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Inequalities for f^*-vectors of Lattice Polytopes

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Abstract. The Ehrhart polynomial $\mathrm{ehr}_P(n)$ of a lattice polytope P counts the number of integer points in the n-th dilate of P. The f^*-vector of P, introduced by Felix Breuer in 2012, is the vector of coefficients of $\mathrm{ehr}_P(n)$ with respect to the binomial coefficient basis $\left\{\binom{n-1}{0}, \binom{n-1}{1}, \ldots, \binom{n-1}{d}\right\}$, where $d = \dim P$. Similarly to h/h^*-vectors, the f^*-vector of P coincides with the f-vector of its unimodular triangulations (if they exist). We present several inequalities that hold among the coefficients of f^*-vectors of polytopes. These inequalities resemble striking similarities with existing inequalities for the coefficients of f-vectors of simplicial polytopes; e.g., the first half of the f^*-coefficients increases and the last quarter decreases. Even though f^*-vectors of polytopes are not always unimodal, there are several families of polytopes that carry the unimodality property. We also show that for any polytope with a given Ehrhart h^*-vector, there is a polytope with the same h^*-vector whose f^*-vector is unimodal.

Keywords: Lattice polytope, Ehrhart polynomial, Gorenstein polytope, f^*-vector, h^*-vector, unimodality.

1 Introduction

For a d-dimensional *lattice polytope* $P \subset \mathbb{R}^d$ (i.e., the convex hull of finitely many points in $\mathbb{Z}^d$) and a positive integer n, let $\mathrm{ehr}_P(n)$ denote the number of integer lattice points in nP. Ehrhart's famous theorem [12] says that $\mathrm{ehr}_P(n)$ evaluates to a polynomial in n. Similar to the situations with other combinatorial polynomials, it is useful to express $\mathrm{ehr}_P(n)$ in different bases; here we consider two such bases consisting of binomial coefficients:

$$\mathrm{ehr}_P(n) \;=\; \sum_{k=0}^{d} h_k^* \binom{n+d-k}{d} \;=\; \sum_{k=0}^{d} f_k^* \binom{n-1}{k}. \tag{1.1}$$

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We call $(f_0^*, f_1^*, \dots, f_d^*)$ the *f^*-vector* and $(h_0^*, h_1^*, \dots, h_d^*)$ the *h^*-vector* of P. Stanley [16] proved that the h^*-vector of any lattice polytope is nonnegative (whereas the coefficients of $\mathrm{ehr}_P(n)$ written in the standard monomial basis can be negative). Breuer [10] proved that the f^*-vector of any lattice polytopal complex is nonnegative (whereas the h^*-vector of a complex can have negative coefficients); his motivation was that various combinatorially-defined polynomials can be realized as Ehrhart polynomials of complexes and so the nonnegativity of the f^*-vector yields a strong constraint for these polynomials. The f^*- and h^*-vector can also be defined through the *Ehrhart series* of P:

$$\mathrm{Ehr}_P(z) := 1 + \sum_{n \ge 1} \mathrm{ehr}_P(n)\, z^n = \frac{\sum_{k=0}^d h_k^*\, z^k}{(1-z)^{d+1}} = 1 + \sum_{k=0}^d f_k^* \left(\frac{z}{1-z}\right)^{k+1} .$$

It is thus sometimes useful to add the definition $f_{-1}^* := 1$. The polynomial $\sum_{k=0}^d h_k^*\, z^k$ is the *h^*-polynomial* of P, and its degree is the *degree* of P. For general background on Ehrhart theory, see, e.g., [4]. The f^*- and h^*-vectors share the same relation as f- and h-vectors of polytopes/polyhedral complexes, namely

$$\sum_{k=0}^d h_k^*\, z^k = \sum_{k=0}^{d+1} f_{k-1}^*\, z^k (1-z)^{d-k+1} \tag{1.2}$$

$$h_k^* = \sum_{j=-1}^{k-1} (-1)^{k-j-1} \binom{d-j}{k-j-1} f_j^* \tag{1.3}$$

$$f_k^* = \sum_{j=0}^{k+1} \binom{d-j+1}{k-j+1} h_j^* \, . \tag{1.4}$$

The (very special) case that P admits a unimodular triangulation yields the strongest connection between f^*/h^*-vectors and f/h-vectors: in this case the f^*/h^*-vector of P equals the f/h-vector of the triangulation, respectively.

