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Pizza and 2-structures

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Abstract. Let $\mathcal{H}$ be a Coxeter hyperplane arrangement in n-dimensional Euclidean space. Assume that the negative of the identity map belongs to the associated Coxeter group W. Furthermore assume that the arrangement is not of type A_1^n. Let K be a measurable subset of the Euclidean space with finite volume which is stable by the Coxeter group W and let a be a point such that K contains the convex hull of the orbit of the point a under the group W. In a previous article the authors proved the generalized pizza theorem: that the alternating sum over the chambers T of $\mathcal{H}$ of the volumes of the intersections $T \cap (K + a)$ is zero. In this paper we give a dissection proof of this result. In fact, we lift the identity to an abstract dissection group to obtain a similar identity that replaces the volume by any valuation that is invariant under affine isometries. This includes the cases of all intrinsic volumes. Apart from basic geometry, the main ingredient is a previous theorem of the authors where we relate the alternating sum of the values of certain valuations over the chambers of a Coxeter arrangement to similar alternating sums for simpler subarrangements called 2-structures, introduced by Herb to study discrete series characters of real reduced groups.

Résumé. Soit $\mathcal{H}$ un arrangement d'hyperplans de Coxeter dans un espace euclidien V de dimension n. On suppose que $\mathcal{H}$ n'est pas de type A_1^n et que l'isométrie $x \mapsto -x$ sur V appartient à son groupe de Coxeter W. Soit K un sous-ensemble mesurable de mesure finie de V qui est stable par W, et soit a un point de V tel que K contienne l'enveloppe convexe de l'orbite de a sous W. Dans un article précédent, nous avons prouvé le "théorème de la pizza généralisé", qui dit que la somme alternée sur les chambres T de $\mathcal{H}$ du volume de $T \cap (K + a)$ est nulle. Dans cet article, nous donnons une preuve géométrique par dissection de ce résultat. Plus précisément, nous relevons l'identité ci-dessus dans un groupe de dissection abstrait, ce qui implique des identités similaires lorsque le volume est remplacé par n'importe quelle valuation (mesure

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finiment additive) invariante par isométries affines. Ceci inclut en particulier le cas des volumes intrinsèques sur les ensembles convexes compacts. Les ingrédients principaux de la preuve sont de la géométrie élémentaire et un théorème précédent des auteurs qui calcule la somme alternée des valeurs d'une valuation sur les chambres d'un arrangement de Coxeter en fonction de sommes alternées similaires associées à des sous-arrangements plus simples appelés 2-structures, introduits à l'origine par Herb pour étudier les caractères des séries discrètes des groupes réductifs réels.

Keywords: Coxeter arrangements, 2-structures, dissections, Pizza Theorem, reflection groups, intrinsic volumes, pseudo-root systems, Bolyai-Gerwien theorem

1 Introduction

The 2-dimensional pizza theorem is the following result: Given a disc in the plane, choose a point on this disc and cut the disc by $2k$ equally spaced lines passing through the point, where $k \geq 2$. The alternating sum of the areas of the resulting slices is then equal to zero. This was first proved by Goldberg [8]. Frederickson [7] gave a dissection proof based on dissection proofs of Carter-Wagon in the case $k = 2$ (see [4]) and of Allen Schwenk (unpublished) in the cases $k = 3, 4$. Frederickson deduced dissection proofs of a similar sharing result for the pizza crust and of the so-called calzone theorem, which is the analogue of the pizza theorem for a ball in $\mathbb{R}^3$ that is cut by one horizontal plane and by $2k$ equally-spaced vertical planes all meeting at one point in the ball.

To generalize the pizza problem, consider a finite central hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$ and fix a base chamber of this arrangement. Each chamber T has a sign $(-1)^T$ determined by the parity of the number of hyperplanes separating it from the base chamber. If K is a measurable subset of $\mathbb{R}^n$ of finite volume, what can we say about the pizza quantity $\sum_T (-1)^T \operatorname{Vol}(T \cap K)$, where the sum runs over all the chambers T of $\mathcal{H}$? The original pizza theorem is the case where $n = 2$, $\mathcal{H}$ has the type of the dihedral arrangement $I_2(2k)$ and K is a disc containing the origin. The calzone theorem is the case where $n = 3$, $\mathcal{H}$ has the type of the product arrangement $I_2(2k) \times A_1$ and K is a ball containing the origin.

The following generalization of the pizza and the calzone was proved in [6, Corollary 6.9] by analytic means. We also recently learned that Brailov had proved independently the same theorem for a ball and an arrangement of type B_n in [3], by somewhat similar methods.

