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Birational rowmotion over noncommutative rings

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Abstract. We study the dynamics of birational rowmotion over an arbitrary noncommutative ring $\mathbb{K}$. This generalizes the birational rowmotion map in the commutative setting, which itself lifts the well-studied combinatorial rowmotion map on a finite poset. When the underlying poset P is a rectangle (i.e., a product of two chains), this operation has "twisted periodicity" and "reciprocity" properties, surprisingly similar to the commutative setting. We briefly outline proofs of these results (details are on the arXiv) and discuss extensions and variants. In particular, we conjecture similar results for the case when P is a Δ- or ∇-shaped triangle or a trapezoid. We also conjecture that the results remain valid when $\mathbb{K}$ is a semiring. We further prove some elementary properties of birational rowmotion for general P, and (for the sake of exposition) discuss connections to the octahedron recurrence and Zamolodchikov periodicity (which are not new, but deserve better circulation).

Keywords: birational rowmotion, birational combinatorics, toggles, dynamical algebraic combinatorics, rowmotion, Zamolodchikov periodicity, posets

1 Introduction

Let P be a finite poset, and $\mathcal{J}(P)$ be the set of order ideals of P. *Combinatorial rowmotion* is an invertible map $\rho : \mathcal{J}(P) \to \mathcal{J}(P)$ which takes each ideal $I \in \mathcal{J}(P)$ to the order ideal generated by the minimal elements of the complement of I in P. It can also be viewed as a map on the antichains $\mathcal{A}(P)$ of a poset, via the well-known bijection $\mathcal{J}(P) \to \mathcal{A}(P)$ that sends an order ideal to its set of maximal elements. Rowmotion in this sense was studied in a sporadic sequence of papers by Brouwer and Schrijver [2], and Cameron and Fon-der-Flaass [3], who realized it as a product of "toggling" involutions, one for each element of the poset. Among the core results of these works were formulas for the *order* of ρ for special types of posets, such as products of chains.

In 2012, Striker and Williams [18] defined a related notion of *order-ideal promotion* π, which generalizes some cases of *Schützenberger promotion* on semistandard tableaux. Both ρ and π live in the *toggle group* of P (the group of permutations of $\mathcal{J}(P)$ generated by the "toggling" involutions), and are conjugate in it, which allows properties of π to

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be derived from properties of ρ and vice versa. Armstrong, Stump, and Thomas [1] identified rowmotion for *root posets* with the *Kreweras complementation map* studied by Lie theorists. This rekindled interest in ρ; a flurry of activity followed, opening up what is now known as *dynamical algebraic combinatorics*. We refer to [18, 17, 10] for some snapshots of progress in this field.

A major avenue of study have been "liftings" (i.e., generalizations) of ρ and π. The map ρ has been lifted three times successively:

- first, to a *piecewise-linear rowmotion* $\rho_{\mathcal{P}}$ on the *order polytope* $\mathcal{O}(P) := \{f : P \to [0,1] : f \text{ is order-preserving}\}$ (see [4]),
- then by "detropicalization" to a *birational rowmotion* map R (also known as $\rho_{\mathcal{B}}$) defined on vertex-labelings of P by rational functions (in commuting variables), as detailed by Einstein and Propp [4],
- and finally to *noncommutative birational rowmotion*, where the rational functions are replaced by elements of an arbitrary ring.

A priori, there was no reason to expect periodicity at all for these lifted maps; indeed, already $\rho_{\mathcal{P}}$ can have infinite order even if the poset P is rather "nice". However, in [7, 6, 16], it was shown that for a few classes of posets (e.g., root posets and minuscule posets associated with finite-dimensional Lie algebras and *skeletal posets*, built up inductively as a special kind of series-parallel poset), the order of birational rowmotion R is finite and surprisingly small — generally the **same** as for combinatorial rowmotion ρ. For the case where $P = [p] \times [q]$ is a product of two chains, proofs of periodicity (i.e., finite order) of R took significant effort [6, 15]. The result was extended to all minuscule posets by Okada [16] in a type-by-type way (with some computer algebra), and the conceptual reasons for the periodicity remain a mystery, although Garver, Patrias and Thomas [5] have found an explanation for the periodicity of the (less general) $\rho_{\mathcal{P}}$ using quiver representations. Periodicity of R for all root posets of coincidental types is still an open question [9, Conj. 4.40], as a proof for "trapezoid-shaped" posets [6, Conj. 75] has not been found.[1]

The noncommutative realm remained unexplored until recently. First results were obtained by Joseph and Roby in [13, 14], but (as with any of the three levels of generality) some of the good behavior of R is lost when allowing noncommutativity. Our initial conjecture from 2014 was that the periodicity property of R on $[p] \times [q]$ remains true, up to a (predictable and simple) "twist". This resisted a number of different proof attempts until we finally resolved it [8].