Example 1. Let P be the 2-dimensional cube $[-1,1]^2$. The unimodular triangulation of P shown in Figure 1, has f-vector $(f_0, f_1, f_2) = (9, 16, 8)$, as f_i counts its i-dimensional faces. Equivalently,

$$f^*(P) = (9, 16, 8) \, ,$$

and one easily checks that (1.1) yields the familiar Ehrhart polynomial $\mathrm{ehr}_P(n) = (2n+1)^2$.

Example 2. The f^*-vector of a d-dimensional unimodular simplex Δ equals

$$\left[\binom{d+1}{1}, \binom{d+1}{2}, \dots, \binom{d+1}{d+1} \right],$$

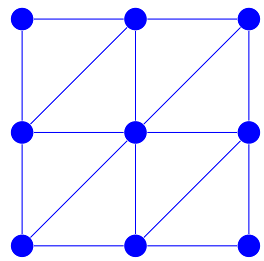

Figure 1: A (regular) unimodular triangulation of the cube $[-1,1]^2$.

coinciding with the f-vector of Δ considered as a simplicial complex. If we append this vector by $f^*_{-1} = 1$, it gives the only instance of a *symmetric* f^*-vector of a lattice polytope P, since the equality $f^*_{-1} = f^*_d$ implies that $h^*_i = 0$ for all $1 \le i \le d$.

There has been much research on (typically linear) constraints for the h^*-vector of a given lattice polytope (see, e.g., [17, 18]). On the other hand, f^*-vectors seem to be much less studied, and our goal is to rectify that situation. Our motivating question is how close the f^*-vector of a given lattice polytope is to being *unimodal*, i.e., the f^*-coefficients increase up to some point and then decrease. Our main results are as follows.

Theorem 3. *Let $d \ge 2$ and let P be a d-dimensional lattice polytope. Then*

(a) $f^*_0 < f^*_1 < \cdots < f^*_{\lfloor \frac{d}{2} \rfloor - 1} \le f^*_{\lfloor \frac{d}{2} \rfloor}$;

(b) $f^*_{\lfloor \frac{3d}{4} \rfloor} > f^*_{\lfloor \frac{3d}{4} \rfloor + 1} > \cdots > f^*_d$;

(c) $f^*_k \le f^*_{d-1-k}$ *for* $0 \le k \le \frac{(d-3)}{2}$.

Examples 1 and 2 yield cases of polytopes for which the inequalities $f^*_{\lfloor \frac{3d}{4} \rfloor - 1} < f^*_{\lfloor \frac{3d}{4} \rfloor}$ and $f^*_{\lfloor \frac{d}{2} \rfloor} > f^*_{\lfloor \frac{d}{2} \rfloor + 1}$ hold, respectively, and thus the strings of inequalities in ((a)) and ((b)) can, in general, not be extended further. We record the following immediate consequence of Theorem 3.

Corollary 4. *Let P be a d-dimensional lattice polytope. Then for $0 \le k \le d$,*

$$f^*_k \ \ge \ \min\{f^*_0, f^*_d\}\,.$$

We remark that one can prove that if P is of degree $d \ge 2$, equal to its dimension, then $f^*_0 \le f^*_d$, except when the h^*-vector of P has entries $h^*_2 = \cdots = h^*_d = 1$.

