridge's [9] work on enriched P-partitions implies that the peak set and peak number are shuffle-compatible. Gessel and Zhuang coined the term "shuffle-compatibility" and initiated the study of shuffle-compatibility per se; in [4], they developed an algebraic framework for shuffle-compatibility based on their notion of the shuffle algebra of a shuffle-compatible statistic, whose multiplication encodes the distribution of the statistic over sets of shuffles.

Gessel's [3] quasisymmetric functions serve as natural generating functions for P-partitions, and for a family of statistics called "descent statistics", one can use quasisymmetric functions to characterize shuffle algebras and prove shuffle-compatibility results. Notably, the multiplication rule for fundamental quasisymmetric functions shows that the descent set is shuffle-compatible and that its shuffle algebra is isomorphic to the algebra QSym of quasisymmetric functions. One of Gessel and Zhuang's main results is a necessary and sufficient condition for shuffle-compatibility of descent statistics which implies that the shuffle algebra of any shuffle-compatible descent statistic is isomorphic to a quotient algebra of QSym.

In the past few years, shuffle-compatibility has become an active topic of research. Most relevant here are the recent papers of Adin–Gessel–Reiner–Roichman [1] and Liang [5] on cyclic quasisymmetric functions and toric $[\vec{D}]$-partitions, and of Domagalski–Liang–Minnich–Sagan–Schmidt–Sietsma [2] which defined and investigated a notion of shuffle-compatibility for cyclic permutations.

1.1 Cyclic permutations, statistics, and shuffles

Given a linear permutation $\pi = \pi_1\pi_2\cdots\pi_n$, let $[\pi]$ be the equivalence class of π under cyclic rotation, that is,

$$[\pi] := \{\pi_1\pi_2\cdots\pi_n,\ \pi_n\pi_1\cdots\pi_{n-1},\ \ldots,\ \pi_2\cdots\pi_n\pi_1\}.$$

The sets $[\pi]$ are called *cyclic permutations*. Let the *length* of a cyclic permutation $[\pi]$ be the length of π, which makes sense as all linear permutations in $[\pi]$ have the same length.

Let us define a *cyclic permutation statistic* to be a function cst on cyclic permutations such that $\operatorname{cst}[\pi] = \operatorname{cst}[\sigma]$ whenever π and σ have the same relative order. Two examples of cyclic permutation statistics are the cyclic descent set cDes and the cyclic descent number cdes. First, define the *cyclic descent set* of a linear permutation $\pi \in \mathfrak{P}_n$ by

$$\operatorname{cDes}\pi := \{\, i \in [n] : \pi_i > \pi_{i+1} \text{ where } i \text{ is considered modulo } n \,\};$$

the elements of $\operatorname{cDes}\pi$ are called *cyclic descents* of π. Then the *cyclic descent set* of a cyclic permutation $[\pi]$ is the multiset

$$\operatorname{cDes}[\pi] := \{\{\, \operatorname{cDes}\bar{\pi} : \bar{\pi} \in [\pi] \,\}\},$$

i.e., the distribution of the linear statistic cDes over all linear permutation representatives of $[\pi]$. For example, we have $\operatorname{cDes}[279358] = \{\{\, \{3,6\}^2, \{1,4\}^2, \{2,5\}^2 \,\}\}$. Note that $\operatorname{cDes}[\pi]$ can also be characterized as the multiset of cyclic shifts of $\operatorname{cDes}\pi$. More precisely, given $S \subseteq [n]$ and an integer i, define the *cyclic shift* $S + i$ by

$$S + i := \{\, s + i : s \in S \,\}$$

where the values are considered modulo n; then $\operatorname{cDes}[\pi] = \{\{\, \operatorname{cDes}\pi + i : i \in [n] \,\}\}$.

The *cyclic descent number* on linear and cyclic permutations are defined by

$$\operatorname{cdes}\pi := |\operatorname{cDes}\pi| \quad \text{and} \quad \operatorname{cdes}[\pi] := \operatorname{cdes}\pi,$$

respectively; note that the latter is well-defined because all linear permutations in $[\pi]$ have the same number of cyclic descents. The cyclic peak set cPk and cyclic peak number cpk can be defined in an analogous way, and we will state their definitions in Section 4.