[1] To keep the story simple, we are limiting our discussion to ρ (as opposed to π) and to periodicity (as opposed to other properties, such as *homomesies*). The promotion map π has also been lifted (along with the entire toggle group) to the birational and even to the noncommutative realm [12, 13, 14]. Much of the work has been focusing on the antichain variant of rowmotion, which however is equivalent to the order-ideal variant ρ via the *transfer map*.

By well-understood reductions, our proof perforce establishes periodicity of $\rho_{\mathcal{P}}$ and ρ as well (for $P = [p] \times [q]$). We feel that this proof is more elementary and simpler than any that we have seen up until now at any level above the combinatorial. This abstract summarizes the results of [8] and briefly outlines the main steps of their proofs, sets them in context and adds some newer observations.

2 Definitions and examples

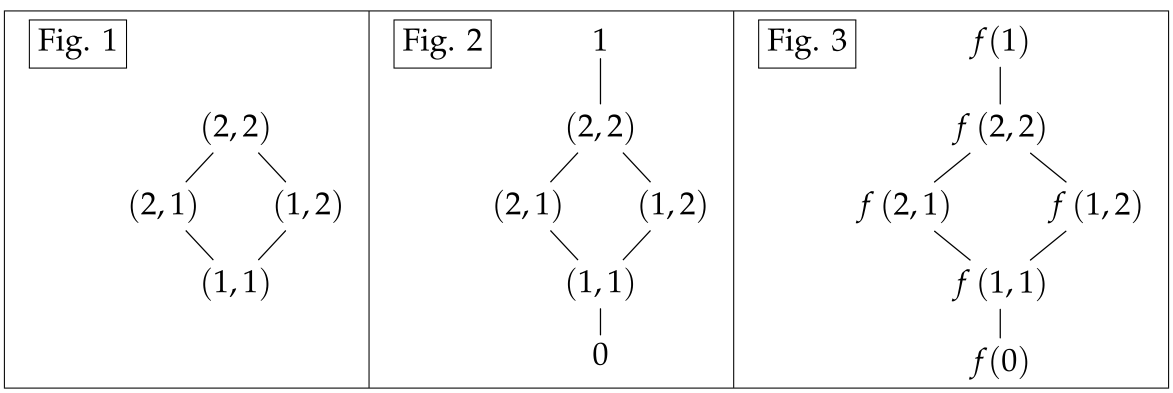
Throughout this work, P denotes a finite poset, and $\mathbb{K}$ denotes a ring (associative and with 1, but not necessarily commutative). We use the notation $\overline{x}$ for the (multiplicative) inverse x^{-1} of an element $x \in \mathbb{K}$.

The simplest type of poset is a *k-chain* (for $k \in \mathbb{N}$) – i.e., the set $\{1, 2, \ldots, k\}$ equipped with its usual total order. We denote it by $[k]$.

Our favorite poset will be the $p \times q$*-rectangle* (for $p, q \in \mathbb{N}$) – i.e., the Cartesian product $[p] \times [q]$ with entrywise partial order. Its Hasse diagram looks like a rectangular grid.

We let $\widehat{P}$ denote the poset P with two new elements 0 and 1 adjoined to it, arranged so that every $p \in \widehat{P}$ satisfies $0 \leq p \leq 1$ in $\widehat{P}$.

A $\mathbb{K}$-*labelling* (or, short, *labelling*) of P means a map $f : \widehat{P} \to \mathbb{K}$. Its values $f(p)$ are called its *labels* at the "points" $p \in \widehat{P}$, and we draw it by overlaying these labels at the respective points on the Hasse diagram of $\widehat{P}$. For example, if P is the 2×2-rectangle $[2] \times [2]$, then the following pictures show P itself (Fig. 1), the extended poset $\widehat{P}$ (Fig. 2) and a labelling of P (Fig. 3):



When u and v are two elements of $\widehat{P}$, we shall use the notation "$u \lessdot v$" (or, equivalently, "$v \gtrdot u$") for the statement "$u < v$, and there is no $w \in \widehat{P}$ such that $u < w < v$".

