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The isomorphism problem for cominuscule Schubert varieties

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Abstract. Cominuscule flag varieties generalize Grassmannians to other Lie types. Schubert varieties in cominuscule flag varieties are indexed by posets of roots labeled long/short. These labeled posets generalize Young diagrams. We prove that Schubert varieties in potentially different cominuscule flag varieties are isomorphic as varieties if and only if their corresponding labeled posets are isomorphic, generalizing the classification of Grassmannian Schubert varieties using Young diagrams by the last two authors. Our proof is type-independent.

Keywords: Schubert variety, flag variety, cominuscule

1 Introduction

Cominuscule flag varieties correspond to algebraic varieties that admit the structure of a compact Hermitian symmetric space and have been studied extensively due their shared properties with Grassmannians [1, 3, 14, 22, 8, 19, 4, 20, 5, 18, 6]. These varieties come in four infinite families and two exceptional types and are determined by a pair $(\mathcal{D}, \gamma)$ of a Dynkin diagram $\mathcal{D}$ of a reductive Lie group and a cominuscule simple root γ. See Table 1 for a classification of cominuscule flag varieties. Let X denote the cominuscule flag variety corresponding to $(\mathcal{D}, \gamma)$ and R denote the root system of the Dynkin diagram $\mathcal{D}$. Set $\mathcal{P}_X := \{\alpha \in R : \alpha \geq \gamma\}$ with the partial order $\alpha \leq \beta$ if $\beta - \alpha$ is a non-negative sum of simple roots, and give $\mathcal{P}_X$ a labeling of long/short roots. By [7, Theorem 2.4], Schubert varieties in X are indexed by lower order ideals in $\mathcal{P}_X$, generalizing the fact that Schubert varieties in a Grassmannian are indexed by Young diagrams.

Our main result Theorem 1 is a combinatorial criterion for distinguishing isomorphism classes of Schubert varieties coming from cominuscule flag varieties.

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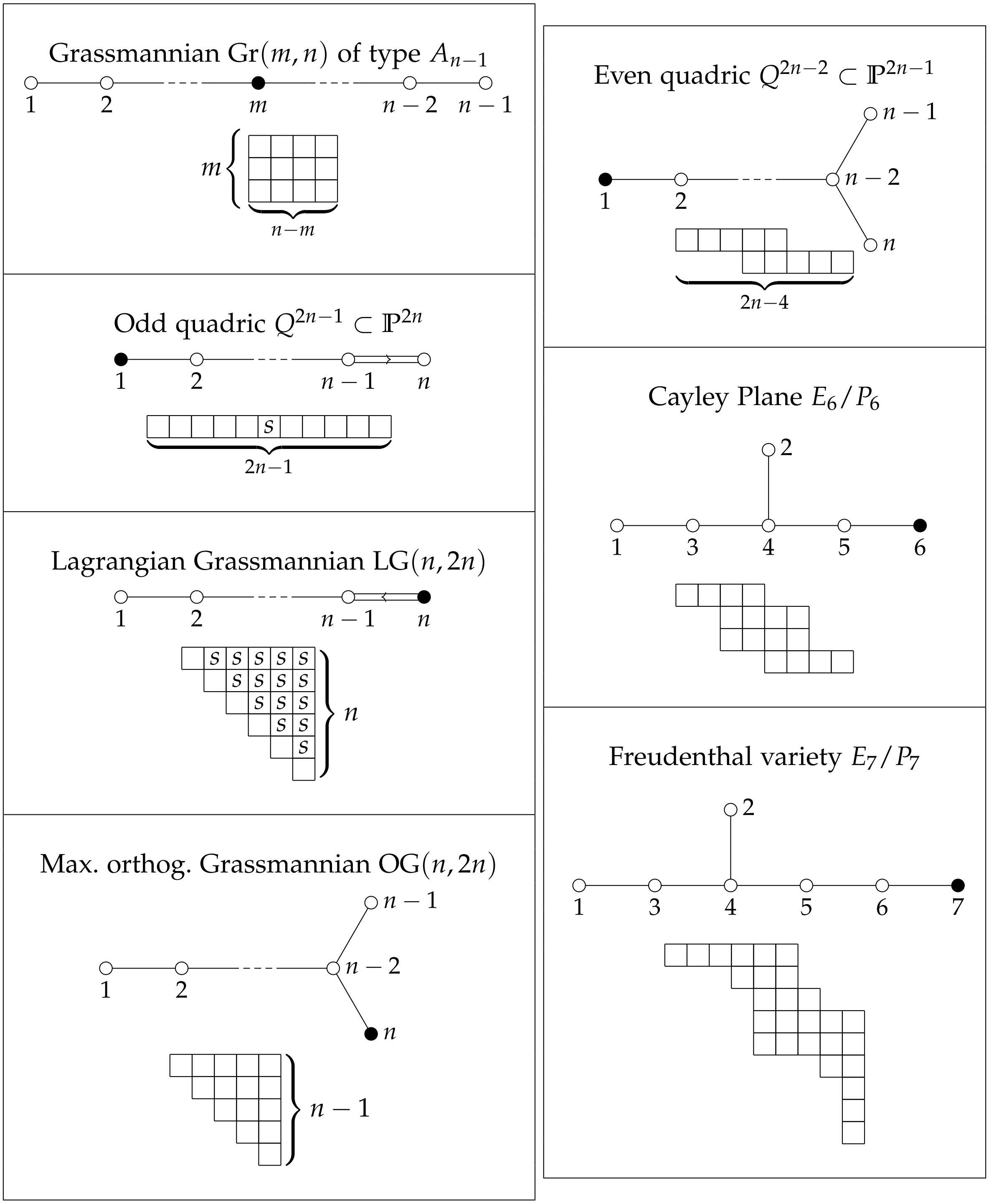
Table 1: The labeled poset $\mathcal{P}_X$ for a cominuscule flag variety X. Each element in $\mathcal{P}_X$ is drawn as a box, and boxes decorated with an "s" correspond to short roots. The partial order on boxes is given by $\alpha \leq \beta$ if and only if α is weakly north-west of β. This table is a modification of [6, Table 1].