Theorem 1.1 (Ehrenborg, Morel and Readdy)**.** *Let $\mathcal{H}$ be a Coxeter arrangement with Coxeter group W that contains the negative of the identity map, denoted by $-\operatorname{id}$. Assume that $\mathcal{H}$ is not of type A_1^n. Let K be a set of finite measure that is stable by the group W. Then for every point $a \in \mathbb{R}^n$ such that K contains the convex hull of $\{w(a) : w \in W\}$, we have*

$$\sum_T (-1)^T \operatorname{Vol}(T \cap (K + a)) = 0.$$

The proof of this result uses an expression for $\sum_T (-1)^T \operatorname{Vol}(T \cap (K+a))$ as an alternating sum of pizza quantities over subarrangements of $\mathcal{H}$ of the form $\{e_1^\perp, \dots, e_n^\perp\}$ with $(e_1, \dots, e_n)$ an orthonormal basis of $\mathbb{R}^n$, in other words, subarrangements that have type A_1^n.

In the paper [5], we study a different sum $\sum_T (-1)^T \nu(\overline{T})$, where ν is a valuation defined on closed convex polyhedral cones of $\mathbb{R}^n$ that takes integer values. Under the same condition that $\mathcal{H}$ is a Coxeter arrangement, we rewrite this quantity as an alternating sum of similar quantities for certain subarrangements of $\mathcal{H}$ that are products of rank 1 and rank 2 arrangements [5, Theorem 3.2.1], and then deduce an expression for it. These subarrangements, called 2-*structures*, were introduced by Herb [11] to study characters of discrete series of real reductive groups. In fact, the identity of [5, Theorem 3.2.1] is valid for any valuation and its proof uses only basic properties of Coxeter systems and closed convex polyhedral cones.

In this paper we use the setting of 2-structures and [5, Corollary 3.2.4] (recalled in Theorem 2.5) to obtain a dissection proof of the higher-dimensional pizza theorem of [6, Corollary 6.9] that is independent of the results and methods of [6] (and of [3]):

Theorem 1.2 (Abstract pizza theorem; see Theorem 3.4.)**.** *With the notation and hypotheses of Theorem 1.1, we have*

$$\sum_T (-1)^T [\overline{T} \cap (K+a)] = \sum_T (-1)^T [T \cap (K+a)] = 0,$$

where the brackets denote classes in the abstract dissection group of Definition 3.1.

As we take into account lower-dimensional sets when defining our abstract dissection group, this result implies generalizations of the higher-dimensional pizza theorem to all the intrinsic volumes (if K is convex). The idea of the proof of Theorem 1.2 is the following: by expanding the expression using 2-structures, we can reduce to a sum where each term is a similar expression for an arrangement that is a product of arrangements of types A_1 and $I_2(2^k)$. We then adapt the dissection proof of Frederickson to an arrangement of type $I_2(2m) \times \mathcal{H}'$. We also explain how to keep track of lower-dimensional regions of the dissection. If our product arrangement contains at least one dihedral factor, then its contribution is zero, and we immediately get a dissection proof of the result. However, if all the product arrangements that appear are of type A_1^n, then their individual contributions are not zero. We need one extra step in the proof to show that the contributions cancel. This step uses a slight refinement of the Bolyai-Gerwien theorem presented in Section 4. An interesting point to note is that the shape of the pizza plays absolutely no role in this proof, as long as it has the same symmetries as the arrangement and contains the convex hull of $\{w(a) : w \in W\}$. In particular, we no longer need to assume that it is measurable and of finite volume.

Let us mention some interesting questions that remain open:

(1) The paper [6] proves the pizza theorem for more general arrangements (the condition is that the arrangement $\mathcal{H}$ is a Coxeter arrangement and that its number of hyperplanes is greater than the dimension n and has the same parity as that dimension), but only in the case of the ball; see [6, Corollary 7.6]. Is it possible to give a dissection proof of this result?

(2) Mabry and Deiermann [16] show that the two-dimensional pizza theorem does not hold for a dihedral arrangement having an odd number of lines. More precisely, they determine the sign of the quantity $\sum_T (-1)^T \operatorname{Vol}(T \cap K)$, where K is a disc containing the origin, and show that it vanishes if and only if the center of K lies on one of the lines. Their methods are analytic. As far as we know, there exists no dissection proof of this result either. The higher-dimensional case where $\mathcal{H}$ is a Coxeter arrangement and the number of its hyperplanes does not have the same parity as n also remains wide open.