Given disjoint $\pi \in \mathfrak{P}_m$ and $\sigma \in \mathfrak{P}_n$, we say that $[\tau]$ is a *cyclic shuffle* of $[\pi]$ and $[\sigma]$ if $\tau \in \mathfrak{P}_{m+n}$ and there exist $\bar{\pi} \in [\pi]$ and $\bar{\sigma} \in [\sigma]$ such that $\tau \in \bar{\pi} \sqcup \bar{\sigma}$. Let $[\pi] \sqcup [\sigma]$ denote the set of cyclic shuffles of $[\pi]$ and $[\sigma]$. For instance, we have $[63] \sqcup [24] = \{[6324], [6234], [6243], [6342], [6432], [6423]\}$.

A cyclic permutation statistic cst is called *cyclic shuffle-compatible* if the distribution of cst over all cyclic shuffles of $[\pi]$ and $[\sigma]$ depends only on $\operatorname{cst}[\pi]$, $\operatorname{cst}[\sigma]$, and the lengths of $[\pi]$ and $[\sigma]$. That is, cst is cyclic shuffle-compatible if we have $\operatorname{cst}([\pi] \sqcup [\sigma]) = \operatorname{cst}([\pi'] \sqcup [\sigma'])$ whenever $\operatorname{cst}[\pi] = \operatorname{cst}[\pi']$, $\operatorname{cst}[\sigma] = \operatorname{cst}[\sigma']$, $|\pi| = |\pi'|$, and $|\sigma| = |\sigma'|$.

The first results in cyclic shuffle-compatibility were implicit in the work of Adin et al. [1], which introduced toric $[\vec{D}]$-partitions (a toric poset analogue of P-partitions) and cyclic quasisymmetric functions (which are natural generating functions for toric $[\vec{D}]$-partitions). In particular, Adin et al. established a multiplication formula for fundamental cyclic quasisymmetric functions which implies that the cyclic descent set cDes is cyclic shuffle-compatible, as well as another formula which implies that the cyclic descent number cdes is cyclic shuffle-compatible.

In [2], Domagalski et al. formally defined cyclic shuffle-compatibility and proved a result called the "lifting lemma", which allows one (under certain conditions) to prove that a cyclic statistic is cyclic shuffle-compatible from the shuffle-compatibility of a related linear statistic. They then used the lifting lemma to prove the cyclic shuffle-compatibility of all four statistics cDes, cdes, cPk, and cpk. Most recently, Liang [5] defined and studied enriched toric $[\vec{D}]$-partitions, an analogue of enriched P-partitions for toric posets, whose generating functions are "cyclic peak quasisymmetric functions". She derived a multiplication formula for these cyclic peak quasisymmetric functions which gives a different proof for the cyclic shuffle-compatibility of the cyclic peak set cPk.

The lifting lemma of Domagalski et al. is purely combinatorial, but the work of Adin et al. and Liang suggest that there is an algebraic framework for cyclic shuffle-compatibility à la Gessel and Zhuang, in which the role of quasisymmetric functions is replaced by cyclic quasisymmetric functions. This extended abstract is based on a forthcoming paper by the present authors in which we develop this algebraic framework.

1.2 Outline

We organize this extended abstract as follows. In Section 2, we review the definition of the shuffle algebra of a shuffle-compatible permutation statistic, define the cyclic shuffle algebra of a cyclic shuffle-compatible statistic, and state one of our main results (Theorem 2.1), which allows one to construct cyclic shuffle algebras from linear shuffle algebras.

In Section 3, we review the role of quasisymmetric functions in the theory of shuffle-compatibility and then summarize our analogous theory in the cyclic realm. Theorem 2.1 is used to construct the non-Escher subalgebra cQSym^- of cyclic quasisymmetric functions from the algebra QSym of quasisymmetric functions, which gives another proof that cDes is cyclic shuffle-compatible with cyclic shuffle algebra isomorphic to cQSym^-. We then give a necessary and sufficient condition for cyclic shuffle-compatibility of cyclic descent statistics which implies that the cyclic shuffle algebra of any cyclic shuffle-compatible cyclic descent statistic is isomorphic to a quotient algebra of cQSym^-.

The theory from Section 3 is used to give explicit descriptions of the cyclic shuffle algebras of cPk, $(\mathrm{cpk}, \mathrm{cdes})$, cpk, and cdes, which yields algebraic proofs for their cyclic shuffle-compatibility. We end in Section 4 by describing these cyclic shuffle algebras.

