*Séminaire Lotharingien de Combinatoire* **89B** (2023)
Article #53, 12 pp.

*Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)*

# 0-Hecke Modules, Quasiysmmetric Functions, and Peak Functions in Type B

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**Abstract.** We extend the recently-introduced *ascent-compatibility framework* to arbitrary Coxeter systems, yielding a uniform method for constructing modules of 0-Hecke algebras. In type B, we apply this method to produce 0-Hecke modules whose type B quasisymmetric characteristics are notable type B quasisymmetric functions. We also construct a 0-Hecke module on the set of standard domino tableaux of a given shape; the type B quasisymmetric characteristic of this module is a certain type B analogue of a Schur function called a *domino function*. We then introduce an analogous function defined on shifted domino tableaux and prove that this function expands positively in the type B analogue of the peak functions. Finally, we introduce a type B variant of the 0-Hecke–Clifford algebra and consider the modules of this algebra induced from the simple type B 0-Hecke modules. We characterize the isomorphism classes of these induced modules in terms of type B peak sets and prove that the type B quasisymmetric characteristics of the restrictions of these induced modules are precisely the type B peak functions.

## 1 Introduction

The 0-Hecke algebra $H_W(0)$ of a Coxeter system $(W, S)$ is a certain deformation of the group algebra of $W$. In type A, the Grothendieck group of finite-dimensional 0-Hecke modules is isomorphic to the Hopf algebra QSym of quasisymmetric functions via the *quasisymmetric characteristic* map [7]. There has been a great deal of recent work devoted to finding 0-Hecke modules whose quasisymmetric characteristics are certain notable quasisymmetric functions, thereby giving representation-theoretic interpretations for those quasisymmetric functions, and studying their properties, e.g., [1], [2], [9], [11], [13], [15]. Recently, a framework for producing such modules that relies on a simple *ascent-compatibility condition* was introduced in [14]; as far as we are aware, all of

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the notable families of quasisymmetric functions for which 0-Hecke modules have been constructed can be obtained via the ascent-compatibility framework.

In this extended abstract, we summarize results from [6]. Our first main result extends the ascent-compatibility condition from type A to all Coxeter systems, thus providing a framework for constructing 0-Hecke modules for any Coxeter system. Most of our attention is directed toward hyperoctahedral groups, where previous researchers have already laid the groundwork to extend much of the story from type A. Indeed, Chow [5] introduced the ring $\mathrm{QSym}^B$ of type B quasisymmetric functions, and Huang [8] recently defined a type B quasisymmetric characteristic map from the Grothendieck group of finite-dimensional type B 0-Hecke modules to $\mathrm{QSym}^B$. We provide several applications in which the ascent-compatibility framework readily produces type B 0-Hecke modules whose quasisymmetric characteristics are interesting type B quasisymmetric functions, thus providing a representation-theoretic interpretation of these functions.

Next, we define an action of the type B 0-Hecke algebra on *standard domino tableaux* of partition shape, thus obtaining a family of type B 0-Hecke modules whose quasisymmetric characteristics are the type B analogues of Schur functions defined in [10]. We also consider the shifted domino tableaux that were introduced and used to provide an expansion formula for products of Schur $Q$-functions in [4]. We define variants of these tableaux whose associated generating functions are in $\mathrm{QSym}^B$, and we prove that these type B analogues of the Schur $Q$-functions expand positively in Petersen's type B peak functions [12], with the expansion indexed by type B peak sets of standard shifted domino tableaux.

Finally, we consider the *0-Hecke–Clifford algebra* obtained by combining the 0-Hecke algebra and the Clifford algebra. The representation theory of this algebra is governed by the peak subalgebra of QSym [3], and type A ascent-compatibility was applied in [14] to construct 0-Hecke–Clifford modules for important families of functions in the peak algebra. We define a type B analogue of the 0-Hecke–Clifford algebra and show that the isomorphism classes of the modules of this algebra induced from the simple type B 0-Hecke modules are indexed by the type B peak sets. We further show that the quasisymmetric characteristics of the restrictions of these modules to the type B 0-Hecke algebra are precisely the type B peak functions [12], thus providing a representation-theoretic interpretation of these functions.