Theorem 1. *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. Then X_λ and Y_μ are algebraically isomorphic if and only if λ and μ are isomorphic as labeled posets.*

For illustrative examples of Theorem 1, see Section 2.

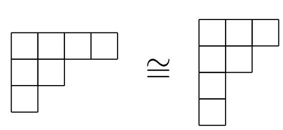
Since Grassmannians are cominuscule flag varieties, Theorem 1 extends the work of Ţarigradschi and Xu in [21], where they prove two Grassmannian Schubert varieties are isomorphic if and only if their Young diagrams are the same or the transpose of each other. Other related works include Richmond and Slofstra's characterization of the isomorphism classes of Schubert varieties coming from complete flag varieties in [17] using Cartan equivalence. However, they also note that Cartan equivalence is neither necessary nor sufficient to distinguish Schubert varieties in partial flag varieties. A class of smooth Schubert varieties in type A partial flag varieties are classified by Develin, Martin, and Reiner in [9]. Yet many Schubert varieties are singular, with the first example being the Schubert divisor in the Grassmannian $\mathrm{Gr}(2, 4)$.

We discuss preliminaries in Section 3, and then in Section 4, we prove Theorem 1 and illustrate it with examples. Our proof is type-independent and employs several new techniques.

2 Examples of Theorem 1

For the following examples, recall that cominuscule Schubert varieties are indexed by lower order ideals in $\mathcal{P}_X$. Examples of $\mathcal{P}_X$ are illustrated in Table 1, where each element in $\mathcal{P}_X$ is drawn as a box, and boxes decorated with an "s" correspond to short roots. The partial order on boxes is given by $\alpha \leq \beta$ if and only if α is weakly north-west of β. In particular, lower order ideals are given by subsets of boxes that are closed under moving to the north and west.

