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Degenerations of the Grassmannian via matroidal subdivisions of the hypersimplex

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Abstract. We study initial degenerations of $\mathrm{Gr}_0(3,8)$—the open part of the Grassmannian $\mathrm{Gr}(3,8)$ defined by the nonvanishing of all Plücker coordinates—using matroid subdivisions of the $(3,8)$–hypersimplex. We show that all initial degenerations are smooth and, outside of a single symmetry class, irreducible. As an application, we show that the Chow quotient of $\mathrm{Gr}(3,8)$ by the diagonal torus of $\mathrm{PGL}(8)$ is the log canonical compactification of the moduli space of 8 lines in the projective plane in linear general position. This fully resolves a conjecture of Hacking, Keel, and Tevelev.

Keywords: matroids, Grassmannians, tropical geometry

1 Introduction

Regular subdivisions of lattice polytopes, as developed by Gelfand, Kapranov, and Zelevinsky [7] with roots in the 1908 work of Voronoi [24], are combinatorial constructions that have deep connections to algebra and geometry. Of particular relevance is the relationship between regular matroidal subdivisions of the hypersimplex and initial degenerations of the Grassmannian. Denote by $\mathrm{Gr}(d,n)$ the Grassmannian of d-planes in $\mathbb{C}^n$ and $\mathrm{Gr}^\circ(d,n)$ the open locus of $\mathrm{Gr}(d,n)$ defined by the nonvanishing of all Plücker coordinates. Given a vector w in the tropicalization $\mathrm{TGr}^\circ(d,n)$ of $\mathrm{Gr}^\circ(d,n)$, we may form its w-initial degeneration $\mathrm{in}_{\mathsf{w}}\,\mathrm{Gr}^\circ(d,n)$. On the combinatorial side, the vector w induces a regular subdivision of the hypersimplex $\Delta(d,n)$ into matroid polytopes, a result which is proved independently by Lafforgue [18] and Speyer [22], see also [11]. From such a subdivision, one may form a finite inverse limit of thin Schubert cells (also known as matroid strata), which we denote by $\mathrm{Gr}(\mathsf{w})$. The first author in [3] shows that there is a closed immersion $\mathrm{in}_{\mathsf{w}}\,\mathrm{Gr}^\circ(d,n) \hookrightarrow \mathrm{Gr}(\mathsf{w})$, which is an isomorphism for $(2,n)$, $(3,6)$ and $(3,7)$.

In this extended abstract we discuss the main results and techniques of *The Grassmannian of 3-planes in* $\mathbb{C}^8$ *is schön* [4]. We study the initial degenerations of $\mathrm{Gr}^\circ(3,8)$ via finite inverse limits of thin Schubert cells as discussed in the previous paragraph.

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Theorem 1.1. *The initial degenerations of* $\mathrm{Gr}^\circ(3,8)$ *are smooth.*

In particular, $\mathrm{Gr}^\circ(3,8)$ is *schön* in the sense of Tevelev [23] (see the characterization in [9]). We also consider connectedness of the initial degenerations of $\mathrm{Gr}^\circ(3,8)$. Surprisingly, we find that they are not all connected.

Theorem 1.2. *There exists an initial degeneration of* $\mathrm{Gr}^\circ(3,8)$ *with two connected components. Up to* $\mathfrak{S}_8$*–symmetry, the remaining initial degenerations are connected.*

Given the size of the Plücker ideal, it is challenging to directly prove that the initial degenerations of $\mathrm{Gr}^\circ(3,8)$ are smooth, even with computer assistance. By contrast, once the regular subdivision is computed, one may determine "by hand" if the corresponding inverse limit is smooth, irreducible, and compute its dimension. We illustrate this in Example 5.4. Of course, this is only useful if the closed immersions $\mathrm{in}_{\mathsf{w}}\,\mathrm{Gr}^\circ(3,8) \hookrightarrow \mathrm{Gr}(\mathsf{w})$ are isomorphisms. It turns out that they are. For example, if $\mathrm{Gr}(\mathsf{w})$ is smooth, irreducible and has dimension 15, then $\mathrm{in}_{\mathsf{w}}\,\mathrm{Gr}^\circ(3,8) \hookrightarrow \mathrm{Gr}(\mathsf{w})$ is an isomorphism. Thus, while these results are geometric, the proofs of Theorems 1.1 and 1.2 use techniques from polyhedral geometry, matroid theory, and commutative algebra.

Denote by $X(d,n)$ the moduli space of n hyperplanes in $\mathbb{P}^{d-1}$ in linear general position, up to projective transformations. Our main application of the previous theorems is to study compactifications of $X(3,8)$. The diagonal torus $H \subset \mathrm{PGL}(n)$ acts freely on $\mathrm{Gr}^\circ(d,n)$, and the quotient $\mathrm{Gr}^\circ(d,n)/H$ is isomorphic to $X(3,8)$; this is the famous Gelfand-MacPherson correspondence. The normalization $\overline{X}(d,n)$ of the Chow quotient $\mathrm{Gr}(d,n)/\!/H$ compactifies $X(d,n)$, but it is in general not the log canonical compactification, failing already for $(3,9)$ and $(4,8)$ [16]. Hacking, Keel, and Tevelev conjecture in [*loc. cit.* , Conjecture 1.6] that $\overline{X}(d,n)$ is the log canonical compactification in the cases $(2,n)$, $(3,6)$, $(3,7)$, and $(3,8)$. This is motivated by the relationship between these spaces and moduli spaces of del Pezzo surfaces as in [8]. The first 3 cases were handled previously by [15], [20], and [3], respectively.