We are now ready to define the main actor of our play: the *birational rowmotion* operation. While it is traditionally defined as a composition of *toggles* (see, e.g., [8, Definition 3.16]), we instead here use a recursive equation (which is equivalent):

Definition 1. *Birational rowmotion* is the partial map $R : \mathbb{K}^{\widehat{P}} \dashrightarrow \mathbb{K}^{\widehat{P}}$ that transforms each labelling f of P into a new labelling Rf whose values are

$$(Rf)(0) = f(0), \qquad (Rf)(1) = f(1), \qquad \text{and}$$

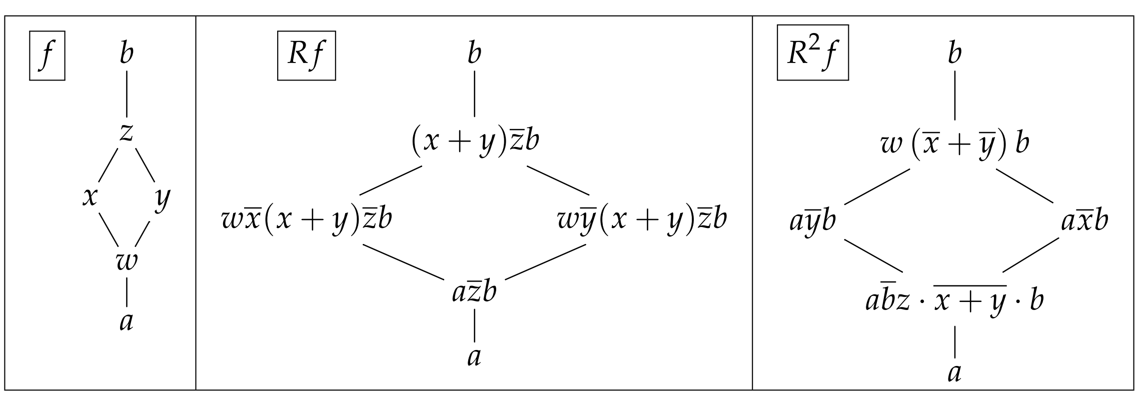
$$(Rf)(v) = \left(\sum_{\substack{u \in \widehat{P}; \\ u \lessdot v}} f(u) \right) \cdot \overline{f(v)} \cdot \overline{\sum_{\substack{u \in \widehat{P}; \\ u \gtrdot v}} \overline{(Rf)(u)}} \qquad \text{for all } v \in P.$$

Note that the formula for $(Rf)(v)$ involves several inverses, which may or may not exist. If they don't, Rf is undefined, which is why R is only a **partial** map. Also, this formula is **recursive**: The value it assigns to $(Rf)(v)$ depends on the values $(Rf)(u)$ for elements $u \in \widehat{P}$ satisfying $u \gtrdot v$. Since the poset $\widehat{P}$ is finite, we can solve this recursion "starting from the top of P" (using $(Rf)(1) = f(1)$ to bootstrap the process). Making this procedure explicit gives precisely the description of R as a composition of toggles that is used as a definition of R in [8]. (See [8, Proposition 3.18] for details.)

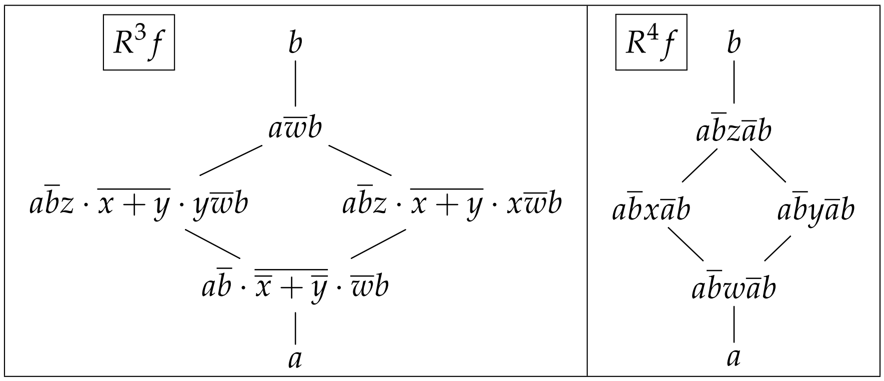
The map R is what we called "noncommutative birational rowmotion" in Section 1 – that is, the third (and ultimate) generalization of combinatorial rowmotion ρ. The second generalization (commutative birational rowmotion) is obtained when $\mathbb{K}$ is a field. The first generalization (piecewise-linear rowmotion $\rho_{\mathcal{P}}$) is recovered from the second by *tropicalization* (i.e., by setting $\mathbb{K}$ to be a tropical semifield[2]). The original rowmotion ρ on $\mathcal{J}(P)$ is the restriction of $\rho_{\mathcal{P}}$ to the vertices of the order polytope $\mathcal{O}(P)$ (which are the indicator functions of order ideals of P). Therefore, any (positive) result about noncommutative birational rowmotion automatically yields results about all lower levels (commutative, piecewise-linear and combinatorial).