Before continuing, we note that not all of our results are included in this extended abstract due to the page limit. For example, we have also shown that if two cyclic permutation statistics are related by a symmetry (such as complementation) and one is cyclic shuffle-compatible, then the other is cyclic shuffle-compatible and they have isomorphic cyclic shuffle algebras. In addition, any linear permutation statistic st induces a multiset-valued cyclic permutation statistic (which we also denote st by a slight abuse of notation) if we let $\mathrm{st}[\pi] := \{\{\, \mathrm{st}\,\bar{\pi} : \bar{\pi} \in [\pi] \,\}\}$, the distribution of the linear statistic st over all linear permutations in $[\pi]$. We have investigated the cyclic shuffle-compatibility of these cyclic statistics induced by various linear statistics. Full details are available in [6].

2 Cyclic shuffle algebras

Let st be a permutation statistic. We say that π and σ are st-*equivalent* if $\mathrm{st}\,\pi = \mathrm{st}\,\sigma$ and $|\pi| = |\sigma|$, and we write the st-equivalence class of π as π_{st}. Let $\mathcal{A}_{\mathrm{st}}$ denote the $\mathbb{Q}$-vector space consisting of formal linear combinations of st-equivalence classes of permutations. If st is shuffle-compatible, then we can define a multiplication on $\mathcal{A}_{\mathrm{st}}$ by

$$\pi_{\mathrm{st}}\sigma_{\mathrm{st}} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} \tau_{\mathrm{st}}$$

for any disjoint representatives $\pi \in \pi_{\mathrm{st}}$ and $\sigma \in \sigma_{\mathrm{st}}$; in fact, this multiplication is well-defined precisely when st is shuffle-compatible. The resulting $\mathbb{Q}$-algebra $\mathcal{A}_{\mathrm{st}}$ is called the (*linear*) *shuffle algebra* of st. Observe that $\mathcal{A}_{\mathrm{st}}$ is graded by length, that is, π_{st} belongs to the nth homogeneous component of $\mathcal{A}_{\mathrm{st}}$ if π has length n.

Our definition of cyclic shuffle algebras will be analogous to that of linear ones. Let cst be a cyclic permutation statistic. Then the cyclic permutations $[\pi]$ and $[\sigma]$ are called cst-*equivalent* if $\mathrm{cst}[\pi] = \mathrm{cst}[\sigma]$ and $|\pi| = |\sigma|$, and we use the notation $[\pi]_{\mathrm{cst}}$ to denote the cst-equivalence class of the cyclic permutation $[\pi]$. We associate to cst a $\mathbb{Q}$-vector space $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ by taking as a basis the set of all cst-equivalence classes of permutations, and then we give this vector space a multiplication by defining

$$[\pi]_{\mathrm{cst}}[\sigma]_{\mathrm{cst}} = \sum_{[\tau] \in [\pi] \sqcup\!\sqcup [\sigma]} [\tau]_{\mathrm{cst}}$$

for any disjoint π and σ with $[\pi] \in [\pi]_{\mathrm{cst}}$ and $[\sigma] \in [\sigma]_{\mathrm{cst}}$; this multiplication is well-defined if and only if cst is cyclic shuffle-compatible. The resulting $\mathbb{Q}$-algebra $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ is called the *cyclic shuffle algebra* of cst, and is also graded by length.

The next theorem—one of our main results—allows us to construct a cyclic shuffle algebra from an associated linear shuffle algebra.

Theorem 2.1. *Let* cst *be a cyclic permutation statistic and let* st *be a shuffle-compatible* (*linear*) *permutation statistic. Given a cyclic permutation* $[\pi]$, *let*

$$v_{[\pi]} = \sum_{\bar{\pi} \in [\pi]} \bar{\pi}_{\mathrm{st}} \in \mathcal{A}_{\mathrm{st}}.$$

Suppose that $v_{[\pi]} = v_{[\sigma]}$ *whenever* $[\pi]$ *and* $[\sigma]$ *are* cst*-equivalent, and that* $\{v_{[\pi]}\}$ (*ranging over all* cst*-equivalence classes*) *is linearly independent. Then* cst *is cyclic shuffle-compatible and the map* $\psi_{\mathrm{cst}} \colon \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}} \to \mathcal{A}_{\mathrm{st}}$ *given by* $\psi_{\mathrm{cst}}([\pi]_{\mathrm{cst}}) = v_{[\pi]}$ *extends linearly to a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *to the span of* $\{v_{[\pi]}\}$*, a subalgebra of* $\mathcal{A}_{\mathrm{st}}$.