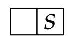
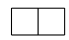
Example 2. As illustrated below, transposing a Young diagram does not change the poset structure. Therefore, two Grassmannian Schubert varieties are isomorphic if their indexing Young diagrams are the transpose of each other. Geometrically, this is related to the isomorphism $\mathrm{Gr}(m, m+k) \cong \mathrm{Gr}(k, m+k)$.



Example 3. Using Table 1, it is not hard to see that if a Grassmannian Schubert variety is isomorphic to a non-type A cominuscule Schubert variety, then they are both isomorphic to a projective space. Indeed, in order to fit inside a $\mathcal{P}_X$ of another type, the lower order ideal is forced to be a chain.

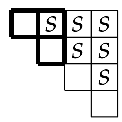
As a special case, we also see that any Schubert curve in any cominuscule flag variety is isomorphic to $\mathbb{P}^1$. In fact, any Schubert curve in any flag variety is isomorphic to $\mathbb{P}^1$, which follows from the more general statements that Schubert varieties are rational normal projective varieties and that $\mathbb{P}^1$ is the only rational normal projective curve.

Example 4. The Schubert divisor in Q^3 is not isomorphic to $\mathbb{P}^2$, because the labeling of their posets does not match:

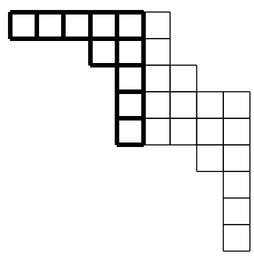
 $\not\cong$  .

We can also see it geometrically, as the Schubert divisor in Q^3 is singular.

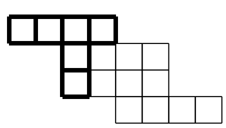
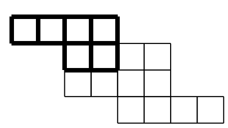
Example 5. The quadric Q^3 embeds in $\mathrm{LG}(n, 2n)$ ($n \geq 3$) as a Schubert variety, as illustrated by

 .

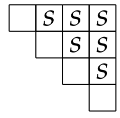
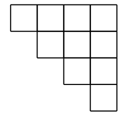
Example 6. The quadric Q^{10} embeds in E_7/P_7 as a Schubert variety, as illustrated by

 .

Example 7. There are two non-isomorphic 6-dimensional Schubert varieties in E_6/P_6, given by the two order ideals illustrated below.

 $\not\cong$ 

Example 8. While $\mathcal{P}_{\mathrm{LG}(n,2n)}$ and $\mathcal{P}_{\mathrm{OG}(n+1,2n+2)}$ are isomorphic as posets, this isomorphism does not preserve the labeling of long/short roots (see illustration below). As a result, $\mathrm{LG}(n, 2n)$ and $\mathrm{OG}(n + 1, 2n + 2)$ do not contain isomorphic Schubert varieties of dimension greater than one.

 $\not\cong$ 

3 Preliminaries

Let G be a complex reductive linear algebraic group. We fix subgroups $T \subset B \subset G$, where T is a maximal torus and B is a Borel subgroup. With this setup, $T \subset G$ determines a root system R of G, with corresponding Weyl group $W := N(T)/T$, and B determines a set of simple roots $\Delta \subseteq R$. The set of roots decomposes into positive and negative roots $R = R^+ \sqcup R^-$, with R^+ being non-negative sums of simple roots. The Weyl group W is generated by the set of simple reflections

$$S := \{s_\alpha : \alpha \in \Delta\}.$$

To each subset $I \subseteq S$ one can associate a Weyl subgroup $W_I := \langle s : s \in I \rangle \subseteq W$, a parabolic subgroup $P_I = BW_IB \subseteq G$ and the corresponding (partial) flag variety $X = G/P_I$. Schubert varieties in X are indexed by W^I, the set of minimal length coset representatives of W/W_I. Explicitly, for $w \in W^I$, the Schubert variety

$$X_w := \overline{BwP_I/P_I}$$

has dimension the Coxeter length of w, denoted $\ell(w)$. Moreover, for any $u \in W^I$, we have $X_u \subseteq X_w$ if and only if $u \leq w$ in Bruhat order.

