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Set-valued tableaux for Macdonald polynomials

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Abstract. Set-valued tableaux formulas play an important role in Schubert calculus. Using the box greedy reduced word for the construction of the Macdonald polynomials, we convert the alcove walk formula to a set-valued tableaux formula for Macdonald polynomials. Our hope is that providing such formulas will help to strengthen the analogies and possible connections between the calculus of Macdonald polynomials and Schubert calculus.

Keywords: Macdonald polynomials, tableaux formulas, Schubert calculus

1 Introduction

The goal of this paper is to give set-valued tableaux formulas for Macdonald polynomials. As evidenced by the work of [2], [9], [11], [14], [17], [18] and others, set-valued tableaux formulas play an important role in Schubert calculus. Although there are many hints from the study of affine Springer fibers [13] and seminfinite flag varieties [8] the concrete connection between Schubert calculus and Macdonald polynomials for general type and general q and t is still elusive. There are many flavors of set-valued tableaux in the Schubert calculus literature and the relationships between them, including the relationship to the set-valued tableaux that we use, remain under-explored. Our hope is to help strengthen the analogies and possible connections between the calculus of Macdonald polynomials and Schubert calculus. See [3] for a version of this paper with additional details and examples.

Our formulas for Macdonald polynomials are derived by making a bijection between set-valued tableaux and alcove walks and converting the alcove walks formula from [16, Theorem 2.2] to a set-valued tableaux formula. We follow the framework of [5, §1 and §2], which gives an exposition of the alcove walks formula in the GL_n context and analyzes the favourite reduced word for the n-periodic permutation u_μ that is used to construct the relative Macdonald polynomial E^z_μ. Then a set-valued tableau formula for the symmetric Macdonald polynomials P_λ, with λ a partition, follows from

$$P_\lambda = (const) \sum_{z \in S_n \lambda} E^z_\lambda \;;$$

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see [6, (1.4), (1.5) and (1.10)].

The set-valued tableaux formulas given in Theorem 2.1 (respectively, Theorem 2.2) are specializable at $q = 0$ and $t = 0$ (respectively, $q = \infty$ and $t = \infty$). The specializations of E^z_μ at $t = 0$ and $t = \infty$ have interpretations in terms of crystals for level 1 and level 0 affine Demazure characters (see, for example, [7, Theorem C and Corollary D] or [12, (2.12), Theorem 1.1 and §2.4]). It would be interesting to write the root operators for these crystals explicitly on the set-valued tableaux (it is likely that the crystals of [18, §4] cover some cases of these crystals).

2 Set-valued tableaux for Macdonald polynomials

We begin with combinatorial definitions necessary for stating the set-valued tableaux formula for Macdonald polynomials. To aid in processing these definitions, we encourage the reader to follow Example 2.1 in tandem.

Fix $n \in \mathbb{Z}_{>0}$ and let S_n denote the symmetric group on n letters. Identify $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ with *the set of boxes in* μ,

$$\mu = \{(r, c) \mid r \in \{1, \ldots, n\} \text{ and } c \in \{1, \ldots, \mu_r\}\}.$$

2.1 Definition of set-valued tableaux

For $(r, c) \in \mu$ define

$$u_\mu(r, c) = \#\{r' \in \{1, \ldots, r-1\} \mid \mu_{r'} < c \leq \mu_r\} + \#\{r' \in \{r+1, \ldots, n\} \mid \mu_{r'} < c-1 < \mu_i\}.$$

The values $u_\mu(r, c)$ play a similar role to the flagging in the use of set-valued tableaux to give formulas for Grothendieck polynomials (see [17, §2.6 and Theorem 7.1]). A *set-valued tableau T of shape μ* is a choice of subset $T(r, c) \subseteq \{1, \ldots, u_\mu(r, c)\}$ for each box $(r, c) \in \mu$. More formally, a set-valued tableau T of shape μ is a function

$$T : \mu \to \{\text{subsets of } \{1, \ldots, n\}\} \qquad \text{such that} \quad T(r, c) \subseteq \{1, \ldots, u_\mu(r, c)\}.$$

Let $|T|$ denote the total number of entries in T.