Theorem 5. *The f^*-vector of a d-dimensional lattice polytope, where $1 \le d \le 13$, is unimodal. On the other hand, there exists a 15-dimensional lattice simplex with nonunimodal f^*-vector.*

Even though f^*-vectors are quite different from f-vectors of polytopes, the above results resemble striking similarities with existing theorems on f-vectors. Namely, Björner [6, 7, 8] proved that the f-vector of a simplicial d-polytope satisfies all inequalities in Theorem 3 (with the *s removed, and the last coordinate dropped). In fact, Björner also showed that in the f-analogue of Theorem 3((b)) the decrease starts from $\lfloor \frac{3(d-1)}{4} \rfloor - 1$ instead of $\lfloor \frac{3d}{4} \rfloor$, and that the inequalities in Theorem 3((a)) and ((b)) cannot be further extended, by constructing a simplicial polytope with f-vector that peaks at f_j, for any $\lfloor \frac{d}{2} \rfloor \le j \le \lfloor \frac{3(d-1)}{4} - 1 \rfloor$. Corollary 4 compares the entries of the f^*-vector with the minimum between the first and the last entry. Note that a similar relation for f-vectors of polytopes was recently proven by Hinman [15], answering a question of Bárány from the 1990s. (Hinman also proved a stronger result, namely certain lower bounds for the ratios $\frac{f_k}{f_0}$ and $\frac{f_k}{f_{d-1}}$.) The f-analogue of Theorem 5 is again older: Björner [6] showed that the f-vector of any simplicial d-polytope is unimodal for $d \le 15$ (later improved to $d \le 19$ by Eckhoff [11]), and he and Lee [5] produced examples of 20-dimensional simplicial polytopes with nonunimodal f-vectors. For a special class of polytopes we can increase the range in Theorem 3((b)). A lattice polytope P is *Gorenstein of index g* if

- nP contains no interior lattice points for $1 \le n < g$,
- gP contains a unique interior lattice point, and
- $\mathrm{ehr}_P(n-g)$ equals the number of interior lattice points in nP, for $n > g$.

This is equivalent to P having degree $d+1-g$ and a symmetric h^*-vector (with respect to its degree).

Theorem 6. *Let P be a d-dimensional Gorenstein polytope of index g. Then*

$$f^*_{k-1} > f^*_k \qquad \text{for } \tfrac{1}{2}\left(d+1+\left\lfloor \tfrac{d+1-g}{2} \right\rfloor\right) \le k \le d\,.$$

Going even further, for a certain class of polytopes we can prove unimodality of the f^*-vector, a consequence of the following refinement of Theorem 3((b)) for polytopes with degree $< \frac{d}{2}$.

Theorem 7. *Let P be a d-dimensional lattice polytope with positive degree $\le s$. Then*

$$f^*_{k-1} > f^*_k \qquad \text{for } \left\lceil \tfrac{d+s}{2} \right\rceil \le k \le d\,,$$

unless the degree of P is 0, i.e., P is a unimodular simplex with f^*-vector as in Example 2.

This theorem implies that lattice d-polytopes of degree s satisfying $s^2 - s - 1 \le \frac{d}{2}$ have a unimodal f^*-vector (see Proposition 9 below for details). One family with asymptotically small degree, compared to the dimension, is given by taking iterated pyramids. Given a polytope $P \subset \mathbb{R}^d$, we denote by $\mathrm{Pyr}(P) \subset \mathbb{R}^{d+1}$ the convex hull of P and the $(d+1)$st unit vector. It is well known that P and $\mathrm{Pyr}(P)$ have the same h^*-vector (ignoring an extra 0), and so we conclude:

Corollary 8. *If P is any lattice polytope then $\mathrm{Pyr}^n(P)$ has unimodal f^*-vector for sufficiently large n.*

2 Selected proofs

The aim of this section is to offer some insight into the proofs of the results stated in Section 1. See [3] for full details of the proofs.