3 Cyclic shuffle-compatibility of cyclic descent statistics

This section is concerned with cyclic shuffle-compatibility of cyclic descent statistics, in which cyclic quasisymmetric functions play a central role. We begin by providing the necessary background on descent compositions and cyclic descent compositions.

3.1 Descent compositions and cyclic descent compositions

Given a subset $S \subseteq [n-1]$ with elements $s_1 < s_2 < \cdots < s_j$, let $\operatorname{Comp} S$ be the composition $(s_1, s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j)$ of n. Also, given a composition $L = (L_1, L_2, \ldots, L_k)$, let $\operatorname{Des} L := \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}$ be the corresponding subset of $[n-1]$. It is straightforward to verify that Comp and Des are inverse bijections. If $\pi \in \mathfrak{P}_n$ has descent set $S \subseteq [n-1]$, then we say that $\operatorname{Comp} S$ is the *descent composition* of π, which we also denote by $\operatorname{Comp} \pi$. By convention, the empty permutation has descent composition $\varnothing$. Conversely, a permutation with descent composition L has descent set $\operatorname{Des} L$.

Let us now extend the notion of descent compositions to cyclic permutations, which shall require a few more definitions. A *cyclic shift* of a composition $L = (L_1, L_2, \ldots, L_k)$ is a composition of the form $(L_j, L_{j+1}, \ldots, L_k, L_1, \ldots, L_{j-1})$. A *cyclic composition* of n is then the equivalence class of a composition of n under cyclic shift. For example, both $[2,1,3] = \{(2,1,3), (1,3,2), (3,2,1)\}$ and $[1,2,1,2] = \{(1,2,1,2), (2,1,2,1)\}$ are cyclic compositions. By convention, we'll also allow the empty set $\varnothing$ to be a cyclic composition.

Let us call S a *non-Escher* subset of $[n]$ if S is the cyclic descent set of some linear permutation of length n. When $n = 0$ or $n = 1$, only the empty set is non-Escher, and when $n \geq 2$, all subsets of $[n]$ are non-Escher except for the empty set and $[n]$ itself. For a non-Escher subset $S = \{s_1 < s_2 < \cdots < s_j\} \subseteq [n]$, define the composition $\operatorname{cComp} S$ by

$$\operatorname{cComp} S := \begin{cases} (s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j + s_1), & \text{if } n \geq 2, \\ (1), & \text{if } n = 1, \\ \varnothing, & \text{if } n = 0. \end{cases}$$

It is easy to see that if S' is a cyclic shift of S, then $\mathrm{cComp}\, S'$ is a cyclic shift of $\mathrm{cComp}\, S$. So, if $[S]$ is the equivalence class of S under cyclic shift, then we can let $\mathrm{cComp}[S]$ be the cyclic composition given by $\mathrm{cComp}[S] := [\mathrm{cComp}\, S]$.

We say that a cyclic composition is *non-Escher* if it is an image of this induced map cComp; then cComp is a bijection from equivalence classes of non-Escher subsets of $[n]$ under cyclic shift to non-Escher cyclic compositions of n. If S is the cyclic descent set of a linear permutation π, then we call $\mathrm{cComp}[S]$ the *cyclic descent composition* of the cyclic permutation $[\pi]$. We denote the cyclic descent composition of $[\pi]$ simply as $\mathrm{cComp}[\pi]$. For example, $\pi = 179624$ has cyclic descent set $S = \{3, 4, 6\}$, so the cyclic descent composition of $[\pi]$ is $\mathrm{cComp}[\pi] = \mathrm{cComp}[S] = [1, 2, 3]$.

3.2 Cyclic quasisymmetric functions

Given a composition L of n, recall that the *fundamental quasisymmetric function F_L* is defined by

$$F_L := \sum_{\substack{i_1 \le i_2 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in \mathrm{Des}\, L}} x_{i_1} x_{i_2} \cdots x_{i_n},$$

and that the set $\{F_L\}$ over all compositions is a basis of QSym. It is sometimes more convenient to index fundamental quasisymmetric functions by sets rather than compositions, in which case we'll use the notation $F_{n,S} := F_{\mathrm{Comp}\, S}$.