2.2 Definition of the shift and height statistics

The minimal length permutation $v_\mu \in S_n$ such that $v_\mu \mu$ is weakly increasing is given by

$$v_\mu(r) = 1 + \#\{r' \in \{1, \ldots, r-1\} \mid \mu_{r'} \leq \mu_r\} + \#\{r' \in \{r+1, \ldots, n\} \mid \mu_{r'} < \mu_r\},$$

for $r \in \{1, \ldots, n\}$. For a box $(r, c) \in \mu$ and $i \in \{1, \ldots, u_\mu(r, c)\}$ define

$$\mathrm{sh}(r, c) = \mu_r - c + 1 \qquad \text{and} \qquad \mathrm{ht}(i, r, c) = v_\mu(r) - i.$$

2.3 Definition of $z^T_{(r,c)}$ and x^T

Using one-line notation, let γ_n be the n-cycle in the symmetric group S_n given by

$$\gamma_n = \begin{pmatrix} n & 1 & 2 & \cdots & n-1 \end{pmatrix}.$$

For positive integers $k_1, \ldots, k_\ell$ such that $k_1 + \cdots + k_\ell = n$, let

$$\gamma_{k_1} \times \cdots \times \gamma_{k_\ell} \qquad \text{be the disjoint product of cycles in} \qquad S_{k_1} \times \cdots \times S_{k_\ell} \subseteq S_n.$$

Fix a permutation $z \in S_n$. Order the boxes of μ down columns and then left to right (i.e., in increasing order of the values $r + nc$). Starting from the permutation z, associate a permutation $z^T_{(r,c)}$ to each box (r,c) of μ as follows:
If $(r,c) \in \mu$ and $T(r,c) = \{m_1, \ldots, m_p\}$ with $1 \le m_1 < \cdots < m_p \le u_\mu(r,c)$, define

$$z^T_{(r,c)} = z^T_{(r',c')}(\gamma_{m_1-0} \times \gamma_{m_2-m_1} \cdots \times \gamma_{m_p-m_{p-1}} \times \gamma_{u_\mu(r,c)+1-m_p} \times (\gamma_1)^{\times(n-u_\mu(r,c)-1)})\gamma_n^{-1},$$

where $(r',c') \in \mu$ denotes the box before (r,c) in μ. In the case that (r,c) is the first box in μ we let $z^T_{(r',c')} = z$. Then define

$$x^T = \prod_{(r,c)\in\mu} x_{z^T_{(r,c)}(n)},$$

where $z^T_{(r,c)}(n)$ is the value of the permutation $z^T_{(r,c)}$ at n.

The permutation $z^T_{\mathrm{init}} = z$ is the *left key* (or *initial direction*) of T and the permutation z^T_{fin} corresponding to the last box in μ is the *right key* (or *final direction*) of T. These left and right keys play important roles in Schubert calculus: for example in [18, §3] and in the statement of Pieri-Chevalley formulas (compare [4, Theorem 1] and [15, Corollary]).

2.4 Definition of the cov and maj statistics.

Keep the notation $T(r,c) = \{m_1, \ldots, m_p\}$ with $m_1 < \ldots < m_p$. Define

$$T^z_<(r,c) = \{m_k \in T(r,c) \mid z^T_{(r,c)}(m_{k-1}) < z^T_{(r,c)}(m_k)\} \qquad \text{and}$$
$$T^z_>(r,c) = \{m_k \in T(r,c) \mid z^T_{(r,c)}(m_{k-1}) > z^T_{(r,c)}(m_k)\},$$

where we make the convention that $m_0 = n$. Then define

$$\mathrm{maj}^z_>(T) = \sum_{(r,c)\in\mu} \mathrm{sh}(r,c) \cdot |T^z_>(r,c)|,$$
$$\mathrm{cov}^z_>(T) = \Big(\sum_{(r,c)\in\mu} \sum_{m \in T^z_>(r,c)} \mathrm{ht}(m,r,c) \Big) + \tfrac{1}{2}\big(\ell(z^T_{\mathrm{fin}}) - \ell(z^T_{\mathrm{init}} v_\mu^{-1}) - |T| \big),$$

where $\ell(w)$ denotes the length of the permutation w in S_n.

