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Robertson's Conjecture in Algebraic Topology

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Abstract. One of the most famous results in graph theory is that of Kuratowski's theorem, which states that a graph G is non-planar if and only if it contains one of $K_{3,3}$ or K_5 as a topological minor. That is, if some subdivision of either $K_{3,3}$ or K_5 appears as a subgraph of G. In this case we say that the question of planarity is determined by a finite set of forbidden (topological) minors. A conjecture of Robertson, whose proof was recently announced by Liu and Thomas, characterizes the kinds of graph theoretic properties that can be determined by finitely many forbidden minors. In this extended abstract we will present a categorical version of Robertson's conjecture, which we have proven in certain cases. We will then illustrate how this categorification, if proven in all cases, would imply many non-trivial statements in the topology of graph configuration spaces.

Keywords: Configuration Spaces, Representations of Categories, Graph Well-Quasi-Orders, Representation Stability

1 Introduction

Let G denote a finite graph. If we view G as a simplicial complex, then one may define its **configuration space on n points** as the topological space

$$\mathcal{F}_n(G) = \{(x_1, \ldots, x_n) \in G^n \mid x_i \neq x_j\}.$$

One also considers the quotient of this space by the coordinate-permuting action of the symmetric group $\mathfrak{S}_n$

$$\mathcal{UF}_n(G) = \mathcal{F}_n(G)/\mathfrak{S}_n.$$

This latter space is called the **unlabeled configuration space on n points**. The study of these spaces has seen a tremendous increase in activity due not only to their connections to physics and robotics [1, 16], but also for their surprising theoretical properties. For the purposes of this work, we take the time to point out the following collection of results from the literature of graph configuration spaces, spanning almost 20 years:

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- (Torsion Boundedness) For any graph G and any $n \geq 1$, the group $H_1(\mathcal{UF}_n(G))$ is either torsion free, or all torsion is 2-torsion. Moreover, it is torsion free if and only if G is planar [9].
- (Universality of generators for trees) If G is a tree, and $i, n \geq 1$, then $H^i(\mathcal{UF}_n(G))$ is generated by products of H^1 classes pushed forward along embeddings of star trees into G [6]. A star tree is a tree with only one vertex of degree above one.
- (Universality of planar generators for H_2) There exists a finite set of graphs, depending only on n, such that for any planar graph G, $H_2(\mathcal{UF}_n(G))$ is generated by push forwards of classes along embeddings of the members of this finite list into G [3].

Each of these results is remarkable in its uniformity. Namely, that one can conclude extremely powerful results about the general "shapes" of these homology and cohomology groups across infinite families of graphs by only understanding much simpler cases.

The purpose of the present abstract is to present a possible framework that unifies and expands upon all of these universality phenomena. This framework accomplishes this by proposing a kind of categorification (see Conjecture 6) of an extremely powerful and delicate theorem arising from structural graph theory - Robertson's Conjecture (see Theorem 2).

The structure of this work proceeds as follows. To start, we outline the necessary graph theory background to describe Robertson's conjecture. Following this, we turn to graph categories, and explain what is meant by a representation of a category as well as what a categorification of Robertson's conjecture would necessarily look like. This second section ends by stating the technical heart of our aforementioned framework, Conjecture 6. Finally, we conclude by showing how this technical theorem can be applied to graph configuration spaces, and why each of the above can be thought as arising from this application.

2 Robertson's conjecture on topological minors

In this section, we outline the necessary background related with the underlying classical graph theory that motivates our study.

Definition 1. In this work, the terminology of **graph** will always refer to a finite *simple* graph (i.e. not permitting loops or multi-edges). A **path** in G is a sequence of vertices of G, $v_1, v_2, \ldots, v_r$ such that no vertex is repeated, and each pair of subsequent vertices is connected by an edge, modulo the involution which reverses the beginning and ending point of the sequence.. If G is a graph we will write V_G, E_G, and P_G for the sets of

vertices, edges, and **paths** of G, respectively. In cases where the graph G is understood from context, we will often drop the subscript in these notation.