The multiplication rule for the fundamental basis is given by the following theorem, which can be proved using P-partitions; see [8, Exercise 7.93].

Theorem 3.1. *Let $A \subseteq [m-1]$ and $B \subseteq [n-1]$. Then*

$$F_{m,A} F_{n,B} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} F_{m+n, \mathrm{Des}\, \tau}$$

where π is any permutation of length m with descent set A and σ is any permutation (disjoint from π) of length n with descent set B.

From Theorem 3.1, we see that the descent set shuffle algebra $\mathcal{A}_{\mathrm{Des}}$ is isomorphic to QSym; this is Corollary 4.2 of [4].

We are now ready to discuss cyclic quasisymmetric functions and their role in cyclic shuffle-compatibility. Given a subset S of $[n]$ where $n \ge 1$, let

$$F^{\mathrm{cyc}}_{n,S} := \sum_{i \in [n]} F_{n,(S+i) \cap [n-1]},$$

and let $F^{\mathrm{cyc}}_{0,\varnothing} := 1$; these are the *fundamental cyclic quasisymmetric functions* introduced by Adin, Gessel, Reiner, and Roichman [1]. It is clear that the $F^{\mathrm{cyc}}_{n,S}$ are invariant under cyclic shift; in other words, if $S' = S + i$ for some integer i, then $F^{\mathrm{cyc}}_{n,S} = F^{\mathrm{cyc}}_{n,S'}$. As such, if

$[S]$ is the equivalence class of the set S under cyclic shift, then it makes sense to define $F^{\mathrm{cyc}}_{n,[S]} := F^{\mathrm{cyc}}_{n,S}$. We can also index fundamental cyclic quasisymmetric functions using compositions; for a composition L of n, let $F^{\mathrm{cyc}}_{L} := F^{\mathrm{cyc}}_{n,\mathrm{cDes}\,L}$ and $F^{\mathrm{cyc}}_{[L]} := F^{\mathrm{cyc}}_{L}$.

Let cQSym^{-} denote the span of $\{F^{\mathrm{cyc}}_{n,[S]}\}$ over all $n \geq 0$ and all equivalence classes of non-Escher subsets $S \subseteq [n]$. The following theorem by Adin et al. [1, Theorem 3.22] gives a multiplication rule for the fundamental cyclic quasisymmetric functions in cQSym^{-}.

Theorem 3.2. *Let $A \subseteq [m]$ and $B \subseteq [n]$ be non-Escher subsets. Then*

$$F^{\mathrm{cyc}}_{m,[A]} F^{\mathrm{cyc}}_{n,[B]} = \sum_{[\tau] \in [\pi] \sqcup [\sigma]} F^{\mathrm{cyc}}_{m+n,\mathrm{cDes}[\tau]}$$

where $[\pi]$ is any cyclic permutation of length m with cyclic descent set $[A]$ and $[\sigma]$ is any cyclic permutation (with σ disjoint from π) of length n with cyclic descent set $[B]$.

Corollary 3.3. *The cyclic descent set* cDes *is cyclic shuffle-compatible, and the linear map on $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cDes}}$ defined by $[\pi]_{\mathrm{cDes}} \mapsto F^{\mathrm{cyc}}_{\mathrm{cComp}[\pi]}$ is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cDes}}$ to* cQSym^{-}.

Adin et al. proved Theorem 3.2 using toric $[\vec{D}]$-partitions, and this theorem implies that cQSym^{-} is a graded $\mathbb{Q}$-subalgebra of QSym. In [6], we give an alternative proof of Theorem 3.2 by constructing cQSym^{-} from QSym using Theorem 2.1.

Adin et al. also show that the span of $\{F^{\mathrm{cyc}}_{0,\varnothing}, F^{\mathrm{cyc}}_{1,\varnothing}, F^{\mathrm{cyc}}_{1,\{1\}}\} \cup \{F^{\mathrm{cyc}}_{n,[S]}\}_{n\geq 2, \varnothing \neq S \subseteq [n]}$, denoted cQSym, is a graded $\mathbb{Q}$-subalgebra of QSym, although this result is less relevant to cyclic shuffle-compatibility. They call cQSym^{-} the *non-Escher subalgebra* of cQSym.