# 2 Background

## 2.1 Quasisymmetric functions

A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a finite sequence of positive integers. The integers $\alpha_i$ are called the *parts* of $\alpha$. If the parts of $\alpha$ sum to $n$, then we say $\alpha$ is a composition of $n$. There is a natural bijection between compositions of $n$ and subsets of $[n-1]$: given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of $n$, its *descent set* $\mathrm{Des}(\alpha)$ is the set $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$.

For $I \subseteq [n-1]$, the *fundamental quasisymmetric function* $F_I$ is defined by

$$F_I = \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n \\ j \in I \implies i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

As $\alpha$ ranges over all compositions, the fundamental quasisymmetric functions $F_{\mathrm{Des}(\alpha)}$ form a basis for the algebra QSym of quasisymmetric functions.

**Example 2.1.** If $n = 4$, then $F_{\{2,3\}} = \sum_{1 \le i_1 \le i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}$.

A type B analogue of the fundamental quasisymmetric functions was introduced by Chow [5]. These involve an additional variable $x_0$. For $I \subseteq [0, n-1]$, the *type B fundamental quasisymmetric function* $F_I^B$ is defined by

$$F_I^B = \sum_{\substack{0 = i_0 \le i_1 \le \cdots \le i_n \\ j \in I \implies i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Note that the variable $x_0$ appears in $F_I^B$ if and only if $0 \notin I$.

**Example 2.2.** If $n = 4$, then

$$F^B_{\{2,3\}} = \sum_{0 \le i_1 \le i_2 < i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4} \quad \text{and} \quad F^B_{\{0,3\}} = \sum_{0 < i_1 \le i_2 \le i_3 < i_4} x_{i_1} x_{i_2} x_{i_3} x_{i_4}.$$

A subset $P \subseteq [n-1]$ is called a *peak set* if $1 \notin I$ and $I$ contains no two consecutive integers. The compositions whose descent sets are peak sets are called *peak compositions*; these are precisely the compositions whose parts are all greater than 1, except possibly the last part. Given a peak set $P \subseteq [n-1]$, the *peak function* $K_P$ is defined as

$$K_P = 2^{|P|+1} \sum_{J \subseteq [n-1] \,:\, P \,\subseteq\, J \triangle (J+1)} F_J,$$

where $J + 1 = \{j + 1 : j \in J\}$ and $\triangle$ denotes symmetric difference.

Petersen introduced type B analogues of the peak functions in [12]. A subset $P \subseteq [n-1]$ is called a *type B peak set* if $P$ contains no two consecutive integers; note that a type B peak set is permitted to contain 1. For each type B peak set $P$, there are two type B peak functions $K_{(0,P)}$ and $K_{(1,P)}$ defined by

$$K_{(0,P)} = 2^{|P|} \sum_{J \subseteq [0,n-1] \,:\, P \,\subseteq\, J \triangle (J+1)} F_J^B \quad \text{and} \quad K_{(1,P)} = 2^{|P|+1} \sum_{0 \in J \subseteq [0,n-1] \,:\, P \,\subseteq\, J \triangle (J+1)} F_J^B,$$

with the proviso that $K_{(1,P)}$ is defined only if $1 \notin P$. For $I \subseteq [0, n-1]$, let $\mathrm{Peak}^B(I) = \{p \in I : p \ge 1 \text{ and } p - 1 \notin I\}$ and define

$$\Delta^B(I) = \begin{cases} K_{(0,\mathrm{Peak}^B(I))} & \text{if } 0 \notin I \\ K_{(1,\mathrm{Peak}^B(I))} & \text{if } 0 \in I. \end{cases}$$

## 2.2 0-Hecke algebras and modules

Given symbols $\gamma, \delta$ and a nonnegative integer $r$, we write $[\gamma \mid \delta]_r$ for the word $\gamma\delta\cdots$ of length $r$ that starts with $\gamma$ and alternates between $\gamma$ and $\delta$. For example, $[\gamma \mid \delta]_5 = \gamma\delta\gamma\delta\gamma$.