2 Review of 2-structures and of the basic identity

Let V be a finite-dimensional real vector space with an inner product $(\cdot,\cdot)$. For every $\alpha \in V$, we denote by H_α the hyperplane $\alpha^\perp$ and by s_α the orthogonal reflection in the hyperplane H_α.

We say that a subset Φ of V is a *normalized pseudo-root system* if:

(a) Φ is a finite set of unit vectors;

(b) for all $\alpha, \beta \in \Phi$, we have $s_\beta(\alpha) \in \Phi$ (in particular, taking $\alpha = \beta$, we get that $-\alpha \in \Phi$).

Elements of Φ are called *pseudo-roots*. The *rank* of Φ is the dimension of its span.

We call such objects *pseudo-root systems* to distinguish them from the root systems that appear in representation theory. If Φ' is a root system then $\Phi = \{\alpha/\|\alpha\| : \alpha \in \Phi'\}$ is a normalized pseudo-root system. Not every normalized pseudo-root system arises in this manner; see for instance the pseudo-root systems of type H_3 and H_4.

We say that a normalized pseudo-root system Φ is *irreducible* if, whenever $\Phi = \Phi_1 \sqcup \Phi_2$ with Φ_1 and Φ_2 orthogonal, we have either $\Phi_1 = \varnothing$ or $\Phi_2 = \varnothing$. Every normalized pseudo-root system can be written uniquely as a disjoint union of irreducible normalized pseudo-root systems. Irreducible normalized pseudo-root systems are classified: they are in one of the infinite families A_n ($n \geq 1$), B_n/C_n ($n \geq 2$), [1] D_n ($n \geq 4$), $I_2(m)$ ($m \geq 3$) or one of the exceptional types E_6, E_7, E_8, F_4, H_3 or H_4, with types $I_2(3)$ and A_2 isomorphic, as well as types $I_2(4)$ and B_2. (See [10, Chapter 5] or Table 1 in [1, Appendix A].)

[1] The pseudo-root systems of types B_n and C_n are identical after normalizing the lengths of the roots.

We say that a subset $\Phi^+ \subset \Phi$ is a *positive system* if there exists a total ordering $<$ on the $\mathbb{R}$-vector space V such that $\Phi^+ = \{\alpha \in \Phi : \alpha > 0\}$ (see [13, Section 1.3]). The *Coxeter group* of Φ is the group of isometries W of V generated by the reflections s_α for $\alpha \in \Phi$. This group preserves Φ by definition of a normalized pseudo-root system, and it acts simply transitively on the set of positive systems by [13, Section 1.4]. In particular, the Coxeter group W is finite.

Let E be a finite set of unit vectors of V such that $E \cap (-E) = \varnothing$. The corresponding *hyperplane arrangement* is the set of hyperplanes $\mathcal{H} = \{H_e : e \in E\}$. A *chamber* of $\mathcal{H}$ is a connected component of $V - \bigcup_{e \in E} H_e$; we denote by $\mathscr{T}(\mathcal{H})$ the set of chambers of $\mathcal{H}$. Fix a chamber T_0 to be the *base chamber*. For a chamber $T \in \mathscr{T}(\mathcal{H})$ we denote by $S(T, T_0)$ the set of $e \in E$ such that the two chambers T and T_0 are on different sides of the hyperplane H_e, and define the *sign* of T to be $(-1)^T = (-1)^{|S(T,T_0)|}$.

We say that $\mathcal{H}$ is a *Coxeter arrangement* if it is stable by the orthogonal reflections in each of its hyperplanes. In that case, the set $\Phi = E \cup (-E)$ is a normalized pseudo-root system. We call its Coxeter group the Coxeter group of the arrangement. The map sending a positive system $\Phi^+ \subset \Phi$ to the set $\{v \in V : \forall \alpha \in \Phi^+ \ (v, \alpha) > 0\}$ is a bijection from the positive systems in Φ to the chambers of $\mathcal{H}$. See for example [2, Chapitre V § 4 No. 8 Proposition 9 p. 99] and the discussion following it. Conversely, if $\Phi \subset V$ is a normalized pseudo-root system with Coxeter group W and $\Phi^+ \subset \Phi$ is a positive system, then $\mathcal{H} = \{H_\alpha : \alpha \in \Phi^+\}$ is a Coxeter hyperplane arrangement, and in that case we always take the base chamber T_0 to be the chamber corresponding to Φ^+.