Theorem 1.3. *The normalization* $\overline{X}(d,n)$ *of the Chow quotient* $\mathrm{Gr}(3,8)/\!/H$ *is the log canonical compactification of* $X(3,8)$*.*

Code

Our results rely on extensive computations using `polymake.jl` [6, 12] and `OSCAR` [5, 21], both of which run using `julia` [2]. The code can be found at the following link:

https://github.com/dcorey2814/Gr38Schoen

Conventions and notation

Given a positive integers $d \le n$, we write $[n] = \{1, \dots, n\}$ and $\binom{[n]}{d}$ for the collection of d-element subsets of $[n]$. For subsets with a small number of elements, we use juxtaposition to denote the subset they form, e.g., if $a, b \in \lambda$, then we write ab for $\{a, b\}$. Given an abelian group A and a field F, we denote by $A_F = A \otimes_{\mathbb{Z}} F$. We write $\mathbf{1} = (1, 1, \dots, 1)$ viewed either in $\mathbb{Z}^m$ or $\mathbb{R}^m$. The symmetric group on $[n]$ is denoted by $\mathfrak{S}_n$.

2 Background

2.1 Regular matroid subdivisions

Given positive integers $d \le n$, a (d, n)–*matroid* is a nonempty subset $\mathsf{Q} \subset \binom{[n]}{d}$ satisfying the *basis-exchange axiom*: given distinct $\lambda_1, \lambda_2 \in \mathsf{Q}$ and $a \in \lambda_1 \setminus \lambda_2$, there is $b \in \lambda_2$ such that $\lambda_1 \setminus a \cup b \in \mathsf{Q}$. Denote by $\epsilon_1, \dots, \epsilon_n$ the standard basis of $\mathbb{R}^n$, and ϵ_λ the indicator vector of $\lambda \subset [n]$. The *polytope* of Q, written $\Delta(\mathsf{Q})$, is the convex hull of the vectors $\epsilon_\lambda \in \mathbb{R}^n$ for $\lambda \in \mathsf{Q}$; in fact, these are the vertices of $\Delta(\mathsf{Q})$. The *hypersimplex* $\Delta(d, n)$ is the polytope of the uniform matroid $\binom{[n]}{d}$, and its vertices are naturally labeled by $\binom{[n]}{d}$.

Given $\mathsf{w} \in \mathbb{R}^{\binom{[n]}{d}}/\langle \mathbf{1} \rangle$, denote by $\mathcal{Q}(\mathsf{w})$ the regular subdivision of $\Delta(d, n)$ induced by w. The subdivision $\mathcal{Q}(\mathsf{w})$ is *matroidal* if each polytope in $\mathcal{Q}(\mathsf{w})$ is the polytope of a matroid. The set of $\mathsf{w} \in \mathbb{R}^{\binom{[n]}{d}}/\langle \mathbf{1} \rangle$ such that $\mathcal{Q}(\mathsf{w})$ is matroidal is called the (d, n)-*Dressian*, denoted $\mathrm{Dr}(d, n)$. There is a fan structure on $\mathrm{Dr}(d, n)$, where two vectors w and w' lie in the relative interior of the same cone if and only if $\mathcal{Q}(\mathsf{w}) = \mathcal{Q}(\mathsf{w}')$; this is called the *secondary fan structure* of $\mathrm{Dr}(d, n)$, which we denote by $\mathcal{S}_{\mathrm{mat}}(d, n)$.

The *tight span* of $\mathcal{Q}(\mathsf{w})$, denoted by $\mathsf{TS}(\mathsf{w})$, is a polyhedral complex dual to $\mathcal{Q}(\mathsf{w})$ that has a k-dimensional (bounded) cell for each $(n - k - 1)$–dimensional cell of $\mathcal{Q}(\mathsf{w})$ meeting the relative interior of $\Delta(d, n)$ [10]. The *dual graph* of $\mathcal{Q}(\mathsf{w})$, denoted by $\Gamma(\mathsf{w})$, is the 1–skeleton of $\mathsf{TS}(\mathsf{w})$, i.e., it has a vertex for each maximal cell of $\mathcal{Q}(\mathsf{w})$, and two vertices share an edge if and only if their corresponding polytopes share a facet.

2.2 Thin Schubert cells of the Grassmannian

As a set, the Grassmannian $\mathrm{Gr}(d, n)$ consists of the d-dimensional linear subspaces of $\mathbb{C}^n$. It is given the structure of a $d(n - d)$–dimensional projective variety via the Plücker embedding $\iota : \mathrm{Gr}(d, n) \hookrightarrow \mathbb{P}^{\binom{[n]}{d}-1}$: concretely, if V is the row-span of a full-rank $d \times n$ matrix A, then $\iota(V)$ is the homogeneous vector of maximal minors of A. The components of $\iota(V)$ are called the *Plücker coordinates* of V; these are only well-defined up to a simultaneous scaling by a nonzero complex number.