If G, G' are two graphs, then a **homeomorphic embedding** from G to G' is a pair $\rho = (\rho_V, \rho_E)$ of maps of sets

$$\rho_V : V_G \to V_{G'}, \qquad \rho_E : E_G \to P_{G'}$$

satisfying the following:

1. ρ_V is an injection;
2. if $e \in E_G$ has endpoints $\{a, b\}$, then $\rho(e)$ is a path connecting a and b;
3. if $e \in E_G$ and $v \in V_G$, $\rho_V(v)$ is a member of $\rho_E(e)$ if and only if v is an end point of e.
4. if $e_1, e_2 \in E_G$ are distinct, then $\rho(e_1)$ and $\rho(e_2)$ are disjoint if e_1 and e_2 do not share an endpoint, and otherwise only intersect at the image of this common endpoint under ρ_V;

Note that a homeomorphic embedding of graphs induces a continuous simplicial embedding between the realizations of the relevant graphs as simplicial complexes. Finally, we write $G \leq_{top} G'$ to indicate that there exists a homeomorphic embedding of G into G'. We refer to the quasi-order $\leq_{top}$ as the **topological minor** relation.

Somewhat notoriously, the topological graph minor relation does permit infinite anti-chains. That is to say, it is not a **well-quasi-order**. This puts it in stark contrast to other orderings on graphs, such as the more commonly used **graph minor** relation, which was very famously proven to not have infinite anti-chains by Robertson and Seymour in [14]. One particular flavor of example of infinite anti-chain will be important enough for us to take some time and describe it in detail.

For $k \geq 1$, let R_k denote the graph which is obtained from a path with $k+1$ vertices by doubling each edge, and then subdividing to do away with each multi-edge. For instance, the graph R_1 is a triangle, whereas R_2 is two triangles joined at a common vertex. Generally, we visualize R_k as a chain with precisely k links. One can easily build infinite anti-chains on the family $\{R_k\}$ by attaching certain graphs to both ends of the chain. For the simplest example, one can attach a trio of leaves to each sides of the chain. The following conjecture of Robertson essentially claims that the anti-chains arising in this way are the basis for all others. A stronger version of this conjecture that allows for vertex labels has recently had a proof announced by Liu and Thomas [10]

Theorem 2 (Robertson's Conjecture, [10])**.** *Fix $k \geq 1$, and let $\{G_i\}_{i\geq 1}$ be any infinite collection of graphs with the property that $R_k \not\leq_{top} G_i$ for any i. Then there exists $j < l$ such that $G_j \leq_{top} G_l$. In particular, for any fixed $k \geq 1$, the restriction of $\leq_{top}$ to the class of graphs not containing R_k as a topological minor is a well-quasi-order.*

Observe that, the condition of not containing R_1 as a topological minor is equivalent to being a tree. Robertson's conjecture therefore specializes to **Kruskal's Tree Theorem** in that case. Moreover, the class of graphs not containing R_2 as a topological minor includes all graphs with maximum vertex degree 3. Robertson's conjecture for $k = 2$ therefore implies a conjecture of Vázsonyi that these subcubic graphs do not admit infinite anti-chains in the topological graph minor relation.

3 Categories of graphs

The purpose of this extended abstract is to lay out a potential categorification of Robertson's conjecture, and illustrate how this categorification can be used to prove statements in the topology of graph configuration spaces. While the most general form of this categorification will remain conjectural, we will spend some time outlining what a proof of this conjecture might look like in principal, and also the cases which have already been proven. This work is very much inspired by similar work on categorifying the graph minor theorem and Kruskal's Tree Theorem [11, 12, 13], as well as other categorification efforts such as the theory of Gröbner categories due to Sam and Snowden [15].

Definition 3. Let $k \geq 1$ be a fixed integer. We write $\mathcal{TG}_k$ to denote the category whose objects are graphs not containing R_k as a topological minor, and whose morphisms are homeomorphic embeddings.