The main ingredient in the proofs of Theorem 3((a)) and ((c)) and Theorem 7 is the nonnegativity of h^*-vectors; we omit the proof details. Corollary 8 is a consequence of Theorem 7 together with the following proposition, whose proof again relies on the nonnegativity of h^*-vectors.

Proposition 9. *Let P be a d-dimensional lattice polytope that has degree at most s for some $s \geq 1$. If $d \geq 2s^2 - 2s - 2$ then the f^*-vector is unimodal with a (not necessarily "sharp") peak at f^*_p, where $\lfloor \frac{d}{2} \rfloor \leq p \leq \lceil \frac{d+s}{2} \rceil - 1$.*

The next proofs use more than just the nonnegativity of h^*-vectors. The first result needs the following elementary lemma on binomial coefficients.

Lemma 10. *Let j, k, n be positive integers such that $k \leq n + 1 - j$. Then, for $n \neq 2k - 1$,*

$$\left| \binom{n}{k} - \binom{n}{k-1} \right| \;\geq\; \left| \binom{n-j}{k} - \binom{n-j}{k-1} \right| .$$

We are now prepared to prove Theorem 3((b)).

Proof of Theorem 3((b)). The inequality $f^*_{d-1} > f^*_d$ holds by Theorem 7. Now, let $\left\lfloor \frac{3d}{4} \right\rfloor + 1 \leq k < d$. By (1.4),

$$f^*_{k-1} - f^*_k \;=\; \sum_{j=0}^{k+1} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \,. \tag{2.1}$$

The difference $\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j}$ is nonnegative whenever $k - j \geq \lfloor \frac{d+1-j}{2} \rfloor$ and negative otherwise. Hence, the difference is nonnegative whenever $j \leq 2k - d$ and negative whenever $j > 2k - d$. Since $2d - 2k < 2k + 1 - d$ for $\lfloor \frac{3d}{4} \rfloor + 1 \leq k$, from (2.1) we obtain

$$f^*_{k-1} - f^*_k \;\geq\; \sum_{j=0}^{2d-2k} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \tag{2.2}$$

$$+ \sum_{j=2k+1-d}^{k+1} \left(\binom{d+1-j}{k-j} - \binom{d+1-j}{k+1-j} \right) h^*_j \tag{2.3}$$

where the differences appearing in (2.2) are nonnegative and the ones in (2.3) are negative. Our aim is to compare the sums in (2.2) and (2.3) to conclude that $f^*_{k-1} - f^*_k$ is positive. Using standard identities for binomial coefficients, the right hand-side of (2.2) equals

$$\sum_{j=0}^{2d-2k}\left(\sum_{l=j}^{2d-2k-1}\left(\binom{d-l}{k-l}-\binom{d-l}{k+1-l}\right)+\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\right)h^*_j$$
$$=\sum_{l=0}^{2d-2k-1}\left(\left(\binom{d-l}{k-l}-\binom{d-l}{k+1-l}\right)\sum_{j=0}^{2d-2k-1-l}h^*_j\right)$$
$$+\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=0}^{2d-2k}h^*_j\,,$$

whence we conclude that right hand-side of (2.2) is bounded below by

$$\left(\binom{d}{k}-\binom{d}{k+1}\right)h^*_0+\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=0}^{2d-2k}h^*_j$$
$$>\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=0}^{2d-2k}h^*_j \tag{2.4}$$

since $\binom{d}{k}-\binom{d}{k+1}>0$ for $\left\lfloor\frac{3d}{4}\right\rfloor+1\le k<d$, and $h^*_0=1, h^*_j\ge 0$ for $j=1,\dots,2d-2k-1$. On the other hand, for the differences appearing in (2.3), using that $2d-2k<j$ and $j\le k+1$, it follows by Lemma 10 that

$$\left|\binom{d+1-(2d-2k)}{d+1-k}-\binom{d+1-(2d-2k)}{d-k}\right|\ge\left|\binom{d+1-j}{d+1-k}-\binom{d+1-j}{d-k}\right|,$$

i.e.,

$$\left|\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right|\ge\left|\binom{d+1-j}{k-j}-\binom{d+1-j}{k+1-j}\right|.$$