Let us define product arrangements. Let V_1 and V_2 be two finite-dimensional real vector spaces equipped with inner products, and suppose that we are given hyperplane arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ on V_1 and V_2 respectively. We consider the product space $V_1 \times V_2$, where the factors are orthogonal. The *product arrangement* $\mathcal{H}_1 \times \mathcal{H}_2$ is then the arrangement on $V_1 \times V_2$ with hyperplanes $H \times V_2$ for $H \in \mathcal{H}_1$ and $V_1 \times H'$ for $H' \in \mathcal{H}_2$. If $\mathcal{H}_1$ is the empty arrangement, then we write $V_1 \times \mathcal{H}_2$ instead of the confusing $\varnothing \times \mathcal{H}_2$. Similarly, if $\mathcal{H}_2$ is the empty arrangement, we write $\mathcal{H}_1 \times V_2$. If the arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$ arise from normalized pseudo-root systems $\Phi_1 \subset V_1$ and $\Phi_2 \subset V_2$, then their product $\mathcal{H}_1 \times \mathcal{H}_2$ arises from the normalized pseudo-root system $\Phi_1 \times \{0\} \cup \{0\} \times \Phi_2 \subset V_1 \times V_2$. We also denote this pseudo-root system by $\Phi_1 \times \Phi_2$.

The notion of 2-structures was introduced by Herb for root systems to study the characters of discrete series representations; see for example the review article [11]. The definition we give here is Definition B.2.1 of [5]. It has been slightly adapted to work for pseudo-root systems.

Definition 2.1. Let Φ be a normalized pseudo-root system with Coxeter group W. A 2-*structure* for Φ is a subset φ of Φ satisfying the following properties:

(a) The subset φ is a disjoint union $\varphi = \varphi_1 \sqcup \varphi_2 \sqcup \cdots \sqcup \varphi_r$, where the φ_i are pairwise orthogonal subsets of φ and each of them is an irreducible pseudo-root system of type A_1, B_2 or $I_2(2^k)$ for $k \geq 3$.

(b) Let $\varphi^+ = \varphi \cap \Phi^+$. If $w \in W$ is such that $w(\varphi^+) = \varphi^+$ then $\det(w) = 1$.

We denote by $\mathcal{T}(\Phi)$ the set of 2-structures for Φ.

Proposition 2.2. *Let Φ be a normalized pseudo-root system with Coxeter group W.*

(i) The group W acts transitively on the set of 2-structures $\mathcal{T}(\Phi)$.

(ii) The pseudo-root system Φ and its 2-structures have the same rank if and only if there exists $w \in W$ whose restriction to $\mathrm{Span}(\Phi)$ is equal to $-\mathrm{id}_{\mathrm{Span}(\Phi)}$.

To each 2-structure $\varphi \subset \Phi$, we can associate a sign $\epsilon(\varphi) = \epsilon(\varphi, \Phi^+)$ (see the start of Section 5 and Lemma 5.1 of [12] and Definition B.2.8 of [5]).

We next introduce the abstract pizza quantity. Let $\mathcal{H}$ be a central hyperplane arrangement on V. Let $\mathcal{C}_{\mathcal{H}}(V)$ be the set of closed convex polyhedral cones in V that are intersections of closed half-spaces bounded by hyperplanes H where $H \in \mathcal{H}$, and let $K_{\mathcal{H}}(V)$ be the quotient of the free abelian group $\bigoplus_{K \in \mathcal{C}_{\mathcal{H}}(V)} \mathbb{Z}[K]$ on $\mathcal{C}_{\mathcal{H}}(V)$ by the relation $[K] + [K'] = [K \cup K'] + [K \cap K']$ for all $K, K' \in \mathcal{C}_{\mathcal{H}}(V)$ such that $K \cup K' \in \mathcal{C}_{\mathcal{H}}(V)$. For $K \in \mathcal{C}_{\mathcal{H}}(V)$, we still denote the image of K in $K_{\mathcal{H}}(V)$ by $[K]$.

Definition 2.3. Suppose that we have fixed a base chamber of $\mathcal{H}$. The *abstract pizza quantity* of $\mathcal{H}$ is

$$P(\mathcal{H}) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [\overline{T}] \in K_{\mathcal{H}}(V).$$

Remark 2.4. By Lemma 3.2.3 of [5], we have

$$P(\mathcal{H}) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [T].$$

We use this alternative definition of $P(\mathcal{H})$ in our proofs.

The following result is Corollary 3.2.4 of [5]. It shows how to evaluate the pizza quantity for a Coxeter arrangement in terms of the associated 2-structures.