Based on everything we saw in the previous section, it should be clear that the categories $\mathcal{TG}_k$ are precisely those that one would want to study in any attempted categorification of Robertson's Conjecture. It therefore only remains to determine what such a categorification would "look like." If $(Q, \leq)$ is any quasi-order, then the condition of being a well-quasi-order is equivalent to saying that the complement of every $\leq$-closed subset of Q has finitely many $\leq$-minimal elements. Put another way, it says that containment in these sets is determined by the appearance, or lack-there-of, of some member of a finite list of elements. When thinking about such a characterization in situations arising in more algebraic contexts, one might draw parallels to some kind of finite generation. The condition of being a well-quasi-order then becomes akin to a kind of Noetherian Property. That is to say, that every submodule of a free module is itself finitely generated.

To make all of this precise, we therefore need to start by understanding what one even means by a module in this context.

Definition 4. For any commutative ring A, a $\mathcal{TG}_k$**-module over** A, or just a $\mathcal{TG}_k$**-module**, is a covariant functor $M : \mathcal{TG}_k \to A$-mod from $\mathcal{TG}_k$ to the category of *finitely generated* A-modules.

We say that a $\mathcal{TG}_k$-module M is **finitely generated** if there is a finite list of objects in $\mathcal{TG}_k$, $\{G_i\}$, such that for any object G of $\mathcal{TG}_k$ the A-module $M(G)$ is generated by the images of the maps $M(G_i) \to M(G)$ induced by maps $G_i \to G$ of $\mathcal{TG}_k$. In this case we call the objects G_i **generators** of M.

More concretely, a $\mathcal{TG}_k$-module is a collection of A-modules, $\{M(G)\}_G$, one for every graph G not containing R_k as a topological minor, such that for any homeomorphic embedding $G \to G'$, one obtains a homomorphism $M(G) \to M(G')$, that respects composition. In slightly different language, a $\mathcal{TG}_k$-module is a representation of the quiver with relations underlying the category $\mathcal{TG}_k$.

The notion of finite generation outlined above can also be viewed more concretely in a similar fashion. In the language of quiver representations it becomes the usual notion of finite generation. Heuristically speaking, saying that a $\mathcal{TG}_k$-module is finitely generated tells one that all of the algebraic content of all of the modules $M(G)$ is determined by only a finite amount of data coming from the modules $M(G_i)$. We will use this heuristic reasoning in the following example.

Example 5. For each object G of $\mathcal{TG}_k$, set $M(G) = \mathbb{Z}$. If $G \to G'$ is a homeomorphic embedding, then we set the map $M(G) \to M(G')$ to be the identity map. Then M is a $\mathcal{TG}_k$-module over $\mathbb{Z}$, and is also seen to be generated by the graph which is just a single vertex.

Now let S be a collection of objects in $\mathcal{TG}_k$ that is closed under the topological minor relation. We may also define a module M_S by the assignments,

$$M_S(G) = \begin{cases} \mathbb{Z} & \text{if } G \notin S \\ 0 & \text{otherwise.} \end{cases}$$

By consequence of the fact that S is minor closed, it follows that M_S is a submodule of M. One may think of M_S as being the submodule of M "generated by" the graphs in the complement of S. The question now becomes whether M_S is also finitely generated.

For any G, the algebraic information encapsulated by $M_S(G)$ precisely encodes whether or not G is a member of S. Therefore, to say that M_S is finitely generated is to say that there is a finite list of graphs that determines containment in S. This is precisely the statement of Robertson's Theorem 2.

We are now ready to state the main technical conjecture of this work.