3.3 A cyclic shuffle-compatibility criterion for cyclic descent statistics

A permutation statistic st is called a *descent statistic* if it depends only on the descent composition, that is, if $\mathrm{Comp}\,\pi = \mathrm{Comp}\,\sigma$ implies $\mathrm{st}\,\pi = \mathrm{st}\,\sigma$. If st is a descent statistic, then two permutations with the same descent composition are necessarily st-equivalent. Thus, the notion of st-equivalence coarsens to an equivalence relation on compositions.

The theorem below, due to Gessel and Zhuang [4, Theorem 4.3], provides a necessary and sufficient condition for shuffle-compatibility of descent statistics in terms of quasisymmetric functions, and implies that the shuffle algebra of any shuffle-compatible descent statistic is a quotient algebra of QSym.

Theorem 3.4. *A descent statistic* st *is shuffle-compatible if and only if there exists a $\mathbb{Q}$-algebra homomorphism $\phi_{\mathrm{st}}\colon \mathrm{QSym} \to A$, where A is a $\mathbb{Q}$-algebra with basis $\{u_\alpha\}$ indexed by* st*-equivalence classes α of compositions, such that $\phi_{\mathrm{st}}(F_L) = u_\alpha$ whenever $L \in \alpha$. In this case, the linear map on $\mathcal{A}_{\mathrm{st}}$ defined by $\pi_{\mathrm{st}} \mapsto u_\alpha$, where $\mathrm{Comp}\,\pi \in \alpha$, is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}_{\mathrm{st}}$ to A.*

A cyclic permutation statistic cst is called a *cyclic descent statistic* if it depends only on the cyclic descent composition, i.e., if $\operatorname{cComp}[\pi] = \operatorname{cComp}[\sigma]$ implies $\operatorname{cst}[\pi] = \operatorname{cst}[\sigma]$. As in the linear setting, the notion of cst-equivalence coarsens to an equivalence relation on non-Escher cyclic compositions when cst is a cyclic descent statistic.

The following is another one of our main results: a cyclic analogue of Theorem 3.4.

Theorem 3.5. *A cyclic descent statistic* cst *is cyclic shuffle-compatible if and only if there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{cst}} \colon \mathrm{cQSym}^{-} \to A$*, where* A *is a* $\mathbb{Q}$*-algebra with basis* $\{v_\alpha\}$ *indexed by* cst*-equivalence classes* α *of non-Escher cyclic compositions, such that* $\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{[L]}) = v_\alpha$ *whenever* $[L] \in \alpha$*. In this case, the linear map on* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *defined by* $[\pi]_{\mathrm{cst}} \mapsto v_\alpha$*, where* $\operatorname{cComp}[\pi] \in \alpha$*, is a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *to* A*.*

Corollary 3.6. *If* cst *is a cyclic shuffle-compatible descent statistic, then* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *is isomorphic to a quotient algebra of* cQSym^{-}*.*

To conclude this section, we state a useful special case of Theorem 3.5 in which the homomorphism ϕ_{cst} is given in terms of the homomorphism ϕ_{st} of a related (linear) descent statistic; c.f. Theorem 2.1.

Theorem 3.7. *Let* cst *be a cyclic descent statistic and let* st *be a shuffle-compatible (linear) descent statistic, so there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{st}} \colon \mathrm{QSym} \to A$ *satisfying the conditions in Theorem 3.4. Define the* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{cst}} \colon \mathrm{cQSym}^{-} \to A$ *by*

$$\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{n,S}) = \sum_{i \in [n]} \phi_{\mathrm{st}}(F_{n,(S+i)\cap[n-1]}).$$

Suppose that $\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{n,S}) = \phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{n,T})$ *whenever* $\operatorname{cComp}[S]$ *and* $\operatorname{cComp}[T]$ *are* cst*-equivalent, so that we can write* $\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{n,S}) = v_\alpha$ *whenever* $\operatorname{cComp}[S] \in \alpha$*, and suppose that* $\{v_\alpha\}$ *is linearly independent. Then* cst *is cyclic shuffle-compatible and the linear map on* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *defined by* $[\pi]_{\mathrm{cst}} \mapsto v_\alpha$*, where* $\operatorname{cComp}[\pi] \in \alpha$*, is a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *to the span of* $\{v_\alpha\}$*, a subalgebra of* A*.*

4 Characterizations of cyclic shuffle algebras

In [6], we use the theory summarized in the previous section to give explicit descriptions of some other cyclic shuffle algebras. After defining the cyclic peak set cPk and the cyclic peak number cpk, we will present our characterizations of the cyclic shuffle algebras of cPk, (cpk, cdes), cpk, and cdes. This yields new proofs for the cyclic shuffle-compatibility of the statistics cPk, cpk, and cdes, as well as the first proof for (cpk, cdes).