Given $V \in \mathrm{Gr}(d,n)$, its matroid is the set of $\lambda \in \binom{[n]}{d}$ such that the λ-th Plücker coordinate of V is nonzero. Given a $\mathbb{C}$-realizable (d,n)–matroid Q, its *thin Schubert cell* is the locally-closed subscheme $\mathrm{Gr}(\mathsf{Q})$ of $\mathrm{Gr}(d,n)$ consisting of those V whose matroid is Q. Alternatively, it is the scheme-theoretic intersection $\mathrm{Gr}(\mathsf{Q}) = \mathrm{Gr}(d,n) \cap (\mathbb{C}^*)^{\mathsf{Q}}$, thus we frequently view $\mathrm{Gr}(\mathsf{Q})$ as a closed subscheme of $(\mathbb{C}^*)^{\mathsf{Q}}$. The thin Schubert cell of the uniform matroid $\binom{[n]}{d}$ is denoted by $\mathrm{Gr}^\circ(d,n)$, i.e., it is the open subvariety defined by the nonvanishing of all Plücker coordinates. The decomposition of the Grassmannian into thin Schubert cells refines the decomposition by Schubert cells, see [17, §1.1] for a nice discussion on this.

2.3 Initial Degenerations

Let M, N be a pair of rank m dual lattices, and let $\langle \mathsf{u}, \mathsf{v} \rangle$ be the natural pairing for $\mathsf{u} \in M$ and $\mathsf{v} \in N$. Write $x^{\mathsf{u}} \in \mathbb{C}[M]$ to denote the monomial corresponding to $\mathsf{u} \in M$. Given $f = \sum c_{\mathsf{u}} x^{\mathsf{u}} \in \mathbb{C}[M]$ and $\mathsf{w} \in N_{\mathbb{R}}$, the w-*initial form* is

$$\mathsf{in}_{\mathsf{w}}(f) = \sum_{\substack{\langle \mathsf{u},\mathsf{w}\rangle \\ \text{is minimal}}} c_{\mathsf{u}} x^{\mathsf{u}}.$$

Denote by $T = N \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^m$, let $I \subset \mathbb{C}[M]$ be an ideal, and $X = V(I) \subset T$ its vanishing locus. The w-initial ideal of I is $\mathsf{in}_{\mathsf{w}} I = \langle \mathsf{in}_{\mathsf{w}}(f) : f \in I \rangle$. The *tropicalization* of X is $\mathrm{Trop}(X) = \{\mathsf{w} \in N_{\mathbb{R}} : \mathsf{in}_{\mathsf{w}} I \neq \langle 1 \rangle\}$; it is the underlying set of a N-rational polyhedral fan. Given $\mathsf{w} \in \mathrm{Trop}(X)$, the w-*initial degeneration* of X is $\mathsf{in}_{\mathsf{w}}(X) = V(\mathsf{in}_{\mathsf{w}} I)$.

In this work we study initial degenerations of $\mathrm{Gr}^\circ(d,n)$. We abbreviate $\mathrm{Trop}\,\mathrm{Gr}^\circ(d,n)$ by $\mathrm{TGr}^\circ(d,n)$. By [11, 22], if $\mathsf{w} \in \mathrm{TGr}^\circ(d,n)$, then $\mathcal{Q}(\mathsf{w})$ is matroidal, and by [1], there is a subfan $\mathcal{S}_{\mathrm{trop}}(3,8)$ of $\mathcal{S}_{\mathrm{mat}}(3,8)$ whose support is $\mathrm{TGr}^\circ(3,8)$. The fan $\mathcal{S}_{\mathrm{trop}}(3,8)$ coarsen the Gröbner fan structure on $\mathrm{TGr}^\circ(3,8)$, see [4, §2.3].

2.4 Inverse Limits

The *face order* on the set of (d,n)–matroids is the relation $\mathsf{Q}_1 \preccurlyeq \mathsf{Q}_2$ whenever $\Delta(\mathsf{Q}_1)$ is a face of $\Delta(\mathsf{Q}_2)$. When $\mathsf{Q}_1 \preccurlyeq \mathsf{Q}_2$, the coordinate projection $(\mathbb{C}^*)^{\mathsf{Q}_2} \to (\mathbb{C}^*)^{\mathsf{Q}_1}$ induces a morphism $\mathrm{Gr}(\mathsf{Q}_2) \to \mathrm{Gr}(\mathsf{Q}_1)$ by [18, Proposition I.6]. Given $\mathsf{w} \in \mathrm{TGr}^\circ(d,n)$, the cells of $\mathcal{Q}(\mathsf{w})$ and the morphisms induced by the face relations form a finite inverse system, and we denote the corresponding inverse limit by $\mathrm{Gr}(\mathsf{w})$. In fact, the inverse limit of the subdiagram induced by the tight span is isomorphic to $\mathrm{Gr}(\mathsf{w})$.