Let $(W, S)$ be a Coxeter system. Thus, $W$ is generated by $S$, and each element of $S$ is an involution. For any distinct $s, t \in S$, we have the relation $[s \mid t]_{m_{st}} = [t \mid s]_{m_{st}}$, where $m_{st} \in \{2, 3, \ldots\} \cup \{\infty\}$ (the relation is not present if $m_{st} = \infty$). The *0-Hecke algebra* $H_W(0)$ of the Coxeter system $(W, S)$ is a certain deformation of the group algebra of $W$; it has a generating set $\{\pi_s : s \in S\}$ satisfying the relations

$$\pi_s^2 = -\pi_s \quad \text{and} \quad [\pi_s \mid \pi_t]_{m_{st}} = [\pi_t \mid \pi_s]_{m_{st}}.$$

The simple $H_W(0)$-modules are all one-dimensional and indexed by subsets $I$ of $S$. The structure of a simple $H_W(0)$-module $S_I$ is given by

$$\pi_j(v) = \begin{cases} -v & \text{if } j \in I \\ 0 & \text{if } j \notin I, \end{cases}$$

where $v$ is a basis element of $S_I$.

We will mainly consider the 0-Hecke algebras where $W$ is the symmetric group $S_n$ or the hyperoctahedral group $B_n$. These are referred to as, respectively, the type A 0-Hecke algebra $H_n(0)$ and the type B 0-Hecke algebra $H_n^B(0)$. The type A 0-Hecke algebra $H_n(0)$ is generated by $\{\pi_1, \ldots, \pi_{n-1}\}$ with relations

$$\begin{aligned} \pi_i^2 &= -\pi_i; \\ \pi_i\pi_j &= \pi_j\pi_i \quad \text{if } |i-j| \geq 2; \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1}, \end{aligned}$$

while the type B 0-Hecke algebra $H_n^B(0)$ is generated by $\{\pi_0, \pi_1, \ldots, \pi_{n-1}\}$ with the same relations as above for $1 \leq i \leq n-1$ and the additional relations

$$\begin{aligned} \pi_0^2 &= -\pi_0; \\ \pi_0\pi_i &= \pi_i\pi_0; \quad \text{if } i \geq 2 \\ \pi_0\pi_1\pi_0\pi_1 &= \pi_1\pi_0\pi_1\pi_0. \end{aligned}$$

For $I \subseteq [n-1]$, let us (slightly abusing notation) write $S_I$ for the simple $H_n(0)$-module corresponding to the set $\{s_i : i \in I\}$ of simple generators of $A_{n-1}$. Similarly, for $I \subseteq [0, n-1]$, write $S_I^B$ for the simple $H_n^B(0)$-module corresponding to the set $\{s_i : i \in I\}$ of simple generators of $B_n$.

The *quasisymmetric characteristic map* is the Hopf algebra isomorphism ch from the Grothendieck group of finite-dimensional type A 0-Hecke modules to QSym defined by $\text{ch}([S_I]) = F_I$ [7]. Huang [8] defined a type B quasisymmetric characteristic map $\text{ch}^B$ from the Grothendieck group of finite-dimensional type B 0-Hecke modules to $\text{QSym}^B$ by $\text{ch}^B([S_I^B]) = F_I^B$.