Conjecture 6 (The Categorical Robertson's Conjecture)**.** *Fix $k \geq 1$. If A is a Noetherian commutative ring, and M is a finitely generated $\mathcal{TG}_k$-module, then any submodule of M is also finitely generated.*

This conjecture represents the next step in a research program that originated in Sam and Snowden's seminal work [15]. That paper outlines a method by which one can convert purely combinatorial theorems about well-quasi-orders into algebraic statements about Noetherian properties arising in the representation theory of certain associated categories. For instance, Sam and Snowden categorified Higman's lemma - that sequences are well-quasi-ordered by the subword relation - in the original work [15], whereas later work of Barter [4] as well as Proudfoot and the second author [12, 13], do something similar for Kruskal's tree theorem. In [11], Miyata and the second author provide a categorical version of a weakened graph minor theorem as well. From this pre-existing literature we can therefore already say the following in relation to our conjecture.

Theorem 7 (Barter [4]; Proudfoot & Ramos [12, 13])**.** *If A is a Noetherian commutative ring, and M is a a finitely generated $\mathcal{TG}_1$-module, then any submodule of M is also finitely generated.*

In the originating work [8], we provide an outline of Sam and Snowden's machinery, and indicate how one would use it to possibly prove our main conjecture. Importantly, this outline argues what one would need to understand about the proof of Theorem 2 to ultimately prove Conjecture 6. To avoid bogging us down with these technicalities in this extended abstract, however, we instead opt to move on to display how the Categorical Robertson's Conjecture has exciting applications to topology.

Finally, before finishing this section, we note that just as Robertson's Theorem 2 has a stronger labeled version, so does the categorical Robertson's Conjecture. It is also the case that this labeled version of the conjecture has already been proven for $k = 1$. Because our primary applications to configuration space do not use the extra data of vertex labels, we have not presented this version here, though it should be noted that [8] use the labeled version of the conjecture to prove a variety of facts about topologies associated to **cographs**. See [8] for more details on this.

4 Graph configuration spaces

In this section we detail some applications of Conjecture 6 to graph configuration spaces, as defined at the beginning of this abstract. The reader should also keep in mind throughout that Conjecture 6 is actually a theorem in the case where $k = 1$, and so everything that follows can be seen as unconditional in that case.

The applications to graph configuration spaces in the present work are fundamentally related with the observation that if $\rho : G \to G'$ is a homeomorphic embedding, then one has a continuous map from the (labeled or unlabeled) configuration space of G to that of G'. Our goal will be to show that these maps are essentially the mechanisms underlying stability phenomena in the homology groups of these spaces.

To start, we recall the following important construction of Abrams.

Definition 8. Let G be a graph. The **discretized configuration space of** G **on** n **points** $D_n(G)$ is the sub-complex of G^n of all cells $\sigma_1 \times \sigma_2 \times \ldots \times \sigma_n$, such that for any $i \neq j$, the endpoints of σ_i are distinct from those of σ_j. As before, $\mathfrak{S}_n$ acts on $D_n(G)$, and we define the **unlabeled discretized configuration space of G on n points** to be the quotient $UD_n(G) = D_n(G)/\mathfrak{S}_n$.

This particular discretization of configuration space is not the only one that has appear in the literature (see, for instance [17]. Generally, the biggest issue with D_n is that it has far more cells than is practical to work with computationally. Because of this, other authors have applied techniques from discrete Morse theory [7] to try to trim it down to something that is more amenable to computer computation. Because our interests are ultimately more on the theoretical side, however, this overabundance of cells will not be an issue for us. Moreover, we note that $D_n(G)$ behaves relatively nicely when one subdivides the edges of G. This will be critical for us. For now we state the following theorem that reinforces our interest in these discretized spaces.

Theorem 9 (Abrams [2])**.** *Fix $n \geq 2$ and let G be a graph satisfying the **path condition**, that for any two vertices of G, x, y, with degree not equal to 2, every path from x to y has length at least $n+1$. Then there exists an $\mathfrak{S}_n$-equivariant homotopy equivalence*

$$D_n(G) \sim \mathcal{F}_n(G).$$

Notice that while the number of cells of $D_n(G)$ is sensitive to subdivision, the configuration space $\mathcal{F}_n(G)$ is not. For this reason the path condition is not going to hinder us too much.