2.5 Set-valued tableaux formulas for E^z_μ.

The relative Macdonald polynomial $E^z_\mu = E^z_\mu(x_1, \ldots, x_n; q, t)$ is what is termed the "permuted basement nonsymmetric Macdonald polynomial" in [1] (see [5, (3.7)] for further references and a definition in terms of Cherednik-Dunkl operators and the double affine Hecke algebra action on polynomials).

Theorem 2.1. *Let $z \in S_n$ and $\mu \in \mathbb{Z}^n_{\geq 0}$. Then the relative Macdonald polynomial E^z_μ is*

$$E^z_\mu = \sum_T q^{\mathrm{maj}^z_>(T)} t^{\mathrm{cov}^z_>(T)} \Big(\prod_{(r,c)\in\mu} \prod_{m \in T(r,c)} \frac{1-t}{1-q^{\mathrm{sh}(r,c)} t^{\mathrm{ht}(m,r,c)}} \Big) x^T,$$

where the sum is over set-valued tableaux T of shape μ.

Because the powers of q and t in formula in Theorem 2.1 are nonnegative integers, this formula is well-suited to the specializations $q = 0$ and/or $t = 0$. The following theorem provides an alternate formula which is better for identifying the specializations of E^z_μ at $q = \infty$ and/or $t = \infty$ (i.e., specializations at $q^{-1} = 0$ and $t^{-1} = 0$).

$$\begin{aligned} \mathrm{maj}^z_<(T) &= \sum_{(r,c)\in\mu} \mathrm{sh}(r,c) \cdot |T^z_<(r,c)|, \\ \mathrm{cov}^z_<(T) &= \Big(\sum_{(r,c)\in\mu} \sum_{m\in T^z_<(r,c)} \mathrm{ht}(m,r,c) \Big) - \tfrac{1}{2}\big(\ \ell(z^T_{\mathrm{fin}}) - \ell(z^T_{\mathrm{init}} v_\mu^{-1}) - |T| \ \big). \end{aligned}$$

Theorem 2.2. *Let $z \in S_n$ and $\mu \in \mathbb{Z}^n_{\geq 0}$. Then the relative Macdonald polynomial E^z_μ is*

$$E^z_\mu = \sum_T q^{-\mathrm{maj}^z_<(T)} t^{-\mathrm{cov}^z_<(T)} \Big(\prod_{(r,c)\in\mu} \prod_{m \in T_{(r,c)}} \frac{1-t^{-1}}{1-q^{-\mathrm{sh}(r,c)} t^{-\mathrm{ht}(i,r,c)}} \Big) x^T,$$

where the sum is over set-valued tableaux T of shape μ.

The proof of Theorem 2.2 is obtained from Theorem 2.1 by multiplying numerator and denominator of each coefficient by

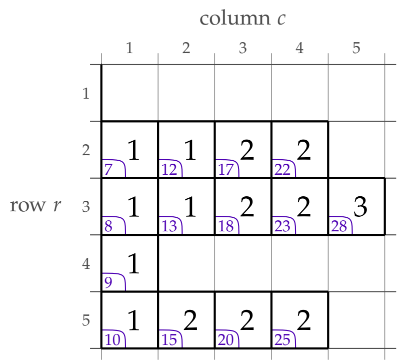
$$\prod_{(r,c)\in\mu} \prod_{i=1}^{u_\mu(r,c)} q^{-\mathrm{sh}(r,c)} t^{-\mathrm{ht}(i,r,c)}.$$

The proof of Theorem 2.1 is obtained by considering the reduced word for the n-periodic permutation

$$u_\mu = \prod_{(r,c)\in\mu} s_{u_\mu(r,c)-1} \cdots s_2 s_1 \pi \qquad \text{(product taken in increasing order of boxes)},$$

studied in [5, Propositions 2.1 and 2.2] (see also [10, (2.4.3)]) and using a straightforward bijection between alcove walks of type u_μ and set-valued tableaux of shape μ (see Section 3.3). This bijection gives a conversion between the alcove walk formula for in E^z_μ in [5, Theorem 1.1(a)] (see also [16, Theorem 2.2] for this alcove walk formula in a root system language) and the set-valued tableaux formula in Theorem 2.1.

Example 2.1. Let $n = 5$ and $\mu = (0, 4, 5, 1, 4)$. Then the box arrangement associated to μ is as follows, with boxes placed on a labeled grid, filled with values $u_\mu(r, c)$, and marked in the lower corner by their cylindrical numbers $r + nc$.