Theorem 2.5. *Let $\Phi \subset V$ be a normalized pseudo-root system. Choose a positive system $\Phi^+ \subset \Phi$ and let $\mathcal{H}$ be the hyperplane arrangement $(H_\alpha)_{\alpha \in \Phi^+}$ on V with base chamber corresponding to the positive system Φ^+. For every 2-structure $\varphi \in \mathcal{T}(\Phi)$, we write $\varphi^+ = \varphi \cap \Phi^+$ and we denote by $\mathcal{H}_\varphi$ the hyperplane arrangement $(H_\alpha)_{\alpha \in \varphi^+}$ with base chamber corresponding to φ^+. Then we have*

$$P(\mathcal{H}) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) P(\mathcal{H}_\varphi).$$

If $\varphi \in \mathcal{T}(\Phi)$ then the closures of the chambers of $\mathcal{H}_\varphi$ are elements of $\mathcal{C}_{\mathcal{H}}(V)$, so $P(\mathcal{H}_\varphi)$ makes sense as an element of $K_{\mathcal{H}}(V)$.

3 A dissection proof of the higher-dimensional pizza theorem

Definition 3.1. Let $\mathcal{C}(V)$ be a non-empty family of subsets of V that is stable by finite intersections and affine isometries and such that, if $C \in \mathcal{C}(V)$ and D is a closed affine half-space of V, then $C \cap D \in \mathcal{C}(V)$. Furthermore, we assume that $\mathcal{C}(V)$ is closed with respect to Cartesian products, that is, if $C_i \in \mathcal{C}(V_i)$ for $i = 0, 1$ then $C_0 \times C_1 \in \mathcal{C}(V_0 \times V_1)$. For example, we could take $\mathcal{C}(V)$ to be the family of all convex subsets of V, or of all closed (or compact) convex subsets, or of all convex polyhedra.

We denote by $K(V)$ the quotient of the free abelian group $\bigoplus_{C \in \mathcal{C}(V)} \mathbb{Z}[C]$ on $\mathcal{C}(V)$ by the three relations: (i) $[\varnothing] = 0$; (ii) $[C \cup C'] + [C \cap C'] = [C] + [C']$ for all $C, C' \in \mathcal{C}(V)$ such that $C \cup C' \in \mathcal{C}(V)$; and (iii) $[g(C)] = [C]$, for every $C \in \mathcal{C}(V)$ and every affine isometry g of V. For $C \in \mathcal{C}(V)$, we still denote the image of C in $K(V)$ by $[C]$.

A *valuation* on $\mathcal{C}(V)$ with values in an abelian group A is a function $\mathcal{C}(V) \longrightarrow A$ that can be extended to a morphism of groups $K(V) \longrightarrow A$.

Remark 3.2. Define $\mathcal{B}(V)$ to be the relative Boolean algebra generated by $\mathcal{C}(V)$, that is, the smallest collection of subsets of V that contains $\mathcal{C}(V)$ and is closed under finite unions, finite intersections and set differences. Groemer's Integral Theorem states that a valuation on $\mathcal{C}(V)$ can be extended to a valuation on the Boolean algebra $B(V)$; see [9] and also [15, Chapter 2]. Applying this to the valuation $C \longmapsto [C]$ with values in $K(V)$, we see that we can make sense of $[C]$ for any $C \in \mathcal{B}(V)$. For instance, we have $[C_1 \cup C_2] = [C_1] + [C_2] - [C_1 \cap C_2]$ and $[C_1 - C_2] = [C_1] - [C_1 \cap C_2]$. Moreover if $\mathcal{C}(V)$ is the set of all convex polyhedra in V, then $\mathcal{B}(V)$ contains all polyhedra (convex or not), and also half-open polyhedra.

Next we have the following straightforward lemma, whose proof we omit, which states that the class symbol is well-behaved with respect to Cartesian products.

Lemma 3.3. *The two class identities $[C_0] = [D_0]$ and $[C_1] = [D_1]$ in $K(V_0)$ and $K(V_1)$ respectively imply that $[C_0 \times C_1] = [D_0 \times D_1]$ in $K(V_0) \times K(V_1)$.*

Let $\mathcal{H}$ be a central hyperplane arrangement on V with fixed base chamber. If $K \in \mathcal{C}(V)$, we have a morphism of groups $e_K : K_{\mathcal{H}}(V) \longrightarrow K(V)$ induced by the map $\mathcal{C}_{\mathcal{H}}(V) \longrightarrow \mathcal{C}(V)$, $C \longmapsto C \cap K$.