Theorem 2.1. *[3] Given $\mathsf{w} \in \mathrm{TGr}^\circ(d,n)$, there exists a closed immersion $\mathsf{in}_{\mathsf{w}}\,\mathrm{Gr}^\circ(d,n) \hookrightarrow \mathrm{Gr}(\mathsf{w})$. If $\mathrm{Gr}(\mathsf{w})$ is smooth, irreducible, and of dimension $d(n-d)$, then this closed immersion is an isomorphism.*

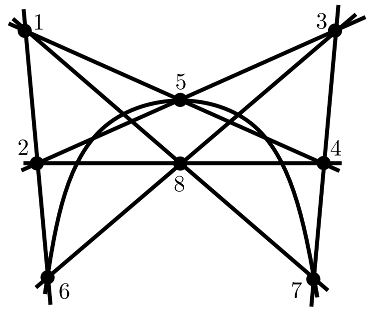

Figure 2.1: The Möbius-Kantor arrangement Q_{mk}

3 A reducible initial degeneration

By Mnëv's universality theorem (see [18, 19]), every singularity type appears in a thin Schubert cell of a rank-3 matroid. However, it is known that the thin Schubert cells of $(2, n)$, $(3, 6)$, and $(3, 7)$–matroids are smooth and irreducible. We show in [4] that the thin Schubert cells of $(3, 8)$–matroids are smooth, and exhibit a matroid whose thin Schubert cell has 2 connected components. We use this matroid to find an initial degeneration of $\mathrm{Gr}^{\circ}(3, 8)$ that is reducible. We sketch these constructions here.

Let Q_{mk} be the $(3, 8)$–matroid whose nonbases are given by the colinearities in Figure 2.1. This appears throughout the literature as the *Möbius-Kantor* or *MacLane* matroid. Since Q_{mk} is connected, we have that $\mathrm{Gr}(\mathsf{Q}_{\mathsf{mk}}) \cong X(\mathsf{Q}_{\mathsf{mk}}) \times H$ as a consequence of the Gelfand-MacPherson correspondence. Up to a projective transformation, the normal vectors to a hyperplane arrangement realizing Q_{mk} can be brought to the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & a & 1 & a \\ 0 & 0 & 1 & 1 & 1 & 0 & 1-a & 1 \end{bmatrix} \tag{3.1}$$

where $a \in \mathbb{C}$ is one of the two solutions to $x^2 - x + 1 = 0$. In fact,

$$X(\mathsf{Q}_{\mathsf{mk}}) \cong \mathrm{Spec}\left(\mathbb{C}[a]/\langle a^2 - a + 1\rangle\right),$$

and so $\mathrm{Gr}(\mathsf{Q}_{\mathsf{mk}})$ is smooth has 2 connected components.

Proposition 3.1. *If* Q *is a* $\mathbb{C}$*-realizable* $(3, 8)$*–matroid, then* $\mathrm{Gr}(\mathsf{Q})$ *is smooth. Additionally, if* Q *is not isomorphic to* Q_{mk}*, then* $\mathrm{Gr}(\mathsf{Q})$ *is irreducible.*

Given distinct $i, j, k \in [8]$, define matroids

$$\mathsf{Q}_{ijk} = \{\lambda \in \tbinom{[8]}{3} \,:\, |ijk \cap \lambda| \geq 2\} \qquad \mathsf{Q}'_{ijk} = \{\lambda \in \tbinom{[8]}{3} \,:\, |ijk \cap \lambda| = 2\}.$$

Denote by $\mathsf{w}_{\mathsf{mk}} \in \mathbb{R}^{\binom{[8]}{3}}$ the vector $(\mathsf{w}_{\mathsf{mk}})_\lambda = 3 - \operatorname{rank}(\lambda)$ for $\lambda \in \binom{[8]}{3}$. The tight span $\mathsf{TS}(\mathsf{w}_{\mathsf{mk}})$ is a star-graph whose central node corresponds to Q_{mk} and the 8 leaves correspond to Q_{ijk} for $ijk \in \binom{[n]}{3} \setminus \mathsf{Q}_{\mathsf{mk}}$. From this, one may show that

$$\operatorname{Gr}(\mathsf{w}_{\mathsf{mk}}) \cong \operatorname{Gr}(\mathsf{Q}_{\mathsf{mk}}) \times_{\prod_{ijk} \operatorname{Gr}(\mathsf{Q}'_{ijk})} \prod_{ijk} \operatorname{Gr}(\mathsf{Q}_{ijk}) \cong X(\mathsf{Q}_{\mathsf{mk}}) \times (\mathbb{C}^*)^{15}.$$

and so $\operatorname{Gr}(\mathsf{w}_{\mathsf{mk}})$ is smooth, 15 dimensional, and has 2 connected components. Denote by $\mathcal{C}_{\mathsf{mk}}$ the closed cone of $\operatorname{Trop}\operatorname{Gr}^\circ(3,8)$ containing w_{mk} in its relative interior. This discussion yields the following theorem.