From now on, X is a cominuscule flag variety. In other words, $I = S \setminus \{s_\gamma\}$, where γ is a cominuscule simple root, i.e., γ appears with coefficient 1 in the highest root of R. Cominuscule roots are illustrated by filled-in circles in Table 1.

Recall that

$$\mathcal{P}_X := \{\alpha \in R : \alpha \geq \gamma\}$$

inherits the usual partial order on roots, i.e., $\alpha \leq \beta$ if $\beta - \alpha$ is a non-negative sum of simple roots, and in addition, we give $\mathcal{P}_X$ a labeling of long/short roots.

In [15], Proctor proves that W^I is a distributive lattice under the induced Bruhat partial order from W. Birkhoff's representation theorem implies there is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$. In particular, the join-irreducible elements of W^I are identified with principal lower order ideals of $\mathcal{P}_X$ and hence with $\mathcal{P}_X$ itself. Explicitly, to each $w \in W^I$ we associate its inversion set

$$\lambda(w) := \{\alpha \in R^+ : w.\alpha < 0\}, \tag{3.1}$$

viewed as a sub-poset of $\mathcal{P}_X$. It is well known that $\ell(w) = |\lambda(w)|$. Moreover, the following proposition was proved in [7, Theorem 2.4 and Corollary 2.6]:

Proposition 9 (Buch–Samuel)**.** *For any $w \in W^I$, the inversion set $\lambda(w)$ is a lower order ideal in $\mathcal{P}_X$. Moreover:*

1. *The map $w \mapsto \lambda(w)$ is a bijection between W^I and the set of lower order ideals in $\mathcal{P}_X$.*
2. *For any $u \in W^I$, we have $u \leq w$ in Bruhat order if and only if $\lambda(u) \subseteq \lambda(w)$.*
3. *If $\alpha \in \lambda(w)$ and $\lambda(w) \setminus \{\alpha\}$ is a lower order ideal, then $ws_\alpha \in W^I$ and $\lambda(ws_\alpha) = \lambda(w) \setminus \{\alpha\}$, where $s_\alpha \in W$ is the reflection corresponding to α.*

Notation 10. Given a lower order ideal $\lambda \subseteq \mathcal{P}_X$, we will write w_λ for the element of W^I corresponding to λ in Proposition 9. We also use $X_\lambda := X_{w_\lambda}$ to denote the corresponding Schubert variety.

In Section 4.2, we will use a map $\delta : \mathcal{P}_X \to \Delta$ defined in [6] as follows.

Definition 11. For $\alpha \in \mathcal{P}_X$, let λ_α be the principal lower order ideal generated by α. Let $\delta(\alpha) = -w_{\lambda_\alpha}.\alpha \in R^+$. Then $s_{\delta(\alpha)} = w_{\lambda_\alpha} s_\alpha w_{\lambda_\alpha}^{-1}$ has length 1 [6, Section 4.1]. Therefore, $\delta(\alpha) \in \Delta$.

The following lemma is a restatement of [7, Corollary 2.10]. See also [6, Section 4.1].

Lemma 12 (Buch–Samuel). *Let $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal and $\beta_1, \beta_2, \ldots, \beta_\ell$ be the boxes it contains written in an increasing order. Then $s_{\delta(\beta_\ell)} \cdots s_{\delta(\beta_2)} s_{\delta(\beta_1)}$ is a reduced decomposition of w_λ. Moreover, every reduced decomposition of w_λ can be obtained in this way.*

4 Proof of Theorem 1

In this section, we prove each direction of Theorem 1 separately.

4.1 Forward direction: the isomorphism class of X_λ determines the labeled poset λ

Let $i_\lambda : X_\lambda \hookrightarrow X$ denote the embedding of a Schubert subvariety into a cominuscule flag variety $X = G/P_I$. The definition of the labeled poset λ depends on the root system of the reductive group G and hence on the embedding $i_\lambda : X_\lambda \hookrightarrow X$. The goal of this section is to show that λ (as a labeled poset) can be constructed using only the variety structure of X_λ and is therefore intrinsic to the isomorphism class of X_λ. We prove the following proposition which states the "forward" direction of Theorem 1.