Partial maps can be composed; in particular, R can be composed with itself. As usual, R^k shall mean the composition $R \circ R \circ \cdots \circ R$ with k many R's.

Example 1. Here are the first four iterations of R for $P = [2] \times [2]$, acting on an arbitrary labelling f:



[2]Of course, a tropical semifield is not a field. However, properties of R that hold for all fields will automatically hold for all semifields (since they reduce to subtraction-free polynomial identities), and thus we can allow $\mathbb{K}$ to be a semifield.



Here, some nontrivial algebra has been used to simplify some of the labels; e.g., the $a\overline{b}y\overline{a}b$ in $R^4 f$ was originally an $a\overline{b} \cdot \overline{\overline{x}+\overline{y}} \cdot \overline{x+y} \cdot y\left(\overline{x}+\overline{y}\right)\left(x+y\right)\overline{a}b$.

3 The main theorems for rectangles

Example 1 illustrates two surprising phenomena that apply to all rectangular posets. One is a "twisted periodicity", saying that the initial labels of f reappear in $R^4 f$ framed by $a\overline{b}$ on the left and $\overline{a}b$ on the right. (Note that this is not a conjugation, since $(a\overline{b})^{-1} = b\overline{a} \neq \overline{a}b$ unless a and b commute.) For a generic poset P, nothing like this holds (instead, the labels of $R^k f$ get increasingly complicated as k grows, even if $\mathbb{K}$ is commutative), but it turns out to be true whenever P is a rectangle:

Theorem 1 (Periodicity for the $p \times q$-rectangle)**.** *Let p and q be two positive integers. Let $P = [p] \times [q]$, and let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling such that $R^{p+q} f$ is well-defined. Set $a = f(0)$ and $b = f(1)$. Then, a and b are invertible, and for any $x \in \widehat{P}$ we have*

$$\left(R^{p+q} f\right)(x) = a\overline{b} \cdot f(x) \cdot \overline{a}b. \tag{3.1}$$

This periodicity actually follows from the following "antipodal reciprocity" property, which is the second phenomenon seen in Example 1. This property generalizes the $a\overline{z}b$ label in Rf, the $a\overline{y}b$ and $a\overline{x}b$ labels in $R^2 f$, and the $a\overline{w}b$ label in $R^3 f$. In its general form, it says that the inverse of any label in $R^k f$ appears as a label in $R^\ell f$ (for some $\ell \geq k$), framed by a on the left and by b on the right. The ℓ depends on the position of the label; the precise statement is as follows:

Theorem 2 (Reciprocity for the $p \times q$-rectangle)**.** *Let p and q be two positive integers. Let $P = [p] \times [q]$. Fix $\ell \in \mathbb{N}$, and let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling such that $R^\ell f$ is well-defined. Set $a = f(0)$ and $b = f(1)$. Let $(i,j) \in P$ satisfy $\ell - i - j + 1 \geq 0$. Then,*

$$\left(R^\ell f\right)(i,j) = a \cdot \overline{\left(R^{\ell-i-j+1} f\right)(p+1-i,\ q+1-j)} \cdot b. \tag{3.2}$$

We conjectured these two theorems in 2014, after proving them in the particular case when $\mathbb{K}$ is commutative ([6, §4–§8]).

4 Connections which do not lift

Strikingly, this noncommutative generalization has resisted all approaches that have previously succeeded in the commutative case. The determinantal computations involved in the proof in [6] can be extended to the noncommutative setting using the *quasideterminants* of Gelfand and Retakh, but it seems impossible to make a rigorous proof out of them (lacking, e.g., any useful notation of Zariski topology in this setting, it is not clear what it means for a $\mathbb{K}$-labelling to be "generic"). The Musiker–Roby proof of commutative periodicity [15] (via a lattice-path formula for $\left(R^i f\right)(v)$) could not be generalized either. Thus the noncommutative case remained an open problem.

At some point, Glick and Grinberg noticed that the Y-variables in the type-AA Zamolodchikov periodicity theorem of Volkov [19] could be written as ratios of labels under iterated birational rowmotion [17, § 4.4]; this allows the periodicity in one setting to be derived from that in the other (with some work). However, for noncommutative $\mathbb{K}$, Zamolodchikov periodicity fails even in small examples such as $r = r' = 2$ (no matter in which of the 120 possible orders we multiply the factors), while noncommutative birational rowmotion continues to exhibit periodicity. This approach is therefore unavailable in the noncommutative case as well.