(2.1)

Box by box, the corresponding factors $\frac{(1-t)}{1-q^{\mathrm{sh}(r,c)}t^{\mathrm{ht}(i,r,c)}}$ are

$\frac{(1-t)}{1-q^{\mathrm{sh}}t^{\mathrm{ht}}} =$

$\frac{(1-t)}{(1-q^4t^2)}$	$\frac{(1-t)}{(1-q^3t^2)}$	$\frac{(1-t)}{(1-q^2t^2)}\frac{(1-t)}{(1-q^2t)}$	$\frac{(1-t)}{(1-qt^2)}\frac{(1-t)}{(1-qt)}$	
$\frac{(1-t)}{(1-q^5t^4)}$	$\frac{(1-t)}{(1-q^4t^4)}$	$\frac{(1-t)}{(1-q^3t^4)}\frac{(1-t)}{(1-q^3t^3)}$	$\frac{(1-t)}{(1-q^2t^4)}\frac{(1-t)}{(1-q^2t^3)}$	$\frac{(1-t)}{(1-qt^4)}\frac{(1-t)}{(1-qt^3)}\frac{(1-t)}{(1-qt^2)}$
$\frac{(1-t)}{(1-qt)}$				
$\frac{(1-t)}{(1-q^4t^3)}$	$\frac{(1-t)}{(1-q^3t^3)}\frac{(1-t)}{(1-q^3t^2)}$	$\frac{(1-t)}{(1-q^2t^3)}\frac{(1-t)}{(1-q^2t^2)}$	$\frac{(1-t)}{(1-qt^3)}\frac{(1-t)}{(1-qt^2)}$	

.

One set-valued tableau of shape λ is

$T =$

$\{1\}$	$\varnothing$	$\{1\}$	$\{2\}$	
$\{1\}$	$\varnothing$	$\{1,2\}$	$\varnothing$	$\{1,3\}$
$\varnothing$				
$\varnothing$	$\varnothing$	$\{1,2\}$	$\varnothing$	

$=$

1		1	2	
1		1 2		1 3
		1 2		

(2.2)

which has size $|T| = 1+1+1+2+2+1+2 = 10$ (we shall often omit the set notation and commas as in the right-most picture above). The product in Theorem 2.1 corresponding to this set-valued tableau T is

$$\prod_{(r,c)\in\mu}\ \prod_{m\in T^z(r,c)} \frac{(1-t)}{1-q^{\mathrm{sh}(r,c)}t^{\mathrm{ht}(i,r,c)}} = \begin{array}{|c|c|c|c|c|}\hline \frac{(1-t)}{(1-q^4t^2)} & 1 & \frac{(1-t)}{(1-q^2t^2)} & \frac{(1-t)}{(1-qt)} \\ \hline \frac{(1-t)}{(1-q^5t^4)} & 1 & \frac{(1-t)}{(1-q^3t^4)}\frac{(1-t)}{(1-q^3t^3)} & 1 & \frac{(1-t)}{(1-qt^4)}\frac{(1-t)}{(1-qt^2)} \\ \hline 1 \\ \hline 1 & 1 & \frac{(1-t)}{(1-q^2t^3)}\frac{(1-t)}{(1-q^2t^2)} & 1 \\ \hline \end{array}\ .$$

Let $z = \mathrm{id}$ be the identity. The *box-by-box permutation sequence of T with initial direction z* (written in 1-line notation $w = (w(1)\ \cdots\ w(n))$) is

$$z^T = \begin{array}{|c|c|c|c|c|}\hline (\underline{2}345\mathbf{1}) & (3245\mathbf{1}) & (\underline{1}425\mathbf{3}) & (2\underline{3}14\mathbf{5}) \\ \hline (\underline{3}451\mathbf{2}) & (3451\mathbf{2}) & (\underline{4}\underline{2}53\mathbf{1}) & (2435\mathbf{1}) & (\underline{5}3\underline{1}4\mathbf{2}) \\ \hline (3512\mathbf{4}) \\ \hline (3124\mathbf{5}) & (3412\mathbf{5}) & (\underline{2}\underline{5}31\mathbf{4}) & (2351\mathbf{4}) \\ \hline \end{array}\ . \tag{2.3}$$