Proposition 13. *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. If X_λ and Y_μ are algebraically isomorphic, then λ and μ are isomorphic as labeled posets.*

Our primary tools come from the intersection theory of algebraic varieties (see [10] for more details). Let $\mathrm{Pic}(X_\lambda)$ and $A_*(X_\lambda)$ denote the Picard and Chow groups of X_λ. It is well known that these groups are algebraic invariants of X_λ. Recall that the k-th Chow group $A_k(Z)$ of a scheme Z is the free abelian group on the k-dimensional subvarieties of Z modulo rational equivalence. When Z is a normal variety, the Picard group $\mathrm{Pic}(Z)$ can be identified with the subgroup of $A_{\dim(Z)-1}(Z)$ generated by classes of locally principal divisors (note that all Schubert varieties are normal). Our aim is to construct the labeled poset λ from the intersection class map or intersection product [10, Definition 2.3]:

$$\mathrm{Pic}(X_\lambda) \times A_*(X_\lambda) \to A_*(X_\lambda).$$

If $(\sigma, \tau) \in \mathrm{Pic}(X_\lambda) \times A_*(X_\lambda)$, we denote the image of the intersection product by $\sigma \cdot \tau$. Next, we consider the effective cone of a scheme:

Definition 14. Let Z be a scheme. The *effective cone* in the Chow group $A_*(Z)$ is the semigroup in $A_*(Z)$ generated by the classes of closed subvarieties of Z.

Since the flag variety X is cominuscule, there is a unique Schubert variety of codimension 1 in X, called the Schubert divisor. Its class, denoted D, is the unique effective generator of the Picard group $\mathrm{Pic}(X) \subseteq A_*(X)$. Recall that $i_\lambda : X_\lambda \hookrightarrow X$ is a closed embedding of varieties. Lemma 15 below follows from [13, Proposition 6] (see also [2, Proposition 2.2.8 part (ii)]).

Lemma 15. *For any non-empty lower order ideal $\lambda \subseteq \mathcal{P}_X$, the map $i_\lambda^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X_\lambda)$ is an isomorphism.*

Since D is effective and generates $\mathrm{Pic}(X)$, Lemma 15 implies that $i_\lambda^* D$ the unique effective generator of $\mathrm{Pic}(X_\lambda)$. Recall from Proposition 9 that we have lower order ideals $\mu \subseteq \lambda$ if and only if $w_\mu \leq w_\lambda$ in Bruhat order. Hence, we have $\mu \subseteq \lambda$ if and only if $X_\mu \subseteq X_\lambda$. We write $[X_\mu]$ for the class of X_μ in $A_*(X_\lambda)$ and in $A_*(X)$, in an abuse of notation. It is well known that the classes $\{[X_\mu]\}_{\mu \subseteq \lambda}$ form an integral basis of $A_*(X_\lambda)$. Lemma 16 below is a special case of [11, Corollary of Thereom 1] and allows us to identify Schubert classes (the effective cone) in $A_*(X_\lambda)$ without referring to the embedding $i_\lambda : X_\lambda \hookrightarrow X$ or using the B-variety structure of X_λ.

Lemma 16 (Fulton–MacPherson–Sottile–Sturmfels)**.** *The Schubert classes $[X_\mu]$ such that $\mu \subseteq \lambda$ are exactly the minimal elements in the extremal rays of the effective cone in $A_*(X_\lambda)$.*

We shall see later that the poset structure of λ can be recovered from the intersection products $i_\lambda^* D \cdot [X_\mu]$. Let $j : X_\mu \hookrightarrow X_\lambda$ be an inclusion of Schubert varieties, where $\mu \subseteq \lambda$. By the definition of intersection product,

$$i_\lambda^* D \cdot [X_\mu] = [j^* i_\lambda^* D] = [(i_\lambda \circ j)^* D] = D \cdot [X_\mu].$$