Theorem 3.2. *For any cone $\mathcal{C}$ of $\mathcal{S}_{\mathrm{trop}}(3,8)$ in the $\mathfrak{S}_8$–orbit of $\mathcal{C}_{\mathsf{mk}}$ and w in the relative interior of $\mathcal{C}$, the initial degeneration $\operatorname{in}_{\mathsf{w}} \operatorname{Gr}^\circ(3,8)$ is smooth and has 2 connected components.*

Remark 3.3. The subdivision $\mathcal{Q}(\mathsf{w}_{\mathsf{mk}})$ is the only matroidal subdivision of $\Delta(3,8)$ that contains $\Delta(\mathsf{Q}_{\mathsf{mk}})$ as a cell.

4 Decomposing tight spans: leaves, branches, and fins

In light of Theorem 2.1, we focus on inverse limits to deduce information about initial degenerations. In most cases, it is convenient to decompose an inverse system into subsystems, and compute the limit in pieces. As we are taking these limits over diagrams coming from tight spans, this amounts to considering subcomplexes of the polytopal complex $\mathsf{TS}(\mathsf{w})$. We describe these pieces for a general polytopal complex Σ.

A *leaf* of Σ is a leaf-vertex of the 1-skeleton of Σ. A *leaf-pair* (v, e) is a leaf v, along with its adjacent edge e. Let Σ_L be the subcomplex of the tight span obtained by removing all leaf-pairs from $\mathsf{TS}(\mathsf{w})$. Observe that $\Sigma_{\mathsf{L}} \subset \Sigma$ may still have leaves. The process of iteratively removing leaf-pairs must terminate, and the result is the subcomplex $\Sigma_{\mathsf{Br}} \subset \Sigma$, which is nonempty provided Σ is not a tree. A maximal connected subgraph of $\Sigma \setminus \Sigma_{\mathsf{Br}}$ is called a *branch* of Σ. Any vertex in a branch is called a *branch vertex* and any edge contained in a branch, or connecting a branch to the rest of Σ, is called a *branch edge*.

Let F be a closed 2-dimensional cell of Σ with $k \geq 3$ vertices. We say that F is a *fin* if its intersection with $\overline{\Sigma \setminus F}$ is a path of edge-length ℓ with $1 \leq \ell \leq k-2$ (see Figure 5.1). We call this the *connecting path* of F. Denote by $V(F)$ and $E(F)$ the vertices and edges of F. We say $v \in V(F)$, respectively $e \in E(F)$, is *exposed* if $v \notin \overline{\Sigma \setminus F}$, respectively $e \notin \overline{\Sigma \setminus F}$. Denote by

$$\mathcal{E}_{\mathrm{vert}}(F) = \{v \in V(F) : \ v \text{ is exposed}\} \quad \mathcal{E}_{\mathrm{edge}} = \{e \in E(F) : \ e \text{ is exposed}\}.$$

Given a collection of fins $\mathfrak{F}$ denote by $\Sigma(\mathfrak{F}) \subset \Sigma$ the subcomplex obtained by removing, for each $F \in \mathfrak{F}$, the relatively open cell F° and the exposed vertices and edges of F.

5 Evaluating inverse limits

5.1 Matrix coordinates for thin Schubert cells

In this section, we give a presentation of the coordinate ring of a thin Schubert cell using *matrix coordinates*. Let Q be a (d,n)–matroid; without loss of generality, suppose $[d] \in \mathsf{Q}$ (see [3, 4] for the more general treatment). Define $\mathbb{C}[x_{ij}] := \mathbb{C}[x_{ij} \,:\, i \in [d], j \in [n-d]]$ and consider the $\mathbb{C}[x_{ij}]$–valued matrix

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 & x_{11} & \dots & x_{1,n-d} \\ 0 & 1 & \dots & 0 & x_{21} & \dots & x_{2,n-d} \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 & x_{d1} & \dots & x_{d,n-d} \end{bmatrix} \tag{5.1}$$

Define

$$B_{\mathsf{Q}} = \mathbb{C}[x_{ij} \,:\, [d]\triangle\{i, d+j\} \in \mathsf{Q}]$$

where $[d]\triangle\{i,d+j\}$ denotes the symmetric difference, and define the projection ring homomorphism

$$\pi_{\mathsf{Q}} : \mathbb{C}[x_{ij}] \to \mathbb{C}[x_{ij}]/\langle x_{ij} \,:\, [d]\triangle\{i,d+j\} \in \tbinom{[n]}{d} \setminus \mathsf{Q}\rangle \cong B_{\mathsf{Q}}.$$

Denote by $\mathsf{col}_i A$ the i-th column of A, and given $\lambda = \{\lambda_1 < \cdots < \lambda_d\} \subset [n]$ define the $d \times d$ submatrix $A_\lambda = [\mathsf{col}_{\lambda_1} A \cdots \mathsf{col}_{\lambda_d} A]$. Define the ideal I_{Q} and multiplicative semigroup S_{Q} by

$$I_{\mathsf{Q}} = \langle \pi_{\mathsf{Q}}(\det A_\lambda) \,:\, \lambda \in \tbinom{[n]}{d} \setminus \mathsf{Q}\rangle, \quad S_{\mathsf{Q}} = \langle \pi_{\mathsf{Q}}(\det A_\lambda) \,:\, \lambda \in \mathsf{Q}\rangle_{\text{semigp}}.$$

The coordinate ring of $\mathrm{Gr}(\mathsf{Q})$ is isomorphic to $S_{\mathsf{Q}}^{-1} B_{\mathsf{Q}} / I_{\mathsf{Q}}$.