We denote by $P(\mathcal{H}, K)$ the image of $P(\mathcal{H})$ by this morphism e_K; in other words, we have

$$P(\mathcal{H}, K) = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [\overline{T} \cap K] = \sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T [T \cap K],$$

where Remark 2.4 implies the second equality.

We state the main theorem of this paper. First for $u, v \in V$ define the half-open line segment $(u, v]$ by $\{(1-\lambda)u + \lambda v : 0 < \lambda \leq 1\}$. Our main result is the following:

Theorem 3.4 (The Abstract Pizza Theorem)**.** *Let $\mathcal{H}$ be a Coxeter hyperplane arrangement with Coxeter group W in an n-dimensional space V such that $-\mathrm{id}_V \in W$. Let $K \in \mathcal{C}(V)$ and $a \in V$. Suppose that K is stable by the group W and contains the convex hull of the set $\{w(a) : w \in W\}$.*

1. *If $\mathcal{H}$ is not of type A_1^n, we have $P(\mathcal{H}, K+a) = 0$ in $K(V)$.*
2. *If $\mathcal{H}$ has type A_1^n, Φ is the normalized pseudo-root system corresponding to $\mathcal{H}$ and $\Phi^+ = \{e_1, \ldots, e_n\}$ where $\Phi^+ \subset \Phi$ is the positive system corresponding to the base chamber of $\mathcal{H}$, then the following identity holds:*

$$P(\mathcal{H}, K+a) = \left[\prod_{i=1}^{n} (0, 2(a, e_i)e_i]\right]. \tag{3.1}$$

Here we are using Remark 3.2 to make sense of the right-hand side of equation (3.1).

The conditions on K are satisfied if for example K is convex, contained in $\mathcal{C}(V)$, stable by W and $0 \in K + a$. Indeed, the last condition implies that $-a \in K$; as $-\mathrm{id}_V \in W$ by assumption, this in turns implies that $a \in K$, hence that K contains the convex hull of the set $\{w(a) : w \in W\}$.

We will give the proof of Theorem 3.4 at the end of the section. This proof does not use Corollary 6.9 of [6], so we obtain a new proof of that result.

Let $V_0, \ldots, V_n$ denote the intrinsic volumes on V (see [18, Section 4.2]).

Lemma 3.5. *Let $(v_1, \ldots, v_k)$ be an orthogonal family of vectors in V. Then $V_i((0, v_1] \times \cdots \times (0, v_k]) = 0$ for $0 \leq i \leq k-1$.*

Corollary 3.6. *We keep the notation and hypotheses of Theorem 3.4. If $\mathcal{H}$ is not of type A_1^n, we have*

$$\sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T V_i(T \cap (K+a)) = 0 \tag{3.2}$$

for every $0 \leq i \leq n$, where K is assumed to be convex if $i \neq n$. If $\mathcal{H}$ has type A_1^n and K is convex then equation (3.2) holds for $0 \leq i \leq n-1$.

Remark 3.7. Theorem 3.4 immediately implies generalizations to our higher-dimensional case of the "thin crust" and "thick crust" results of Confection 3 and Leftovers 1 of [16] for an even number of cuts.

We obtain the "thin crust" result by evaluating the $(n-1)$st intrinsic volume on $P(\mathcal{H}, K+a)$. Note that this result holds for a pizza of any (convex) shape and even in the case where we only make n cuts, where n is the dimension.

To generalize the "thick crust" result, consider two sets $K \subset L$ stable by W and in $\mathcal{C}(V)$. If $a \in V$ is such that K contains the convex hull of the set $\{w(a) : w \in W\}$, then

$$P(\mathcal{H}, (L-K)+a) = P(\mathcal{H}, L+a) - P(\mathcal{H}, K+a) = 0,$$

so in particular

$$\sum_{T \in \mathscr{T}(\mathcal{H})} (-1)^T \operatorname{Vol}(T \cap ((L-K)+a)) = 0.$$

The case where K and L are balls with the same center is the "thick crust" result.

We now state some lemmas that will be used in the proof of Theorem 3.4.