This implies that the product $i_\lambda^* D \cdot [X_\mu]$ on X_λ can be computed on X. By [10, Example 19.1.11], $A_*(X)$ can be identified with the homology group $H_*(X)$. By [10, Proposition 19.1.2] we have that the intersection product $D \cdot [X_\mu]$ can be identified with a cap product. Since X is a smooth complex variety, the Poincaré duality further identifies the intersection product with the cup product of cohomology classes corresponding to D and $[X_\mu]$. This cup product is given by the Chevalley formula [12, Lemma 8.1]. Using these identifications, we restate the Chevalley formula for cominuscule flag varieties (and hence Schubert varieties):

Lemma 17 (Fulton–Woodward)**.** *Let X be a cominuscule flag variety with corresponding cominuscule simple root γ. For any lower order ideals $\mu \subseteq \lambda \subseteq \mathcal{P}_X$, we have*

$$i_\lambda^* D \cdot [X_\mu] = \sum \frac{(\gamma,\gamma)}{(\alpha,\alpha)} [X_{\mu \setminus \{\alpha\}}],$$

the sum over all positive roots α such that $\mu \setminus \{\alpha\}$ is a lower order ideal in $\mathcal{P}_X$. Here $(\cdot,\cdot)$ denotes the usual inner product.

Observe that Lemma 17 reinterprets the Chevalley formula as a degree lowering operator since intersection product with divisors is a map from $A_k(X_\lambda)$ to $A_{k-1}(X_\lambda)$. This is opposite to the standard presentation of the Chevalley formula as a degree raising operator in cohomology.

Example 18. By Lemma 17, the following calculations hold for $X = \mathrm{LG}(3,6)$. We refer to Table 1 for the poset $\mathcal{P}_X$.

$$D \cdot \left[X_{\begin{ytableau} {} & S & S \\ \none & {} \end{ytableau}}\right] = 2\left[X_{\begin{ytableau} {} & S \\ \none & {} \end{ytableau}}\right] + \left[X_{\begin{ytableau} {} & S & S \end{ytableau}}\right], \quad D \cdot \left[X_{\begin{ytableau} {} & S \\ \none & {} \end{ytableau}}\right] = \left[X_{\begin{ytableau} {} & S \end{ytableau}}\right], \quad D \cdot \left[X_{\begin{ytableau} {} & S \end{ytableau}}\right] = 2\left[X_{\begin{ytableau} {} \end{ytableau}}\right].$$

Note that a coefficient 2 occurs whenever the removed box (root) is short.

Proof of Proposition 13. Lemma 15 allows us to identify the unique effective generator $i_\lambda^* D$ of $\mathrm{Pic}(X_\lambda)$, and Lemma 16 allows us to identify Schubert classes in $A^*(X_\lambda)$, or equivalently, their indexing set: the set of lower order ideals in $\mathcal{P}_X$ that are contained in λ. The Bruhat order on this set is recovered from Lemma 17, and its sub-poset of join-irreducible elements recovers the poset λ as discussed in Section 3. Finally, note that if μ is a join-irreducible diagram contained in λ with (unique) maximal box α, then

$$i_\lambda^* D \cdot [X_\mu] = \frac{(\gamma,\gamma)}{(\alpha,\alpha)} [X_{\mu \setminus \{\alpha\}}].$$

Since γ is always long, the coefficient $(\gamma,\gamma)/(\alpha,\alpha)$ is not 1 if and only if α is short. Hence, the labeling of the boxes in λ is also determined. □

4.2 Converse direction: the labeled poset λ determines the isomorphism class of X_λ

Let $X = G/P_I$ be a cominuscule flag variety and $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal. In this section, we prove that the poset λ and its labeling of long/short roots determine the isomorphism class of X_λ. More precisely, we prove the following proposition, which states the "converse" direction of Theorem 1.

Proposition 19. *Let $X_\lambda \subseteq X$ and $Y_\mu \subseteq Y$ be cominuscule Schubert varieties indexed by lower order ideals $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$, respectively. If λ and μ are isomorphic as labeled posets, then X_λ and Y_μ are algebraically isomorphic.*

Our strategy is to embed X_λ in a "minimal" flag variety X' determined by the labeled poset λ.

Recall that S is the set of simple reflections defined in Section 3.