In each box we have underlined the positions specified by the entries of T; these positions are the $m_1, \ldots, m_p$ in Section 2.4. The red highlighted entry indicates $z^T_{(r,c)}(n)$, which is used in the formula for x^T in Section 2.3, and also as $z^T_{(r,c)}(m_0)$ in the definition of $T^z_<(r,c)$ and $T^z_>(r,c)$ in Section 2.4. So we have

$$x^T = \begin{array}{|c|c|c|c|c|}\hline x_1 & x_1 & x_3 & x_5 \\ \hline x_2 & x_2 & x_1 & x_1 & x_2 \\ \hline x_4 \\ \hline x_5 & x_5 & x_4 & x_4 \\ \hline \end{array} = x_1^4 x_2^3 x_3 x_4^3 x_5^3\ ;$$

and, following the definition in Section 2.4 and splitting T into *ascents* and *descents* in the sequence of underlined numbers (preceded by the red number $z^T_{(r,c)}(n)$):

$$T^z_< = \begin{array}{|c|c|c|c|c|}\hline 1 & & & \\ \hline 1 & & & & 1 \\ \hline \ \\ \hline & & 2 & \\ \hline \end{array} \quad \text{and} \quad T^z_> = \begin{array}{|c|c|c|c|c|}\hline & & 1 & 2 \\ \hline & & 2 & & 3 \\ \hline \ \\ \hline & & 1 & \\ \hline \end{array}\ .$$

The initial and final directions of T are

$$z^T_{\text{init}} = \text{id} = (12345) \qquad \text{and} \qquad z^T_{\text{fin}} = z^T_{(3,5)} = (53142),$$

where $z_{(3,5)}$ indicates the permutation in box $(3,5)$ of z^T. Since $v_\mu = (13524)$ then

$$\ell(z^T_{\text{fin}}) = 4+2+0+1 = 7 \quad \text{and} \quad \ell(z^T_{\text{init}} v_\mu^{-1}) = 3,$$

so that

$$\tfrac{1}{2}(\ell(z^T_{\text{fin}}) - \ell(z^T_{\text{init}} v_\mu^{-1}) - |T|) = \tfrac{1}{2}(7-3-10) = -3.$$

Then

$$q^{\text{maj}^z_>(T)} t^{\text{cov}^z_>(T)} = t^{\frac{1}{2}(7-3-10)} \cdot \begin{array}{|c|c|c|c|c|} \\ \hline 1 & 1 & q^2t^2 & qt \\ \hline 1 & 1 & q^3t^3 & 1 & qt^2 \\ \hline 1 \\ \hline 1 & 1 & q^2t^3 & 1 \\ \hline \end{array} = t^{-3}q^9t^{11} = q^9t^8.$$

3 Proof of Theorem 2.1

In this section we describe the conversion from alcove walks to set-valued tableaux. We follow the framework of [5, §1], which gives an exposition of the alcove walks formula in the GL_n context and analyzes the favourite reduced for the n-periodic permutation u_μ that is used to construct the relative Macdonald polynomial E^z_μ.

3.1 Inversions and the box-greedy reduced word

An *n-periodic permutation* is a bijection $w\colon \mathbb{Z} \to \mathbb{Z}$ such that $w(i+n) = w(i)+n$. Since an n-periodic permutation w is determined by the values $w(1), \ldots, w(n)$, a permutation $w \in S_n$ extends uniquely to an n-periodic permutation. For $i \in \{1, \ldots, n-1\}$ let s_i be the transposition in S_n switching i and $i+1$. Define $\pi\colon \mathbb{Z} \to \mathbb{Z}$ by $\pi(i) = i+1$.