Lemma 3.8. *Let $\mathcal{H}_i$ be a hyperplane arrangement on V_i for $i = 0, 1$. Assume furthermore that $\mathcal{H}_1 = \{H_e\}_{e \in E_1}$ has type A_1^r and $\dim(V_1) = r$. Let $E_1 = \{e_1, \ldots, e_r\}$ be the index set of $\mathcal{H}_1$. Let $\mathcal{H}$ and V be the Cartesian products $\mathcal{H}_0 \times \mathcal{H}_1$ and $V_0 \times V_1$ respectively. Then for every $K \in \mathcal{C}(V)$ that is stable under the orthogonal reflections in the hyperplanes $V_0 \times H_{e_1}, \ldots, V_0 \times H_{e_r}$ and for every $a \in V_1$, if $L = K + a$, we have the identity*

$$P(\mathcal{H}, L) = P\left(\mathcal{H}_0 \times V_1, L \cap (V_0 \times (0, 2(a, e_1)e_1] \times \cdots \times (0, 2(a, e_r)e_r])\right),$$

where $\mathcal{H}_0 \times V_1$ is the product of $\mathcal{H}_0$ and the empty hyperplane arrangement on V_1.

Lemma 3.9. *Suppose that we have $V = V_1^{(1)} \times \cdots \times V_1^{(r)} \times V_2^{(1)} \times \cdots \times V_2^{(s)}$, where the factors of the product are pairwise orthogonal, and that $\mathcal{H}$ is a product $\mathcal{H}_1^{(1)} \times \cdots \times \mathcal{H}_1^{(r)} \times \mathcal{H}_2^{(1)} \times \cdots \times \mathcal{H}_2^{(s)}$, where each $\mathcal{H}_i^{(j)}$ is a hyperplane arrangement on $V_i^{(j)}$. Suppose further that:*

(a) If $1 \le j \le r$ then $V_1^{(j)}$ is 1-dimensional, and we have a unit vector $e^{(j)}$ in $V_1^{(j)}$ yielding the hyperplane arrangement $\mathcal{H}_1^{(j)} = \{0\}$.

(b) If $1 \le j \le s$ then $V_2^{(j)}$ is 2-dimensional, and the arrangement $\mathcal{H}_2^{(j)}$ is of type $I_2(2m^{(j)})$ for some $m^{(j)} \ge 2$.

Let $a \in V$ and $K \in \mathcal{C}(V)$. Suppose that K is stable by the Coxeter group W and contains the convex hull of the set $\{w(a) : w \in W\}$. Then the following two statements hold:

(i) If $s \ge 1$ we have $P(\mathcal{H}, K+a) = 0$ in $K(V)$.

(ii) If $s = 0$ we have in $K(V)$ the identity

$$P(\mathcal{H}, K+a) = \left[(0, 2(a, e^{(1)})e^{(1)}] \times \cdots \times (0, 2(a, e^{(r)})e^{(r)}]\right].$$

Proof of Theorem 3.4. Statement (ii) is exactly Lemma 3.9(ii). We now prove statement (i), so we assume that $\mathcal{H}$ is not of type A_1^n. By Theorem 2.5, we have

$$P(\mathcal{H}, K + a) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) P(\mathcal{H}_\varphi, K + a).$$

By definition, any 2-structure for Φ is of type $A_1^r \times \prod_{k \geq 2} I_2(2^k)^{s_k}$ with $\sum_{k \geq 2} s_k$ finite and, as W acts transitively on the set of 2-structures (Proposition 2.2(i)), the integers r and s_k, for $k \geq 2$, do not depend on the 2-structure but only on Φ. Also, by Proposition 2.2(ii), we have $\dim V = r + \sum_{k \geq 2} 2 s_k$, so we are in the situation of Lemma 3.9. Suppose that $\sum_{k \geq 2} s_k \geq 1$. Then by Lemma 3.9(i) we have $P(\mathcal{H}_\varphi, K + a) = 0$ for every $\varphi \in \mathcal{T}(\Phi)$ and hence $P(\mathcal{H}, K + a) = 0$. Assume now that $\sum_{k \geq 2} s_k = 0$, that is, $s_k = 0$ for every k. Statement (ii) of the same lemma implies that

$$P(\mathcal{H}, K + a) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \left[\prod_{e \in \varphi \cap \Phi^+} (0, 2(a, e)e] \right]. \tag{3.3}$$

This is an alternating sum of classes of half-open rectangular parallelotopes in V. So we can apply Theorem 4.1 to prove that $P(\mathcal{H}, K + a) = 0$ in $K(V)$. We know that $V_i(P(\mathcal{H}, K + a)) = 0$ if $0 \leq i \leq n - 1$ by Lemma 3.5, so it remains to prove that $V_n(P(\mathcal{H}, K + a)) = 0$, that is, that the alternating sum of the volumes of the parallelotopes $\prod_{e \in \varphi \cap \Phi^+} (0, 2(a, e)e]$ is equal to zero. This follows from Corollary 6.9 of the paper [6]. However we can now give a direct proof (that does not use analysis) using the method of that corollary. Let $f : V \longrightarrow \mathbb{R}$ be the function defined by

$$f(a) = \sum_{\varphi \in \mathcal{T}(\Phi)} \epsilon(\varphi) \prod_{e \in \varphi \cap \Phi^+} 2(a, e).$$