Definition 20. The *support* of λ is defined as

$$S(\lambda) := \{s \in S : s \leq w_\lambda\}.$$

Equivalently, $S(\lambda)$ is the set of simple reflections appearing in any reduced decomposition of w_λ.

Every reduced decomposition of w_λ, and in particular, $S(\lambda)$, can be read out from the poset λ [6, Section 4]. The variety X' is constructed using $S(\lambda)$ as follows. Let G' be the reductive subgroup of $P_{S(\lambda)}$ with Weyl group $W' := W_{S(\lambda)}$ and $P' := G' \cap P_I$ be the reductive subgroup of G' corresponding to $I' := I \cap S(\lambda)$. Set $X' := G'/P'$. Note that $w_\lambda \in W'^{I'}$.

Lemma 21 below is a restatement of [16, Lemma 4.8].

Lemma 21 (Richmond–Slofstra)**.** *The inclusion $X' \hookrightarrow X$ induces an isomorphism $X'_{w_\lambda} \to X_\lambda$.*

Let Y be another cominuscule flag variety and $\mu \subseteq \mathcal{P}_Y$ be a lower order ideal. Next, we show that a labeled poset isomorphism between λ and μ induces an isomorphism between X' and Y', which restricts to an isomorphism between X_λ and Y_μ. We shall see that X' and Y' are cominuscule and that this isomorphism is given by an isomorphism of their Dynkin diagrams.

In the following, let $\mathcal{D}_X$ be the Dynkin diagram of X with vertex set Δ_X.

Definition 22. The diagram $\mathcal{D}_X^\lambda$ is defined to be the full subgraph of $\mathcal{D}_X$ with vertex set

$$\Delta_X^\lambda := \{\alpha \in \Delta_X : s_\alpha \in S(\lambda)\}.$$

Definition 23. Let a *Dynkin chain* in $\mathcal{P}_X$ be a chain $\pi \subseteq \mathcal{P}_X$ such that:

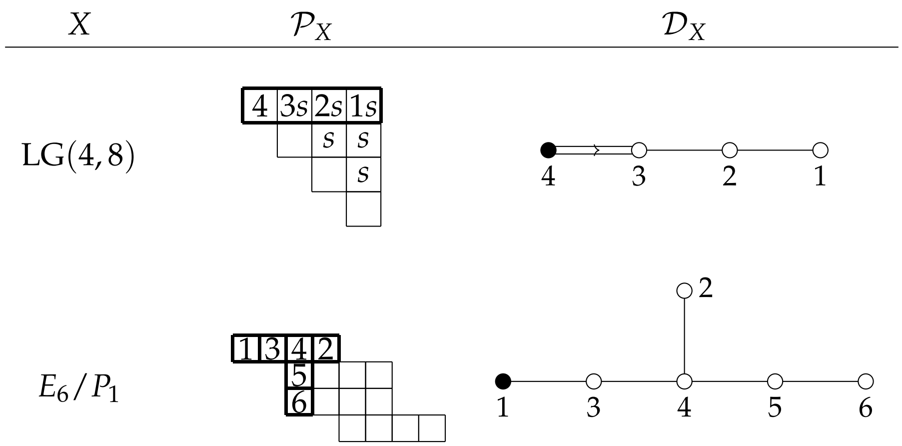
1. the set π is a lower order ideal;
2. the lengths of roots in π are weakly decreasing.

The lower order ideal $\mathcal{P}_X^\Delta \subseteq \mathcal{P}_X$ is defined to be the union of all Dynkin chains in $\mathcal{P}_X$.

In fact, Dynkin chains in $\mathcal{P}_X$ correspond to paths in $\mathcal{D}_X$ starting from the cominuscule root γ. Examples of $\mathcal{P}_X^\Delta$ are illustrated in Figure 1.

The proof of Lemma 24 and Corollary 25 below are left to the reader.

Figure 1: We highlight $\mathcal{P}_X^\Delta \subset \mathcal{P}_X$ with a bold border and label its boxes by the images of δ.