An *inversion* of an n-periodic permutation w is an element of the set

$$\text{Inv}(w) = \{(i,k) \mid i \in \{1,\ldots,n\}, k \in \mathbb{Z}, i < k \text{ and } w(i) > w(k)\}.$$

If $i, j \in \{1, \ldots, n\}$ with $i < j$ and $\ell \in \mathbb{Z}$, then the *shift* and *height* of $(i, j+\ell n)$ are defined to be

$$\text{sh}(i, j+\ell n) = \ell \qquad \text{and} \qquad \text{ht}(i, j+\ell n) = j - i.$$

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ and let u_μ be the n-periodic permutation defined by

$$u_\mu(i) = v_\mu^{-1}(i) + n\mu_i, \qquad \text{for } i \in \{1, \ldots, n\}. \tag{3.1}$$

Let $u_\mu(r,c)$ be as defined in Section 2.1. Following [5, Prop. 2.2(a)], the *box-greedy reduced word for* u_μ (an abstract word in symbols $s_1, \ldots, s_{n-1}$ and π) is

$$u_\mu^\square = \prod_{\text{boxes } (r,c) \text{ in } \mu} (s_{u_\mu(r,c)} \cdots s_2 s_1 \pi), \tag{3.2}$$

where the product is taken in increasing order of the values $r + nc$. As in [5, (2.12)], this reduced word induces a partition of the elements of $\mathrm{Inv}(u_\mu)$ according to the boxes (r,c) of μ, and an ordering of the inversions in each box. For $(r,c) \in \mu$ and $i \in \{1, \ldots, u_\mu(r,c)\}$,

$$\text{the } i\text{th inversion in the box } (r,c) \text{ is} \qquad \beta(i,r,c) = (v_\mu(r), i + n \cdot (\mu_r - c + 1)).$$

The inversion $\beta(i,r,c)$ has shift and height

$$\mathrm{sh}(r,c) = \mathrm{sh}(\beta(i,r,c)) = \mu_r - c + 1 \quad \text{and} \quad \mathrm{ht}(i,r,c) = \mathrm{ht}(\beta(i,r,c)) = v_\mu(r) - i. \tag{3.3}$$

Example 3.1. If $\mu = (0,4,5,1,4)$ then $n = 5$, the box-greedy reduced word for u_μ is

$$u_\mu^\square = (s_1\pi)^6 (s_2 s_1 \pi)^7 (s_3 s_2 s_1 \pi) = \begin{array}{|c|c|c|c|c} \\ \hline s_1\pi & s_1\pi & s_2s_1\pi & s_2s_1\pi & \\ \hline s_1\pi & s_1\pi & s_2s_1\pi & s_2s_1\pi & \multicolumn{1}{|c|}{s_3s_2s_1\pi} \\ \hline s_1\pi & & & & \\ \hline s_1\pi & s_2s_1\pi & s_2s_1\pi & s_2s_1\pi & \\ \hline \end{array} \tag{3.4}$$

(again multiplying boxes in order by cylindrical numbers) and the inversion set of u_μ is

$$\mathrm{Inv}(u_\mu) = \begin{array}{|c|c|c|c|c} \\ \hline (3,1+4\cdot5) & (3,1+3\cdot5) & \begin{matrix}(3,1+2\cdot5)\\(3,2+2\cdot5)\end{matrix} & \begin{matrix}(3,1+1\cdot5)\\(3,2+1\cdot5)\end{matrix} & \\ \hline (5,1+5\cdot5) & (5,1+4\cdot5) & \begin{matrix}(5,1+3\cdot5)\\(5,2+3\cdot5)\end{matrix} & \begin{matrix}(5,1+2\cdot5)\\(5,2+2\cdot5)\end{matrix} & \multicolumn{1}{|c|}{\begin{matrix}(5,1+1\cdot5)\\(5,2+1\cdot5)\\(5,3+1\cdot5)\end{matrix}} \\ \hline (2,1+1\cdot5) & & & & \\ \hline (4,1+4\cdot5) & \begin{matrix}(4,1+3\cdot5)\\(4,2+3\cdot5)\end{matrix} & \begin{matrix}(4,1+2\cdot5)\\(4,2+2\cdot5)\end{matrix} & \begin{matrix}(4,1+1\cdot5)\\(4,2+1\cdot5)\end{matrix} & \\ \hline \end{array}$$

3.2 Alcove walks and permutation sequences

Let $z \in S_n$ and choose a subset F of the s_i factors in $u_\mu^\square = w_1 w_2 \cdots w_r$ to cross out. The corresponding *alcove walk of type* $(z, u_\mu^\square)$ is the sequence $p(F) = (p_0, p_1, \ldots, p_r)$ of elements of W given by

$$p_0 = z \quad \text{and} \quad p_k = \begin{cases} p_{k-1}\pi, & \text{if } w_k = \pi, \\ p_{k-1}, & \text{if } w_k \in F \text{ (so that the factor } w_k \text{ is crossed out),} \\ p_{k-1}w_k, & \text{if } w_k \notin F \text{ (so that the factor } w_k \text{ is not crossed out).} \end{cases}$$