Another recent proof of periodicity and reciprocity for commutative $\mathbb{K}$ (Johnson and Liu, [11]) uses a connection to the octahedron recurrence; this connection, too, does not seem to generalize readily to the noncommutative case (although a proper analysis of all options has not been made yet), and anyway the proof in [11] uses determinants, which are not available for general $\mathbb{K}$.

5 Proof outline

We shall now outline the main steps of our proofs of Theorems 1 and 2. Full details are available in [8], including justifications of all the inverses being taken (in the following, we just assume they all exist) and proofs of all lemmas.

Step 1: *Observe that Theorem 1 follows from Theorem 2.* Indeed, assume that Theorem 2 is proved. Now, to prove Theorem 1, assume WLOG that $x \in P$ (as opposed to $x = 0$ or $x = 1$, which cases are easy), and write $x = (u, v)$. Now, apply Theorem 2 first to $(\ell, i, j) = (p + q,\ u,\ v)$. Then, apply Theorem 2 again to $(\ell, i, j) = (p + q - u - v + 1,\ p + 1 - u,\ q + 1 - v)$ in order to rewrite the right hand side.

It thus remains to prove Theorem 2.

Step 2: *Introduce notations:* Fix a $\mathbb{K}$-labelling f of $P = [p] \times [q]$. For any element $x = (i, j) \in P$, we define its *antipode* $x^\sim \in P$ by $x^\sim := (p + 1 - i,\ q + 1 - j)$. For instance,

$(1,1)^{\sim} = (p,q)$. For any $x \in \widehat{P}$ and $\ell \in \mathbb{N}$, we write

$$x_\ell := \left(R^\ell f\right)(x). \tag{5.1}$$

Informally, we think of our subscripts as indicating the "time" of the label (i.e., number of times R was applied to f). For instance, in Example 1, we have $(1,2)_2 = \left(R^2 f\right)(1,2) = a\overline{x}b$.

In this notation, Theorem 2 claims that

$$x_\ell = a \cdot \overline{\widetilde{x}_{\ell-i-j+1}} \cdot b \tag{5.2}$$

for all $x = (i,j) \in P$ and all $\ell \geq i+j-1$. Clearly, it suffices to prove this for $\ell = i+j-1$.

For each $v \in P$ and $\ell \in \mathbb{N}$, we define the two *slack values*[3]

$$A^v_\ell := v_\ell \cdot \overline{\sum_{u \lessdot v} u_\ell} \qquad \text{and} \qquad V^v_\ell := \overline{\sum_{u \gtrdot v} \overline{u_\ell}} \cdot \overline{v_\ell}.$$

Furthermore, when $v \in \{0,1\}$, we set $A^v_\ell := 1$ and $V^v_\ell := 1$ for all $\ell \in \mathbb{N}$. For instance, in Example 1, we have $A^{(2,2)}_0 = z \cdot \overline{y+x}$ and $V^{(1,1)}_0 = \overline{\overline{y} + \overline{x}} \cdot \overline{w}$ and $V^{(1,1)}_1 = \overline{\overline{w\overline{y}\,(x+y)\,\overline{z}b} + \overline{w\overline{x}\,(x+y)\,\overline{z}b}} \cdot \overline{a\overline{z}b} = w \cdot \overline{a}$ (after simplifications).

A *path* will mean a sequence $(v_0, v_1, \ldots, v_k)$ of elements of $\widehat{P}$ satisfying $v_0 \gtrdot v_1 \gtrdot \cdots \gtrdot v_k$. We will call it a *path from v_0 to v_k*. For instance, there are two paths from $(2,2)$ to $(1,1)$, namely $((2,2),(1,2),(1,1))$ and $((2,2),(2,1),(1,1))$.