Lemma 24. *The restriction $\delta : \mathcal{P}_X^\Delta \to \Delta_X$ is a bijection.*

Corollary 25. *Let $\lambda \subseteq \mathcal{P}_X$ be a lower order ideal. Then $\delta(\lambda \cap \mathcal{P}_X^\Delta) = \Delta_X^\lambda$, and $\mathcal{D}_X^\lambda$ is a connected Dynkin diagram.*

Remark 26. Corollary 25 implies that X' is the cominuscule flag variety given by $(\mathcal{D}_X^\lambda, \gamma)$.

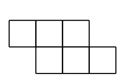
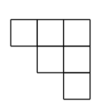
The last ingredient is Proposition 27, a purely combinatorial result. Geometrically, it implies that the "minimal" cominuscule flag varieties for Schubert varieties with isomorphic labeled posets are isomorphic.

Proposition 27. *Let $\lambda \subseteq \mathcal{P}_X$ and $\mu \subseteq \mathcal{P}_Y$ be lower order ideals. Then every labeled poset isomorphism between λ and μ induces a graph isomorphism between $\mathcal{D}_X^\lambda$ and $\mathcal{D}_Y^\mu$ that identifies reduced decompositions of w_λ and w_μ.*

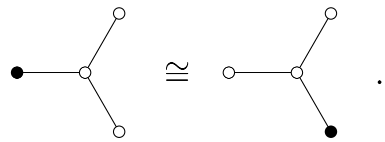
Proof of Proposition 19. Let γ_X and γ_Y denote the cominuscule simple roots corresponding to the cominuscule flag varieties X and Y. Let X' and Y' denote the cominuscule flag varieties given by the pairs $(\mathcal{D}_X^\lambda, \gamma_X)$ and $(\mathcal{D}_Y^\mu, \gamma_Y)$. Proposition 27 implies the cominuscule flag varieties X' and Y' are isomorphic. By Lemma 21, this isomorphism restricts to an isomorphism between X_λ and Y_μ, upon identifying them with Schubert varieties in X' and Y', respectively. □

We illustrate the above process with Example 28 and Example 29 below.

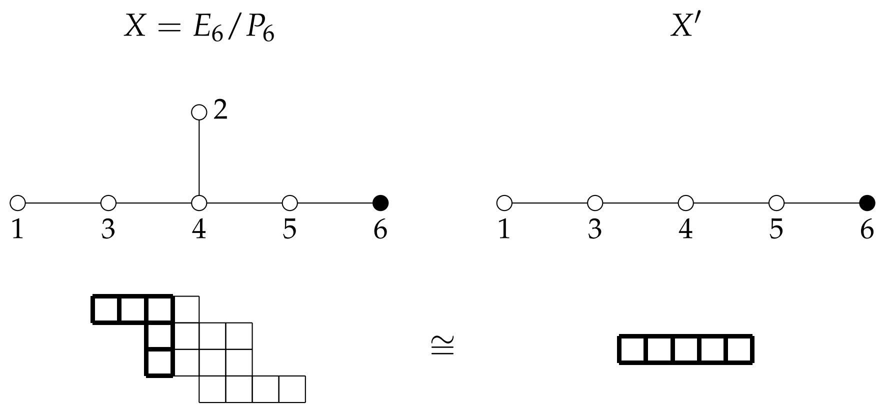
Example 28.

$\mathcal{P}_{Q^6} =$  $\cong$  $= \mathcal{P}_{\mathrm{OG}(4,8)}$,

therefore, $Q^6 \cong \mathrm{OG}(4,8)$. This isomorphism comes from a symmetry of the D_4 Dynkin diagram:



Example 29. Let $X = E_6/P_6$ and λ be the lower order ideal depicted on the left below. Then $S(\lambda) = \{s_1, s_3, s_4, s_5, s_6\}$, where s_i is the simple reflection corresponding to the simple root labeled by i. Therefore, the pair $(\mathcal{D}_X^\lambda, \gamma)$ is as depicted on the right, isomorphic to that of $\mathbb{P}^5$, showing $X_\lambda \cong X' \cong \mathbb{P}^5$.



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