Following [5, (1.14)], the *permutation sequence of* $p(F)$ is the sequence of permutations in S_n given by

$$\overline{p}(F) = (\overline{p_0}, \overline{p_1}, \ldots, \overline{p_r}), \qquad \text{where} \quad \overline{w}(i) = w(i) \bmod n.$$

Note that $\overline{w_1 w_2} = \overline{w_1}\,\overline{w_2}$ and $\overline{\pi} = c_n = s_1 \cdots s_{n-1}$ and $\overline{s_j} = s_j$ for $j \in \{1, \ldots, n-1\}$.

3.3 Converting to set-valued tableaux

The bijection between alcove walks and set-valued tableaux is

$$\begin{array}{ccc} \{\text{alcove walks of type } u_\mu^\square\} & \longleftrightarrow & \{\text{set-valued tableaux of shape } \mu\} \\ p(F) & \longmapsto & T \end{array}$$

where, in the set-valued tableau T corresponding to the alcove walk $p(F)$, the set $T(r,c)$ in box (r,c) specifies which factors are crossed out in that box: if a box in T contains i, then delete s_i in the corresponding word.

Example 3.2. In the case $\mu = (0,4,5,1,4)$, where the box-greedy reduced word for u_μ is as given in (3.4), there are $6 + 2 \cdot 7 + 3 = 23$ factors of the form s_i in $u_\mu^\square$ and so there are a total of 2^{23} alcove walks of type $(z, u_\mu^\square)$ (for any fixed permutation $z \in S_n$). Each alcove walk corresponds to a choice of the s_i factors in $u_\mu^\square$ to cross out.

The tableau T in Example 2.1 corresponds to the subset

$$F = \begin{array}{|c|c|c|c|c|} \\ \hline \xcancel{s_1}\pi & s_1\pi & s_2\xcancel{s_1}\pi & \xcancel{s_2}s_1\pi \\ \hline \xcancel{s_1}\pi & s_1\pi & \xcancel{s_2}\xcancel{s_1}\pi & s_2s_1\pi & \xcancel{s_3}s_2\xcancel{s_1}\pi \\ \hline s_1\pi \\ \hline s_1\pi & s_2s_1\pi & \xcancel{s_2}\xcancel{s_1}\pi & s_2s_1\pi \\ \hline \end{array}\,, \qquad (3.5)$$

which has alcove walk

$$p(F) = (p_0, p_1, \ldots, p_{37}) = (z, z, z\pi, z\pi, z\pi^2, z\pi^2 s_1, z\pi^2 s_1\pi, z\pi^2 s_1\pi s_1, z\pi^2 s_1\pi s_1\pi, \ldots)$$

(there is a repeat entry in $p(F)$ each time there is an s_i crossed out in F). Using one line notation $w = (w(1)\ w(2)\ \cdots w(n))$ for n-periodic permutations (underlining 2-digit terms for emphasis), a box-by-box formulation in the case that $z = \mathrm{id} = (12345)$ is

$p(F) =$

(1 2 3 4 5) (2 3 4 5 6)	(6 3 7 9 10) (3 7 9 10 11)	(3 11 9 12 15) (3 11 9 12 15) (11 9 12 15 8)	(12 15 8 16 14) (15 12 8 16 14) (12 8 16 14 20)	
(2 3 4 5 6) (3 4 5 6 7)	(7 3 9 10 11) (3 9 10 11 12)	(11 9 12 15 8) (11 9 12 15 8) (9 12 15 8 16)	(12 16 8 14 20) (16 12 8 14 20) (12814 20 21)	(12 8 20 21 19) (12 20 8 21 19) (12 20 8 21 19) (20 8 21 19 17)
(4 3 5 6 7) (3 5 6 7 9)				
(5 3 6 7 9) (3 6 7 9 10)	(3 10 9 11 12) (10 3 9 11 12) (3 9 11 12 15)	(9 12 15 8 16) (9 12 15 8 16) (12 15 8 16 14)	(12 14 8 20 21) (14 12 8 20 21) (12 8 20 21 19)	