For any path $\mathbf{p} = (v_0, v_1, \ldots, v_k)$ and any $\ell \in \mathbb{N}$, we define the *slack products*

$$A^{\mathbf{p}}_\ell := A^{v_0}_\ell A^{v_1}_\ell \cdots A^{v_k}_\ell \qquad \text{and} \qquad V^{\mathbf{p}}_\ell := V^{v_0}_\ell V^{v_1}_\ell \cdots V^{v_k}_\ell$$

If u and v are elements of $\widehat{P}$, and if $\ell \in \mathbb{N}$, then we define the *slack sums*

$$A^{u \to v}_\ell := \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} A^{\mathbf{p}}_\ell \qquad \text{and} \qquad V^{u \to v}_\ell := \sum_{\mathbf{p} \text{ is a path from } u \text{ to } v} V^{\mathbf{p}}_\ell.$$

For instance,

$$A^{(2,2)\to(1,1)}_\ell = A^{((2,2),(1,2),(1,1))}_\ell + A^{((2,2),(2,1),(1,1))}_\ell = A^{(2,2)}_\ell A^{(1,2)}_\ell A^{(1,1)}_\ell + A^{(2,2)}_\ell A^{(2,1)}_\ell A^{(1,1)}_\ell.$$

The following formula connects the slack values at "time ℓ" with those at "time $\ell - 1$":

Proposition 1 (Transition equation in A-V-form)**.** *Let $v \in \widehat{P}$ and $\ell \geq 1$. Then, $V^v_\ell = A^v_{\ell-1}$.*

Proof. This is just the formula for $(Rf)(v)$ in Definition 1, rewritten. □

[3] In both sums, u ranges over elements of $\widehat{P}$.

Step 3: *The path formulas.* The following formulas allow us to recover the labels u_ℓ from the slack values A_ℓ^v or V_ℓ^v:

Theorem 3 (path formulas for rectangle)**.** *Let $\ell \in \mathbb{N}$. Then, each $u \in P$ satisfies*

$$u_\ell = \overline{V_\ell^{1 \to u}} \cdot b = \overline{V_\ell^{(p,q) \to u}} \cdot b \tag{5.3}$$

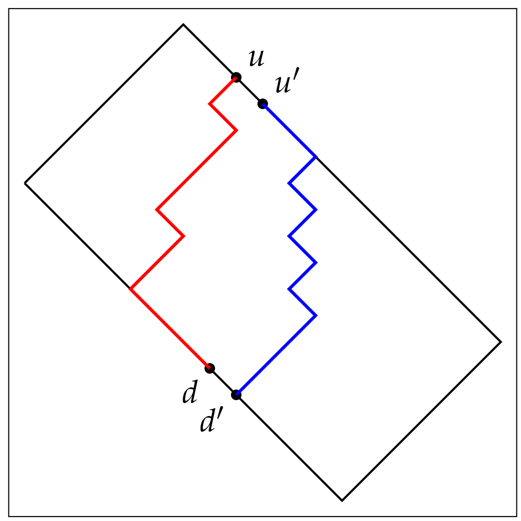
and

$$u_\ell = A_\ell^{u \to 0} \cdot a = A_\ell^{u \to (1,1)} \cdot a. \tag{5.4}$$

Proof. Induction on u (downwards for (5.3), upwards for (5.4)). □

Using (5.4) and (5.3) and Proposition 1, we can already prove that $(1,1)_1 = a\overline{(p,q)_0}b$, which confirms (5.2) for $x = (1,1)$. The general case, however, requires more work.

Step 4: *The conversion lemma.* The following lemma is at the core of our proof:[4]



Lemma 1 (Conversion lemma)**.** *Let u and u' be two elements of the northeastern edge of P satisfying $u \gtrdot u'$ (that is, $u = (k,q)$ and $u' = (k-1,q)$ for some $k \in \{2,3,\ldots,p\}$). Let d and d' be two elements of the southwestern edge of P satisfying $d \gtrdot d'$ (that is, $d = (i,1)$ and $d' = (i-1,1)$ for some $i \in \{2,3,\ldots,p\}$). Then, for each $\ell \in \mathbb{N}$, we have*

$$A_\ell^{u \to d} = V_\ell^{u' \to d'}. \tag{5.5}$$

Note that both sides of this equality are slack sums at "time ℓ" (i.e., defined purely in terms of the labeling $R^\ell f$). Thus, this lemma does not involve birational rowmotion. Nevertheless, its proof is far from trivial. It involves "interpolating" between $A_\ell^{u \to d}$ and $V_\ell^{u' \to d'}$ through a "mixed slack sum". The latter is a sum over "*path-jump-paths*", which are like paths but allow a "jump" in the middle. We refer to [8, §9] for the details.