5.2 Coordinate rings and subcomplexes

Let $\mathsf{w} \in \mathrm{TGr}^\circ(d,n)$. For a face C of $\mathsf{TS}(\mathsf{w})$, denote by Δ_C the cell of $\mathcal{Q}(\mathsf{w})$ corresponding to C. In particular, if v is a vertex, then Δ_v is a maximal cell of $\mathsf{TS}(\mathsf{w})$. A connected subcomplex $\Sigma \subset \mathsf{TS}(\mathsf{w})$ is *vertex-intersecting* if

$$\bigcap_{v \in V(\Sigma)} \Delta_v \neq \emptyset.$$

where $V(\Sigma)$ denotes the vertex set of Σ. In other words, the matroids of the maximal cells in $\mathcal{Q}(\mathsf{w})$ corresponding to $V(\Sigma)$ share a common basis. The subcomplex Σ is *vertex-connecting* if, for each vertex x of $\Delta(d,n)$, the subcomplex of Σ consisting of those cells C such that $x \in \Delta_C$ is empty or connected.

Let $\Sigma \subset \mathsf{TS}(\mathsf{w})$ be a vertex-intersecting connected subcomplex. Given a vertex $v \in \mathsf{TS}(\mathsf{w})$, denote by Q_v the matroid of the polytope Δ_v. Then we have a basis common to all matroids Q_v with $v \in V(\Sigma)$. Without loss of generality, suppose that this basis is $[d]$. Define a polynomial subring B_Σ of $\mathbb{C}[x_{ij}]$ by

$$B_\Sigma = \mathbb{C}\left[x_{ij} : [d] \triangle \{i, d+j\} \in \mathsf{Q}_v \text{ for some } v \in V(\Sigma)\right]$$

Next, define the ideal and multiplicative semigroup

$$I_\Sigma = \sum_{v \in V(\Sigma)} I_{\mathsf{Q}_v} \cdot B_\Sigma \qquad \text{and} \qquad S_\Sigma = \langle S_{\mathsf{Q}_v} : v \in V(\Sigma) \rangle_{\text{semigp}};$$

note that we may view each S_{Q_v} as a subset of B_Σ under the inclusion $B_{\mathsf{Q}_v} \subset B_\Sigma$. Finally, set

$$R_\Sigma = (S_\Sigma)^{-1} B_\Sigma / I_\Sigma \tag{5.2}$$

Proposition 5.1. *If Σ is vertex-intersecting and vertex-connecting, then the coordinate ring of the inverse limit $\varprojlim_\Sigma \mathrm{Gr}$ is isomorphic to R_Σ.*

5.3 A combinatorial sieve of diagrams

In this section we discuss the approach used to prove Theorems 1.1 and 1.2 from the introduction.

Theorem 5.2. *Let $\mathsf{w} \in \mathrm{TGr}^\circ(d, n)$.*

1. *The inverse limit $\mathrm{Gr}(\mathsf{w})$ is smooth of dimension 15.*
2. *If w is not in the interior of a cone in the $\mathfrak{S}_8$ orbit of $\mathcal{C}_{\mathsf{mk}}$, then $\mathrm{Gr}(\mathsf{w})$ is irreducible.*
3. *The closed immersion $\mathrm{in}_{\mathsf{w}} \mathrm{Gr}^\circ(3,8) \hookrightarrow \mathrm{Gr}(\mathsf{w})$ is an isomorphism.*

The secondary fan structure of the tropical Grassmannian $\mathrm{TGr}^\circ(3,8)$ is computed in [1]. A *combinatorial type* is a $\mathfrak{S}_8$–orbit cone in $\mathcal{S}_{\text{trop}}(3,8)$, and we label a combinatorial type by a vector lying in the relative interior of a cone representative. There are 57344 combinatorial types of cones in $\mathcal{S}_{\text{trop}}(3,8)$. Fix a combinatorial type w and set $\Sigma = \mathsf{TS}(\mathsf{w})$. Consider the following conditions, ordered by increasing complexity:

1. The tight span $\mathsf{TS}(\mathsf{w})$ is vertex-intersecting.
2. The dual graph $\Gamma(\mathsf{w})$ is a tree.
3. The subcomplex $\Sigma_{\mathsf{L}} \subset \mathsf{TS}(\mathsf{w})$ is vertex-intersecting.
4. The subcomplex $\Sigma_{\mathsf{Br}} \subset \mathsf{TS}(\mathsf{w})$ is vertex-intersecting.
5. The subcomplex of Σ_{L} obtained by removing all fins of Σ_{L} whose connecting path had length 1 is vertex-intersecting.
6. The subcomplex of Σ_{L} obtained by removing all fins of Σ_{L} is a tree.