The *cyclic peak set* of a linear permutation $\pi \in \mathfrak{P}_n$ is defined by

$$\operatorname{cPk}\pi := \{\, i \in [n] : \pi_{i-1} < \pi_i > \pi_{i+1} \text{ where } i \text{ is considered modulo } n \,\}.$$

and the elements of $\operatorname{cPk}\pi$ are called *cyclic peaks* of π. Then the *cyclic peak set* of a cyclic permutation $[\pi]$ is defined to be the multiset

$$\operatorname{cPk}[\pi] := \{\{\, \operatorname{cPk}\bar{\pi} : \bar{\pi} \in [\pi] \,\}\}.$$

For example, we have $\operatorname{cPk}[184756] = \{\{\, \{2,4,6\}^3, \{1,3,5\}^3 \,\}\}$. It is clear from the definitions that, in general, $\operatorname{cPk}[\pi]$ is the multiset consisting of all cyclic shifts of $\operatorname{cPk}\pi$.

Define the *cyclic peak number* on linear and cyclic permutations by

$$\operatorname{cpk}\pi := |\operatorname{cPk}\pi| \quad \text{and} \quad \operatorname{cpk}[\pi] := \operatorname{cpk}\pi,$$

respectively. It is easy to see that cPk and cpk are both cyclic descent statistics.

4.1 The cyclic shuffle algebra of cPk

In [6], we construct the cyclic shuffle algebra $\mathcal{A}^{\text{cyc}}_{\text{cPk}}$ from the linear shuffle algebra $\mathcal{A}_{\text{Pk}}$. The latter is known to be isomorphic to Stembridge's [9] *algebra of peaks* Π, a subalgebra of QSym spanned by the *peak quasisymmetric functions* $K_{n,S}$ where n ranges over all nonnegative integers and S over all possible peak sets of permutations in $\mathfrak{P}_n$.

The analogue of Stembridge's quasisymmetric peak functions in the cyclic setting are Liang's [5] *cyclic peak quasisymmetric functions* $K^{\text{cyc}}_{n,S}$, which can be defined in terms of the $K_{n,S}$. For brevity, let us say that S is a *cyclic peak set* of $[n]$ if S is the cyclic peak set of some permutation of length n. Then, if S is a cyclic peak set of $[n]$, let

$$K^{\text{cyc}}_{n,S} := \sum_{i\in[n]} K_{n,(S+i)\setminus\{1,n\}} = \sum_{\bar{\pi}\in[\pi]} K_{n,\operatorname{Pk}\bar{\pi}}$$

where π is any permutation in $\mathfrak{P}_n$ with cyclic peak set S. We can also write $K^{\text{cyc}}_{n,[S]} := K^{\text{cyc}}_{n,S}$ since the $K^{\text{cyc}}_{n,S}$ are invariant under cyclic shift.

Liang showed that the $K^{\text{cyc}}_{n,[S]}$ span a subalgebra Λ of cQSym called the *algebra of cyclic peaks*. The following theorem—which is equivalent to Equation (5.10) of [5]—gives a multiplication rule for the $K^{\text{cyc}}_{n,[S]}$.

Theorem 4.1. *Let A be a cyclic peak set of $[m]$ and B a cyclic peak set of $[n]$. Then*

$$K^{\text{cyc}}_{m,[A]} K^{\text{cyc}}_{n,[B]} = \sum_{[\tau]\in[\pi]\sqcup[\sigma]} K^{\text{cyc}}_{m+n,\operatorname{cPk}[\tau]}$$

where $[\pi]$ is any cyclic permutation of length m with cyclic peak set $[A]$ and $[\sigma]$ is any cyclic permutation (disjoint from $[\pi]$) of length n with cyclic peak set $[B]$.

Corollary 4.2. *The cyclic peak set* cPk *is cyclic shuffle-compatible, and the linear map on $\mathcal{A}^{\text{cyc}}_{\text{cPk}}$ defined by $[\pi]_{\text{cPk}} \mapsto K^{\text{cyc}}_{|\pi|,\operatorname{cPk}[\pi]}$ is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}^{\text{cyc}}_{\text{cPk}}$ to Λ.*

While Liang's proof of Theorem 4.1 uses enriched toric $[\vec{D}]$-partitions, in [6] we give an alternative proof by constructing Λ from Stembridge's algebra Π using Theorem 3.7.