For the remainder of this section we will present our results entirely in terms of $D_n(G)$, despite the fact that analogous statements will hold for the unordered spaces as well. We do this just for expositional clarity. In all cases, the proofs for the unlabeled case are identical.

Observe that any homeomorphic embedding $\rho : G \to G'$ can be realized as a composition of a map that purely subdivides G, followed by a map that embeds this subdivision into G'. We have already noted that the Abrams model respects subdivision, and it clearly does so for graph embeddings. It follows that if $\rho : G \to G'$ is a homeomorphic embedding, and $n \geq 2$ is fixed, then one has a cellular map between cubical complexes $D_n(G) \to D_n(G')$. In particular, one has induced maps on the level of cubical chains. Writing $\mathcal{D}_{n,G,\bullet}$ for the chain complex whose homology computes the homologies of $D_n(G)$, we have just argued for the following lemma.

Lemma 10. *The assignment*

$$G \mapsto \mathcal{D}_{n,G,i}$$

can be extended to a $\mathcal{TG}_k$ − module over $\mathbb{Z}$. Moreover, varying both i and G, $\mathcal{D}_{n,\star,\bullet}$ defines a complex of $\mathcal{TG}_k$-modules.

Our way forward now becomes clear. All we really need to do is argue that the $\mathcal{TG}_k$-modules $G \mapsto \mathcal{D}_{n,G,i}$ are finitely generated. The conclusion of Conjecture 6 would then imply the same about the homology groups of the discretized configuration spaces essentially for free! Luckily, finite generation is actually not hard to prove in this particular instance.

If we fix $k, n \geq 1$, and $i \geq 0$, and let $G_{i,n-i}$ be the (disconnected) graph which is a disjoint union of i line segments and $n-i$ isolated vertices, then $\mathcal{D}_{n,\star,i}$ is generated by $\mathcal{D}_{n,G_{i,n-i},i}$. Indeed, for any graph G, any i-cell of $D_n(G)$ can be written as $\sigma_1 \times \ldots \times \sigma_n$, where the σ_j are either edges of vertices of G that do not intersect each other at their endpoints. The data of which edges and vertices are present in this cell induces an embedding from $G_{i,n-i}$ to G.

Moving from the homologies of the discretized configuration spaces back to the homologies of the usual configuration spaces is now just a matter of being careful about the path condition.

Theorem 11. *Let $i \geq 0$ and $k, n \geq 1$ be fixed. Assuming Conjecture 6, the $\mathcal{TG}_k$-module $H_i(\mathcal{F}_n(\bullet))$ is finitely generated.*

Proof. Let $\mathcal{H}$ denote the $\mathcal{TG}_k$-submodule of $H_i(D_n(\bullet))$ generated by all $H_i(D_n(G))$, where G is a graph satisfying the path condition. By Conjecture 6, we know that $\mathcal{H}$ must be generated by some finite list of graphs $\{G_j\}$, which we may assume all satisfy the path condition. Then $H_i(\mathcal{F}_n(\bullet))$ is generated by this same finite list, where one smooths away any vertices of degree 2. □

Example 12. As mentioned at the beginning of this section, the above theorem is unconditionally true in the case $k = 1$. The category $\mathcal{TG}_1$ is equivalent to the category of trees with homeomorphic embeddings. In this case, work of Chettih and Lütgehetmann [5] implies that $H_1(\mathcal{F}_n(\bullet))$ is generated by the trees that look like the letters Y and H. They also prove similar statements for the higher homologies as well. Other than this theorem, however, very little is known beyond some partial results. This is especially the case for labeled configuration spaces.

Looking at the unlabeled case, a recent theorem of An and the first author [3] says that if you look at the full subcategory of $\mathcal{TG}_k$ of *planar* graphs, then the module

$$H_2(U\mathcal{F}_3(\bullet))$$

is generated by the following list of graphs:

- two disjoint triangles;
- the disjoint union of a triangle and a graph that looks like the letter Y;
- the disjoint union of two graphs that look like the letter Y;

- The theta graph of two vertices connected by four (subdivided) edges.