While these conditions are not mutually exclusive, we can still use them to separate the combinatorial types into 6 sets: set G_i consists of those combinatorial types satisfying condition i but not j for $j < i$. This sorting is done using `OSCAR`, and we see that

$$|\mathsf{G}_1| = 13641, \quad |\mathsf{G}_2| = 215, \quad |\mathsf{G}_3| = 28227, \quad |\mathsf{G}_4| = 483, \quad |\mathsf{G}_5| = 14389, \quad |\mathsf{G}_6| = 389.$$

In [4, §6], we prove Theorem 5.2 by employing a different strategy for each set G_i. Sets $\mathsf{G}_2, \mathsf{G}_3, \mathsf{G}_4, \mathsf{G}_5$ all follow a similar strategy, with G_5 requiring the most general technique. For this set, we verify Theorem 5.2 using the following proposition.

Proposition 5.3. *Let $\mathfrak{F}$ be a collection of fins of Σ_{L}. If each fin $F \in \mathfrak{F}$ is B-maximal and $\varprojlim_{\Sigma_{\mathsf{L}}(\mathfrak{F})} \mathrm{Gr}$ is smooth and irreducible then $\mathrm{Gr}(\mathsf{w})$ is smooth and irreducible. Furthermore, the dimension of $\mathrm{Gr}(\mathsf{w})$ can be computed using Proposition 6.15 of [4].*

The definition of B-maximality is technical, so we refer the reader to the full paper [4, §4.4]. In the following example we demonstrate how one can compute the inverse limit obtained from a matroid subdivision using Proposition 5.3.

Example 5.4. Let e_{ijk} for $\{i < j < k\} \in \binom{[8]}{3}$ denote the standard basis of $\mathbb{R}^{\binom{[8]}{3}}$ and let

$$\begin{aligned} \mathsf{w} = &\mathsf{e}_{124} + \mathsf{e}_{125} + 2\mathsf{e}_{126} + 2\mathsf{e}_{134} + 3\mathsf{e}_{137} + \mathsf{e}_{145} + \mathsf{e}_{146} + 2\mathsf{e}_{147} + \mathsf{e}_{148} + \mathsf{e}_{156} + 2\mathsf{e}_{235} + 2\mathsf{e}_{237} + \\ &\mathsf{e}_{245} + \mathsf{e}_{246} + \mathsf{e}_{256} + 2\mathsf{e}_{257} + \mathsf{e}_{258} + 3\mathsf{e}_{347} + 2\mathsf{e}_{357} + 2\mathsf{e}_{367} + 2\mathsf{e}_{368} + 2\mathsf{e}_{378} + 2\mathsf{e}_{456} + 3\mathsf{e}_{678}. \end{aligned}$$

The tight span $\Sigma = \mathsf{TS}(\mathsf{w})$ of $\mathcal{Q}(\mathsf{w})$ is given in Figure 5.1. The edges given by (v_2, v_3), (v_3, v_{14}) and (v_{12}, v_{13}) are leaves. Furthermore the face $\mathsf{F} = \langle v_8, v_{11}, v_{12} \rangle$ of Σ is a fin (see Section 4). Note that F is the unique fin of Σ with connecting path of edge-length 1; let $\mathfrak{F} = \{\mathsf{F}\}$. By considering the matroids of the vertices in $\Sigma_{\mathsf{L}}(\mathfrak{F})$ one readily verifies that this complex is vertex-intersecting, so $\mathsf{w} \in \mathsf{G}_5$.

Observe that the pentagonal and hexagonal cells of Σ_{L} are indeed fins by the general definition in Section 4. However, we have that F is B-maximal (a fact we defer to the full paper). Hence, it suffices to show that $\varprojlim_{\Sigma_{\mathsf{L}}(\mathfrak{F})} \mathrm{Gr}$ is smooth and irreducible to apply Proposition 5.3.

Denote by Q_i the matroid corresponding to the vertex v_i of $\Gamma(\mathsf{w})$. When v_i and v_j share an edge, the matroid of the edge is denoted by $\mathsf{Q}_{i,j}$. Direct computation tells us the dimensions of the thin Schubert cells corresponding to the vertices and edges of $\mathsf{TS}(\mathsf{w})$ are given by

$$\dim \mathrm{Gr}(\mathsf{Q}_i) = 7 \text{ for } 1 \leq i \leq 14 \qquad \dim \mathrm{Gr}(\mathsf{Q}_{i,j}) = 6 \text{ for } (v_i, v_j) \in E(\Gamma(\mathsf{w}))$$

Moreover, we have $\dim \mathrm{Gr}(\mathsf{Q}_{\mathsf{F}}) = 5$ for $\mathsf{F} = \langle v_8, v_{11}, v_{12} \rangle$ of Σ_{L}. Finally, we employ the technique described in Section 5.2 to determine $R_{\Sigma_{\mathsf{L}}(\mathfrak{F})}$. We compute

$$B_{\Sigma_{\mathsf{L}}(\mathfrak{F})} = \mathbb{C}[x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{31}, x_{32}, x_{34}, x_{35}].$$