Hence for $j\ge 2k+1-d$,

$$-\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\le\binom{d+1-j}{k-j}-\binom{d+1-j}{k+1-j}.$$

Since both $-\binom{d+1-j}{k-j}+\binom{d+1-j}{k+1-j}$ and h^*_j are nonnegative for $j\ge 2k+1-d$, the sum in (2.3) is bounded below by

$$-\left(\binom{2k-d+1}{3k-2d}-\binom{2k-d+1}{3k-2d+1}\right)\sum_{j=2k+1-d}^{d}h^*_j\,. \tag{2.5}$$

Now (2.4) and (2.5) yield

$$f_{k-1}^* - f_k^* > \left(\binom{2k-d+1}{3k-2d} - \binom{2k-d+1}{3k-2d+1} \right) \left(\sum_{j=0}^{2d-2k} h_j^* - \sum_{j=2k+1-d}^{d} h_j^* \right).$$

Hibi [13] showed that the inequality

$$\sum_{j=0}^{m+1} h_j^* \;\geq\; \sum_{j=d-m}^{d} h_j^* \tag{2.6}$$

holds for $m = 0, ..., \lfloor \frac{d}{2} \rfloor - 1$. Since $2d - 2k - 1 \leq \lfloor \frac{d}{2} \rfloor - 1$ for $\left\lfloor \frac{3d}{4} \right\rfloor + 1 \leq k$, we can use (2.6) to finally obtain

$$f_{k-1}^* - f_k^* > 0\,.$$

□

Proof of Theorem 5. If $d = 1$ or 2, there is nothing to prove. If $3 \leq d \leq 6$, then by Theorem 3, either

$$f_0^* \leq \cdots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \geq f_{\lfloor \frac{3d}{4} \rfloor}^* \geq \cdots \geq f_d^*$$

or

$$f_0^* \leq \cdots \leq f_{\lfloor \frac{d}{2} \rfloor}^* \leq f_{\lfloor \frac{3d}{4} \rfloor}^* \geq \cdots \geq f_d^*.$$

For $7 \leq d \leq 13$, it suffices to check that if $f_i^* \geq f_{i+1}^*$, then $f_{i+1}^* \geq f_{i+2}^*$, for all $\left\lfloor \frac{d}{2} \right\rfloor \leq i \leq \left\lfloor \frac{3d}{4} \right\rfloor - 2$. By Theorem 3, this will imply the unimodality of $(f_0^*, f_1^*, ..., f_d^*)$. Using the inequality

$$\sum_{j=1}^{m+1} (h_j^* - h_{d+1-j}^*) \;>\; 0\,, \tag{2.7}$$

which holds for $m = 0, ..., \lfloor \frac{d}{2} \rfloor - 1$ [17, Remark 1.2], as well as the nonnegativity of h^*-vectors, for each value of d, we were able to verify that $f_{i+1}^* - f_{i+2}^* \geq c(f_i^* - f_{i+1}^*)$ for some nonnegative real c in each case.

To construct a polytope with nonunimodal f^*-vector, we employ a family of simplices introduced by Higashitani [14]. Concretely, denote the jth unit vector by e_j and let

$$\Delta_w \;:=\; \operatorname{conv}\{0, e_1, e_2, ..., e_{14}, w\}$$

where

$$w \;:=\; (\underbrace{1, 1, \ldots, 1}_{7}, \underbrace{131, 131, \ldots, 131}_{7}, 132)\,.$$

It has h^*-vector

$$(1, \underbrace{0, 0, \ldots, 0}_{7}, 131, \underbrace{0, 0, \ldots, 0}_{7})$$

and, via (1.4), f^*-vector

$$(16, 120, 560, 1820, 4368, 8008, 11440, 13001, \\ 12488, 11676, 11704, 10990, 7896, 3788, 1064, 132)\,. \qquad \square$$

We record the following consequence of Theorem 5, which follows by the nonnegativity of h^*-vectors.