Step 5: *The southwestern edge.* If $\ell \geq 1$, then (5.5) can be rewritten as $A_\ell^{u \to d} = A_{\ell-1}^{u' \to d'}$ (by Proposition 1). Applying this equality many times ("gliding down" the rectangle), we see that each $i \in [p]$ satisfies $A_{i-1}^{(p,\, q) \to (i,\, 1)} = A_0^{(p-i+1,\, q) \to (1,\, 1)}$. However, Proposition 1 yields $V_i^{(p,\, q) \to (i,\, 1)} = A_{i-1}^{(p,\, q) \to (i,\, 1)} = A_0^{(p-i+1,\, q) \to (1,\, 1)}$. In view of (5.4) and (5.3), we can readily use this to show that $(i,1)_i = a \cdot \overline{(p+1-i,\ q)_0} \cdot b$. This proves (5.2) for $x = (i,1)$.

[4]The figure on the right illustrates this lemma (where the red path is one of the paths contributing to the sum $A_\ell^{u \to d}$, while the blue path contributes to $V_\ell^{u' \to d'}$).

In other words, (5.2) is proved for all x lying on the southwestern edge of P (speaking in terms of the Hasse diagram). A symmetric argument applies when x lies on the southeastern edge.

Step 6: *Inducting up the poset.* It remains to prove (5.2) for all $x = (i, j)$ with $i, j \geq 2$. This is done by strong induction on $i + 2j$, which allows us to apply the induction hypothesis to the four pairs

$$m := (i, j-1), \qquad u := (i+1, j-1), \qquad s := (i, j-2), \qquad t := (i-1, j-1)$$

in lieu of $x = (i, j)$ (to the extent that these pairs do belong to P). As a result, we obtain

$$m_\ell = a \cdot \overline{m^{\sim}_{\ell-k}} \cdot b, \qquad s_{\ell-1} = a \cdot \overline{s^{\sim}_{\ell-k}} \cdot b, \tag{5.6}$$

$$t_{\ell-1} = a \cdot \overline{t^{\sim}_{\ell-k}} \cdot b, \qquad m_{\ell-1} = a \cdot \overline{m^{\sim}_{\ell-k-1}} \cdot b, \tag{5.7}$$

$$u_\ell = a \cdot \overline{u^{\sim}_{\ell-k-1}} \cdot b \qquad (\text{where } k = i + j - 2). \tag{5.8}$$

On the other hand, the formula for $(Rf)(v)$ in Definition 1 (applied once to $v = m$ and once to $v = m^{\sim}$) yields

$$m_\ell = (s_{\ell-1} + t_{\ell-1}) \cdot \overline{m_{\ell-1}} \cdot \overline{\overline{u_\ell} + \overline{x_\ell}} \qquad \text{and} \tag{5.9}$$

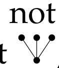
$$m^{\sim}_{\ell-k} = \left(u^{\sim}_{\ell-k-1} + x^{\sim}_{\ell-k-1}\right) \cdot \overline{m^{\sim}_{\ell-k-1}} \cdot \overline{\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}}. \tag{5.10}$$

Taking reciprocals on both sides of (5.10), we obtain

$$\overline{m^{\sim}_{\ell-k}} = \left(\overline{s^{\sim}_{\ell-k}} + \overline{t^{\sim}_{\ell-k}}\right) \cdot m^{\sim}_{\ell-k-1} \cdot \overline{u^{\sim}_{\ell-k-1} + x^{\sim}_{\ell-k-1}}. \tag{5.11}$$

In view of the five equations (5.6)–(5.8), we see that (up to the a and b factors, which cancel out) each label appearing in (5.9) is the inverse of the corresponding label in (5.11), except for x_ℓ and $x^{\sim}_{\ell-k-1}$. Since the equations (5.9) and (5.11) can be uniquely solved for x_ℓ and $x^{\sim}_{\ell-k-1}$, we conclude that x_ℓ must be $a \cdot \overline{x^{\sim}_{\ell-k-1}} \cdot b$. This completes the induction step, proving (5.2) and with it Theorem 2. Recalling Step 1, we have thus proved Theorem 1 too.

6 Other posets

The rectangles $P = [p] \times [q]$ may be the simplest family of posets for which noncommutative birational rowmotion exhibits a (twisted) periodicity.[5] In the commutative case, the "*skeletal posets*" of [7] are even simpler, but their periodicity does not extend to the noncommutative case. For example, if P is the 4-element "claw" poset , then $R^6 = \mathrm{id}$

[5]Apart from the chains $[m]$, which are just $m \times 1$-rectangles, and from the antichains, which are trivial to analyze.

for commutative $\mathbb{K}$, but no periodicity holds for noncommutative $\mathbb{K}$ (not even "twisted"; see [8, §13.4] for details). Experimentally, things look better for "Lie-theoretical" families of posets, although nothing has been proven so far. The family of *type-A positive root posets*, also known as *triangles* $\Delta(p)$, may be within reach.