4.2 The cyclic shuffle algebras of $(\mathrm{cpk}, \mathrm{cdes})$, cpk, and cdes

In [6], we also use Theorem 3.7 to construct the cyclic shuffle algebra $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ from the linear shuffle algebra $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$, for which Gessel and Zhuang [4, Theorem 5.9] gave an explicit description. Below, we write $\mathbb{Q}[[t*]]$ for the $\mathbb{Q}$-algebra of formal power series in t where the multiplication is given by the *Hadamard product* $*$ defined by

$$\Big(\sum_{n=0}^{\infty} a_n t^n\Big) * \Big(\sum_{n=0}^{\infty} b_n t^n\Big) := \sum_{n=0}^{\infty} a_n b_n t^n.$$

Theorem 4.3. *The pair* $(\mathrm{cpk}, \mathrm{cdes})$ *is cyclic shuffle-compatible. Furthermore, let*

$$\begin{aligned} v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k} = [&j(y+t)(1+yt)(1+y+t+yt) \\ &\qquad + ((k-j)(1+yt) + (n-j-k)(y+t))t(1+y)^2] \\ &\times \frac{t^j(y+t)^{k-j-1}(1+yt)^{n-j-k-1}(1+y)^{2j-1}}{(1-t)^{n+1}} x^n. \end{aligned}$$

Then the linear map on $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ *defined by*

$$[\pi]_{(\mathrm{cpk},\mathrm{cdes})} \mapsto \begin{cases} v^{(\mathrm{cpk},\mathrm{cdes})}_{|\pi|,\mathrm{cpk}[\pi],\mathrm{cdes}[\pi]}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra homomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ *to the span of*

$$\Big\{\frac{1}{1-t}, \frac{t(1+y)}{(1-t)^2}x\Big\} \bigcup \{v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k}\}_{n\geq 2,\, 1\leq j\leq \lfloor n/2\rfloor,\, j\leq k\leq n-j},$$

a subalgebra of $\mathbb{Q}[[t*]][x,y]$.

Finally, we present the cyclic shuffle algebras $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ and $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cdes}}$, which are homomorphic images of $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ obtained by setting $y=1$ and $y=0$, respectively. Alternative characterizations of both cyclic shuffle algebras are given in [6].

Theorem 4.4. *The cyclic peak number* cpk *is cyclic shuffle-compatible. Furthermore, let*

$$v^{\mathrm{cpk}}_{n,j} = \frac{(j(1+t)^2 + 2(n-2j)t)(4t)^j(1+t)^{n-2j-1}}{(1-t)^{n+1}} x^n.$$

Then the linear map on $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ *defined by*

$$[\pi]_{\mathrm{cpk}} \mapsto \begin{cases} v^{\mathrm{cpk}}_{|\pi|,\mathrm{cpk}[\pi]}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ *to the span of*

$$\left\{\frac{1}{1-t}, \frac{tx}{(1-t)^2}\right\} \bigcup \{v^{\mathrm{cpk}}_{n,j}\}_{n\geq 2,\, 1\leq j\leq \lfloor n/2\rfloor},$$

a subalgebra of $\mathbb{Q}[[t*]][x]$.

Theorem 4.5. *The cyclic descent number* cdes *is cyclic shuffle-compatible. Furthermore, the linear map on* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cdes}}$ *defined by*

$$[\pi]_{\mathrm{cdes}} \mapsto \begin{cases} \dfrac{\mathrm{cdes}[\pi]t^{\mathrm{cdes}[\pi]} + (|\pi| - \mathrm{cdes}[\pi])t^{\mathrm{cdes}[\pi]+1}}{(1-t)^{|\pi|+1}} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cdes}}$ *to the span of*

$$\left\{\frac{1}{1-t}, \frac{tx}{(1-t)^2}\right\} \bigcup \left\{\frac{kt^k + (n-k)t^{k+1}}{(1-t)^{n+1}} x^n\right\}_{n\geq 2,\, 1\leq k\leq n-1},$$

a subalgebra of $\mathbb{Q}[[t*]][x]$.

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