Corollary 11. *Every lattice polytope of degree at most* 5 *has unimodal* f^**-vector.*

Proof of Theorem 6. Let $s := d + 1 - g$. We first consider the case that s is odd; the case s even will be similar. Since $h^*_j = 0$ for $j > s$ and $h^*_j = h^*_{s-j}$,

$$\begin{aligned} f^*_{k-1} - f^*_k &= \sum_{j=0}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j + \sum_{j=\lfloor \frac{s}{2} \rfloor+1}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j \\ &= \sum_{j=0}^{\lfloor \frac{s}{2} \rfloor} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \right) h^*_j \,. \end{aligned}$$

Because we assume $k \geq \frac{1}{2}(d + 1 + \lfloor \frac{s}{2} \rfloor)$,

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \; > \; 0$$

for $0 \leq j \leq \lfloor \frac{s}{2} \rfloor$. The inequality

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} > 0$$

follows directly if $\binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \geq 0$ or $k - s + j + 1 < 0$. Otherwise, Lemma 10 implies that, for the same range of j,

$$\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \; \geq \; 0\,.$$

In fact, the last inequality is strict for $k \geq \frac{1}{2}(d + 1 + \lfloor \frac{s}{2} \rfloor)$ (the proof is analogous to that of Lemma 10). Finally we use that $h^*_j \geq 0$ and $h^*_0 = 1$ to deduce that $f^*_{k-1} - f^*_k > 0$. The computations in the case s even is very similar. Now we write

$$\begin{aligned}
f^*_{k-1} - f^*_k &= \sum_{j=0}^{s} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} \right) h^*_j \\
&= \sum_{j=0}^{\frac{s}{2}-1} \left(\binom{d-j+1}{k-j} - \binom{d-j+1}{k-j+1} + \binom{d-s+j+1}{k-s+j} - \binom{d-s+j+1}{k-s+j+1} \right) h^*_j \\
&\quad + \left(\binom{d-\frac{s}{2}+1}{k-\frac{s}{2}} - \binom{d-\frac{s}{2}+1}{k-\frac{s}{2}+1} \right) h^*_{\frac{s}{2}}
\end{aligned}$$

and use the same argumentation as in the case s odd. □

3 Concluding Remarks

There are many avenues to explore f^*-vectors, e.g., along analogous studies of h^*-vectors, and we hope the above results form an enticing starting point. The techniques in our proof of Theorem 5 do not offer much insight in the case of 14-dimensional lattice polytopes as there are candidates f^*-vectors with corresponding h^*-vectors that satisfy all inequalities discussed in [17]. It is unknown, though, if such polytopes exist.

Higashitani [14, Theorem 1.1] provided examples of d-dimensional polytopes with nonunimodal h^*-vector for all $d \geq 3$. Therefore, by Theorem 5 we have examples of polytopes that have such an h^*-vector but their f^*-vector is unimodal. It would be interesting to know if the opposite can be true, that is, if there exist polytopes with unimodal h^*-vector and nonunimodal f^*-vector. By Corollary 11, such polytopes would need to have degree at least 6.

Whenever one detects that a given given is unimodal, it is natural to ask about the stronger property that the polynomial is log concave or, even stronger, real rooted. Our methods do not yield these properties but it would be interesting if one could extend, e.g., Corollary 8 or Proposition 9 along these lines.

Finally, starting with Stapledon's work [17], there has been much recent attention to symmetric decompositions of h- and h^*-polynomials; see, e.g., [1, 2] and, in particular, [9] where analogous decompositions for f-vectors are discussed. We believe this line of research is worthy of attention with regards to understanding f^*-vectors and the inequalities that hold among their coefficients.

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