Note that f is a polynomial homogeneous of degree n on V. Furthermore equation (3.3) implies that $\mathrm{Vol}(P(\mathcal{H}, K + a)) = f(a)$ for every convex subset K of V of finite volume that is stable by W and every $a \in V$ such that $0 \in K + a$. The polynomial f satisfies $f(w(a)) = \det(w) f(a)$ for every $w \in W$ and every $a \in V$ (this is easy to see; see for example Corollary 2.3 of [6]), so it vanishes on every hyperplane of $\mathcal{H}$. But if $f \neq 0$, then the vanishing set of f must be of degree at most n, which contradicts the fact that, as $\mathcal{H}$ is not of type A_1^n, we have $|\mathcal{H}| > n$. Hence we must have $f = 0$, and this gives the desired result. $\square$

4 The Bolyai-Gerwien theorem for parallelotopes

The classical Bolyai-Gerwien theorem states that two polygons are scissors congruent if and only if they have the same area. There is also a well-known generalization in higher dimensions that applies to parallelotopes; it follows from the characterization of translational scissors congruences in arbitrary dimensions, and was proved independently by Jessen-Sah and Sah, see the beginning of Section 7 of [14] or Theorem 1.1 in Chapter 4 of [17]. In this section, we state a slight refinement of this generalization, Theorem 4.1, that keeps track of lower-dimensional faces; in other words, we do not want ignore the boundaries.

As in the previous sections, let V be an n-dimensional real vector space with an inner product $(\cdot,\cdot)$. If $(v_1,\ldots,v_r)$ is a linearly independent list of elements of V, we define the parallelotope $P(v_1,\ldots,v_r) = \{\sum_{i=1}^{r} a_i v_i : 0 \leq a_i \leq 1\}$. We denote by $\mathcal{P}(V)$ the set of all convex polytopes in V (including lower-dimensional ones) and by $\mathcal{Z}(V)$ the subfamily of polytopes that are translates of parallelotopes of the form $P(v_1,\ldots,v_r)$. The set $\mathcal{P}(V)$ satisfies the conditions of Definition 3.1, so we can define an abelian group $K_{\mathcal{P}}(V)$ as in that definition. We denote by $K_{\mathcal{Z}}(V)$ the subgroup of $K_{\mathcal{P}}(V)$ generated by the classes $[P]$ for $P \in \mathcal{Z}(V)$. Remark 3.2 implies that, if $\mathcal{P}_{\mathrm{ext}}(V)$ is the relative Boolean algebra generated by $\mathcal{P}(V)$, then we can define the class $[P]$ in $K_{\mathcal{P}}(V)$ of any element P in $\mathcal{P}_{\mathrm{ext}}(V)$. We denote by $\mathcal{Z}_{\mathrm{ext}}(V)$ the set of elements P of $\mathcal{P}_{\mathrm{ext}}(V)$ such that $[P] \in K_{\mathcal{P}}(V)$ is in the subgroup $K_{\mathcal{Z}}(V)$. For example, the set $\mathcal{Z}_{\mathrm{ext}}(V)$ contains $\mathcal{Z}(V)$, and it also contains all half-open parallelotopes.

Let $V_0,\ldots,V_n$ be the intrinsic volumes on V; see [18, Section 4.2]. These are valuations on the set of all compact convex subsets of V, and in particular on $\mathcal{P}(V)$, so they induce morphisms of groups from $K_{\mathcal{P}}(V)$ to $\mathbb{R}$, that we still denote by $V_0,\ldots,V_n$. Note that V_0 is the Euler-Poincaré characteristic with compact support, so the image of $K_{\mathcal{P}}(V)$ is $\mathbb{Z}$.

The main result of this section is the following isomorphism.

Theorem 4.1. *The morphism* $(V_0,V_1,\ldots,V_n) : K_{\mathcal{Z}}(V) \longrightarrow \mathbb{Z}\times\mathbb{R}^n$ *is an isomorphism. In particular, if* $P,P' \in \mathcal{Z}_{\mathrm{ext}}(V)$ *are such that* $V_i(P) = V_i(P')$ *for every* $0 \leq i \leq n$*, then* $[P] = [P']$ *in* $K_{\mathcal{Z}}(V)$.

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