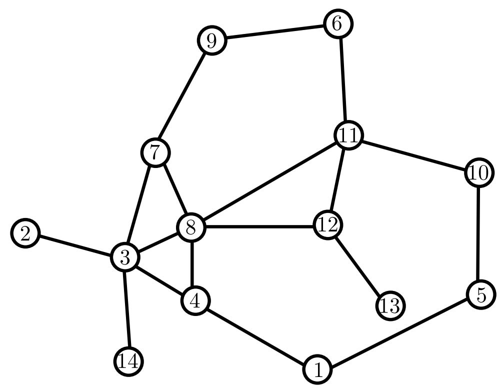

Figure 5.1: Tight span of $\mathcal{Q}(\mathsf{w})$

The ideal is

$$I_{\Sigma_{\mathsf{L}}(\mathfrak{F})} = \langle x_{13}x_{25} - x_{23}x_{15},\ x_{11}x_{32}x_{25} + x_{31}x_{22}x_{15},\ x_{11}x_{32}x_{23} + x_{31}x_{22}x_{13}\rangle,$$

and the semigroup $S_{\Sigma_{\mathsf{L}}(\mathfrak{F})}$ contains all monomials of $B_{\Sigma_{\mathsf{L}}(\mathfrak{F})}$. Let f_1, f_2, f_3 denote the generators of $I_{\Sigma_{\mathsf{L}}(\mathfrak{F})}$. The relations $f_i \equiv 0 \mod I_{\Sigma_{\mathsf{L}}(\mathfrak{F})}$ (for $i = 1, 2$) yield

$$x_{13} \equiv \frac{x_{23}x_{15}}{x_{25}} \quad \text{and} \quad x_{32} \equiv -\frac{x_{31}x_{22}x_{15}}{x_{25}x_{11}} \mod I_{\Sigma_{\mathsf{L}}(\mathfrak{F})}.$$

After these substitutions, the last relation disappears, and so the coordinate ring of $\varprojlim_{\Sigma(\mathfrak{F})} \mathrm{Gr}$ is isomorphic to a Laurent polynomial ring in 12 variables (after possibly inverting a finite number of elements). So we have that $\varprojlim_{\Sigma_{\mathsf{L}}(\mathfrak{F})} \mathrm{Gr}$ is smooth, irreducible and of dimension 12. By B-maximality of F, we have that $\mathrm{Gr}(\mathsf{w})$ is smooth and irreducible by Proposition 5.3. One may use Equation 6.5 in [4] to verify that $\dim \mathrm{Gr}(\mathsf{w}) = 15$. Hence, by Theorem 2.1, $\mathsf{in}_{\mathsf{w}} \mathrm{Gr}^{\circ}(3, 8)$ is smooth and irreducible.

6 The Chow quotient of the Grassmannian

Recall that $X(d, n)$ is the moduli space n hyperplanes in $\mathbb{P}^{d-1}$ in linear general position up to projective transformation. The normalization of the Chow quotient of $\mathrm{Gr}(d, n)$ by the diagonal torus $H \subset \mathrm{PGL}(n)$, which we denote by $\overline{X}(d, n)$, is a compactification of $X(d, n)$. In this section, we sketch the proof of Theorem 1.3, that $\overline{X}(3, 8)$ is the log canonical compactification of $X(3, 8)$, which fully resolves [16, Conjecture 1.6]. For details, see [4, §7].

We follow an approach outlined by Hacking, Keel, and Tevelev in their construction of the log canonical compactification of moduli spaces of marked del Pezzo surfaces [8]. The first key fact we need is that $\overline{X}(d, n)$ is the closure of $X(d, n)$ in the toric variety of the fan $\mathcal{S}_{\text{trop}}(3,8)/L$, where L is the lineality space of $\mathcal{S}_{\text{trop}}(3,8)$, see [14, 13]. By [8, Theorems 3.1, 9.1], to show that the log-canonical divisor of $X(3,8) \subset \overline{X}(3,8)$ is ample, it suffices to prove the following.

- The moduli space $X(3,8)$ is schön.
- The fan $\mathcal{S}_{\text{trop}}(3,8)/L$ is convexly disjoint in the sense of [8, Definition 1.15].

Schönness of $X(3,8)$ follows from Theorem 1.1 and the fact that $\mathsf{in}_{\mathsf{w}}\,\mathrm{Gr}^{\circ}(3,8) \cong \mathsf{in}_{\overline{\mathsf{w}}}X(3,8) \times H$ (where $\overline{\mathsf{w}}$ is the image of w under the projection $\mathbb{R}^{\binom{[8]}{3}}/\langle \mathbf{1} \rangle \to \mathbb{R}^{\binom{[8]}{3}}/L$). For the second point, we show that the secondary fan structure on the Dressian (see [11]) $\mathrm{Dr}(3,8)/L$ is convexly disjoint (we do this for a general (d,n)), which shows that the subfan $\mathcal{S}_{\text{trop}}(3,8)/L$ is convexly disjoint.

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