We can easily define these triangles as pieces of a $p \times p$-rectangle. Indeed, let p be a positive integer. Define two subsets $\Delta(p)$ and $\nabla(p)$ of the $p \times p$-rectangle $[p] \times [p]$ by

$$\Delta(p) = \{(i,j) \in [p] \times [p] \mid i+j > p+1\};$$
$$\nabla(p) = \{(i,j) \in [p] \times [p] \mid i+j < p+1\}.$$

These two subsets $\Delta(p)$ and $\nabla(p)$ inherit partial orders from $[p] \times [p]$. Now, we claim:

Conjecture 1 (Periodicity for $\Delta(p)$ and $\nabla(p)$)**.** *Let $p \geq 2$ be an integer. Assume that P is the poset $\Delta(p)$ or $\nabla(p)$. Let $f \in \mathbb{K}^{\widehat{P}}$ be a $\mathbb{K}$-labelling such that $R^p f$ is well-defined. Let $a = f(0)$ and $b = f(1)$. Let $x \in \widehat{P}$. Let $x' \in \widehat{P}$ be defined as follows: If $x = (i,j)$, then we set $x' := (j,i)$; otherwise (i.e., if $x = 0$ or $x = 1$), we set $x' := x$. Then,*

$$(R^p f)(x) = a\overline{b} \cdot f\left(x'\right) \cdot \overline{a}b.$$

Further posets P for which R is conjecturally periodic (with a twist) are "right-half triangles" [8, §13.2] and trapezoids [8, §13.3].

7 General posets

As already mentioned, periodicity and reciprocity are features of certain classes of posets, even though the complete identification of these classes is still unsolved even at the piecewise-linear level[6]. Nevertheless, the $(i,j) = (1,1)$ case of Theorem 2 can be generalized to arbitrary posets in the following form ([8, §14]):

Proposition 2. *Let P be any finite poset. Let $f \in \mathbb{K}^{\widehat{P}}$ be a labeling of P such that Rf is well-defined and all labels $(Rf)(u)$ are invertible. Let $a = f(0)$ and $b = f(1)$. Then,*

$$b \cdot \sum_{\substack{u \in \widehat{P};\\ u \gtrdot 0}} \overline{(Rf)(u)} \cdot a = \sum_{\substack{u \in \widehat{P};\\ u \lessdot 1}} f(u).$$

This adds to the (so far rather meager) list of properties of R that have been observed for general posets P. (The rest of the list are results by Joseph and Roby [13, §5] that connect R with *birational antichain rowmotion* and the *birational toggle group*.)

Note that Proposition 2 can be used to generalize the w-tuple periodicity [7, Proposition 37] to the case of noncommutative $\mathbb{K}$ (again with a twist).

[6]See [10, Conjecture 5.7] for a sequence of posets $V(n)$ for which $\rho_{\mathcal{P}}$ is conjectured to have finite order, but R seems to have infinite order even when $\mathbb{K}$ is commutative.

8 The semiring case

The definition of R is *subtraction-free*: It involves only addition, multiplication and division in $\mathbb{K}$. Thus, we can replace $\mathbb{K}$ by an arbitrary *semiring* (i.e., a "ring without additive inverses"). It is thus natural to wonder:

Question 1. *Do Theorems 1 and 2 still hold if* $\mathbb{K}$ *is only a semiring?*

This question is subtler than it may seem. When $\mathbb{K}$ is commutative, the answer is positive for general reasons: A polynomial identity true over all commutative rings is automatically true over all commutative semirings. However, such automatic transfer principles do not exist in the noncommutative realm. Case in point: If $\mathbb{K}$ is a ring, then the identity $a \cdot \overline{a+b} \cdot b = b \cdot \overline{a+b} \cdot a$ holds for all $a, b \in \mathbb{K}$ for which $a+b$ is invertible; but this is no longer true if $\mathbb{K}$ is merely a semiring (David Speyer, MathOverflow #401273). A negative answer to Question 1 would thus not be entirely unexpected.

The proofs of Theorems 1 and 2 sketched above rely on subtraction in Step 6, so they cannot be applied to semirings unless $\min\{p, q\} \leq 2$ (in which case there is nothing to subtract). We suspect that the smallest "interesting" case is $[3] \times [3]$ for Theorem 2 and $[3] \times [4]$ for Theorem 1. Noncommutative semirings that cannot be embedded in rings yet have many multiplicative inverses do exist[7] , but automated computation inside them is not straightforward, which complicates experimentation. We nevertheless believe that progress can be made in this direction.

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