The permutation sequence is

$$\overline{p}(F) = (\overline{p_0}, \overline{p_1}, \ldots, \overline{p_{37}}) = (z, z, zc_n, zc_n, zc_n^2, zc_n^2 s_1, zc_n^2 s_1 c_n, zc_n^2 s_1 c_n s_1, zc_n^2 s_1 c_n s_1 c_n, \ldots)$$

(obtained by replacing each π by the n-cycle $c_n = s_1 \cdots s_{n-1} = \gamma_n^{-1}$). Namely, $\overline{p}(F)$ is obtained by reducing all the values in $p(F)$ mod n (and $n = 5$ for $\mu = (0, 4, 1, 5, 4)$).

$\overline{p}(F) =$ (3.6)

(**12**345) (23451)	(13245) (32451)	(31425) (**31**425) (14253)	(2**53**14) (52314) (23145)	
(**23**451) (34512)	(23451) (34512)	(1**42**53) (**14**253) (42531)	(21345) (12345) (24351)	(23**51**4) (25314) (**25**314) (53142)
(43512) (35124)				
(53124) (31245)	(35412) (53412) (34125)	(4**25**31) (**42**531) (25314)	(24351) (42351) (23514)	

The *box-by-box permutation sequence* of F is obtained by recording the last permutation in each box, which is z^T in (2.3). The elements $z^T_{(r,c)}$ in each box of z^T are also obtained from $z = z^T_{\text{init}}$ and T by the formula in Section 2.3.

We have used the blue highlighted numbers in (3.6) to record the positions i, $i+1$ when there is a repeat entry in the sequence $\overline{p}(F)$ coming from a crossed out s_i. These record the information of F. This same information is translated into the underlines in z^T in (2.3), and the underlines exactly specify the entries of T. The set-valued tableau indicating the positions of the underlined numbers in each box is the tableau in (2.2).

3.4 Conversion of the statistics

The proof of Theorem 2.1 is completed by matching up the statistics in Theorem 2.1 with the statistics that appear in [16, Theorem 3.1]. In Example 3.2, we have used the blue highlighted numbers in (3.6) and the underlines in (2.3) to illustrate how the information of the 'folds' in [16, Theorem 3.1] is equivalent to the information of the set-valued tableau T. In the context of [16, (2.36)], the set $T(r,c)$ exactly records the folds coming from the box (r,c). By (3.3), the factor $\prod_{(r,c)\in\mu}\prod_{m\in T(r,c)} \frac{(1-t)}{1-q^{\text{sh}(r,c)}t^{\text{ht}(i,r,c)}}$ which appears in the product in Theorem 2.1 corresponds to

$$\text{the factors} \quad \left(\prod_{k\in f^+(p)} \frac{(1-t_{\beta_k^\vee})}{1-q^{\langle -\beta_k^\vee,\rho_c\rangle}}\right)\left(\prod_{k\in f^-(p)} \frac{(1-t_{\beta_k^\vee})}{1-q^{\langle -\beta_k^\vee,\rho_c\rangle}}\right) \quad \text{in [16, Theorem 3.1].}$$

The sets $T^z_<$ and $T^z_>$ correspond to the sets $f^+(p)$ and $f^-(p)$ of positive and negative folds in [16, Theorem 3.1]. The permutation z^T_{fin} is the permutation denoted $\varphi(p)$ in [16, Theorem 3.1] (denoted z_r in [5, (1.14)]), and the permutation $z^T_{\text{init}}v_\mu^{-1}$ is the permutation m of [16, Theorem 3.1]. With these conversions, the last term $\frac{1}{2}\big(\ \ell(z^T_{\text{fin}}) - \ell(z^T_{\text{init}}v_\mu^{-1}) - |T|\ \big)$ in the definition of $\text{cov}^z_>(T)$ in Section 2.4 corresponds to the combination of the factors $t^{\frac{1}{2}}_{\varphi(p)}$ and $t^{-\frac{1}{2}}_{\beta_k^\vee}$ in [16, Theorem 3.1] and the factor $t^{\frac{1}{2}}_m$ in [16, Remark 3.2].

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