Importantly, one should observe that all of these generators can already be found in $\mathcal{TG}_2$. In other words, one does not obtain new exotic generators as k increases beyond this point. This observation motivates the next direction of our study: To what extent does the presence of the Robertson chains impact the generating set of the homology of graph configuration space?

To make sense of this question, we must first view our categories $\mathcal{TG}_k$ not as separate entities, but rather as members in a coherent family.

Definition 13. Let $\mathcal{TG}$ denote the category of all graphs with homeomorphic embeddings, and for each $k \geq 1$ let ι_k denote the embedding $\iota_k : \mathcal{TG}_k \to \mathcal{TG}$. The functor ι_k induces a pair of adjoint functors

$$(\iota_k)_! : \mathcal{TG}_k\text{-mod} \to \mathcal{TG}\text{-mod},$$
$$(\iota_k)^* : \mathcal{TG}\text{-mod} \to \mathcal{TG}_k\text{-mod}.$$

To be more specific, $(\iota_k)^*$ is the **restriction** functor, while $(\iota_k)_!$ is **extension**.

The kth *Robertson submodule* of $H_i(\mathcal{F}_n(\bullet))$, denoted $R_kH_i(\mathcal{F}_n(\bullet))$, is defined by

$$(\iota_k)_!\iota_k^*H_i(\mathcal{F}_n(\bullet)) \to H_i(\mathcal{F}_n(\bullet)).$$

In other words, $R_kH_i(\mathcal{F}_n(G))$ is the subgroup of $H_i(\mathcal{F}_n(G))$ obtained by pushing forward classes from the topological minors of G containing no embedded copy of R_k.

For instance, in the simplest case, the group $R_1H_i(\mathcal{F}_n(G))$ is that which is generated by pushforwards of homology classes coming from the sub-trees of G.

We refer to the resulting filtration of $H_i(\mathcal{F}_n(\bullet))$ as the *Robertson filtration*. For any fixed graph G, the associated filtration of $H_i(\mathcal{F}_n(G))$ is exhaustive as it will stabilize no later than when k matches the largest Robertson chain present in G. What is more interesting is that, for each fixed homological index i, all evidence suggests that the entire Robertson filtration stabilizes.

Conjecture 14. *For every $i \geq 0$, there exists an integer $g_i \geq 0$ such that for any $n \geq 1$ we have $R_jH_i(\mathcal{F}_n(\bullet)) = R_{j+1}H_i(\mathcal{F}_n(\bullet))$ whenever $j > g_i$.*

In the originating work [8], the authors provide a proof of this fact conditional on a categorical version of the graph minor theorem. There is also some hope of an unconditional argument that instead uses well known spectral sequences relating configuration spaces of a graph to that of the graphs obtained through vertex explosion. In either case, an unconditional proof of the above conjecture would be a critical step in understanding the structure of the homology groups of graph configuration spaces.

As one nice consequence of the boundedness of the Robertson filtration, we can see where the bounded torsion theorem from the introduction is coming from.

Theorem 15. *Fix $i, n \geq 1$. Assuming Conjectures 6 and 14, there exists an integer $d_{i,n}$, such that for any graph G, any torsion appearing in $H_i(\mathcal{F}_n(G))$ has order dividing $d_{i,n}$.*

Proof. To start, consider the $\mathcal{TG}_k$-module $H_i(\mathcal{F}_n(\bullet))$. One may take the submodule generated by all torsion classes appearing among all of the homology groups. This forms a $\mathcal{TG}_k$-submodule, as the torsion classes are sent to torsion classes by the induced maps. By the Categorical Robertson's Conjecture, this submodule must be finitely generated. By taking the least common multiple of the exponents of the generating homologies, we obtain an integer $d_{i,n,k}$ that all torsion appearing in the $\mathcal{TG}_k$-module must have order dividing. To remove the dependence on k, one can apply Conjecture 14. □

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