## 3 Ascent-compatibility and applications in type B

Let $(W, S)$ be a Coxeter system. A simple generator $s \in S$ is an *ascent* of an element $w \in W$ if $\ell(sw) > \ell(w)$; otherwise, $s$ is a *descent* of $w$. The set of descents of $w$ is denoted $\mathrm{Des}(w)$. Fix $X \subseteq W$, and let $\mathbf{N}_X$ be the complex vector space with basis $X$. For $s \in S$, define a linear operator $\pi_s : \mathbf{N}_X \to \mathbf{N}_X$ by

$$\pi_s(x) = \begin{cases} -x & \text{if } s \in \mathrm{Des}(x) \\ 0 & \text{if } s \notin \mathrm{Des}(x) \text{ and } sx \notin X \\ sx & \text{if } s \notin \mathrm{Des}(x) \text{ and } sx \in X \end{cases}$$

for all $x \in X$. We say $X$ is *ascent-compatible* if for all $v, w \in X$ and all $s, t \in S$ such that $s$ is an ascent of $v$, $t$ is an ascent of $w$, and $v^{-1}sv = w^{-1}tw$, we have that $sv \in X$ if and only if $tw \in X$.

**Theorem 3.1.** *Let $(W, S)$ be a Coxeter system. If $X$ is an ascent-compatible subset of $W$, then the operators $\{\pi_s : s \in S\}$ define an action of the 0-Hecke algebra of $W$ on $\mathbf{N}_X$.*

The type A version of Theorem 3.1 was proved in [14].

**Example 3.2.** Consider the type A Coxeter group $S_3$, which has two simple generators $s_1, s_2$. Let

$$X = \{(1,2,3), (2,1,3), (2,3,1), (1,3,2)\} \subseteq S_3$$

where the elements of $X$ are written in one-line notation. This set $X$ is not ascent-compatible, because $(1,2,3)^{-1}s_1(1,2,3)$ and $(2,3,1)^{-1}s_2(2,3,1)$ are equal, but $s_1(1,2,3) = (2,1,3) \in X$ whereas $s_2(2,3,1) = (3,2,1) \notin X$.

On the other hand, the set

$$X' = \{(2,3,1), (3,1,2), (1,3,2)\} \subseteq S_3$$

is ascent-compatible.

We now provide a family of ascent-compatible sets across all Coxeter groups. Given a Coxeter system $(W, S)$, the *left weak order* $\leq_\mathsf{L}$ is the partial order on $W$ defined by $x \leq_\mathsf{L} y$ if there is a reduced word for $y$ that contains a reduced word for $x$ as a suffix. If $x \leq_\mathsf{L} y$, then the *interval* between $x$ and $y$ in the left weak order is the set $[x, y]_\mathsf{L} = \{z \in W : x \leq_L z \leq_\mathsf{L} y\}$. A set $X \subseteq W$ is called *convex* (in the left weak order) if for all $x, y \in X$ with $x \leq_\mathsf{L} y$, we have $[x, y]_\mathsf{L} \subseteq X$.

**Theorem 3.3.** *Let $(W, S)$ be a Coxeter system, and let $X \subseteq W$. If $X$ is convex in the left weak order and has a unique maximal element under the left weak order, then $X$ is ascent-compatible.*

In type A, Theorem 3.3 generalizes the result that intervals in left weak order on $S_n$ are ascent-compatible [14]. The corresponding 0-Hecke modules were studied in [9] and shown to be very useful for interpreting and relating many previously-studied families of 0-Hecke modules in type A.

For 0-Hecke algebras for which a quasisymmetric characteristic map is defined, one can obtain a formula for the characteristic of $[\mathbf{N}_X]$. In particular, in types A and B, we have the following (the type A version of this theorem appeared in [14]).

**Theorem 3.4.** *Let $X$ be an ascent-compatible subset of $S_n$ and $Y$ an ascent-compatible subset of $B_n$. Then*

$$\mathrm{ch}([\mathbf{N}_X]) = \sum_{x\in X} F_{\mathrm{Des}(x)} \qquad \textit{and} \qquad \mathrm{ch}^B([\mathbf{N}_Y]) = \sum_{y\in Y} F^B_{\mathrm{Des}(y)}.$$

**Example 3.5.** Let $\mathbf{N}_{X'}$ be the 0-Hecke module defined on the set $X'$ from Example 3.2. Then

$$\mathrm{ch}([\mathbf{N}_{X'}]) = F_{\{1\}} + 2F_{\{2\}}.$$

Given a function expressed as sums of type A or type B fundamental quasisymmetric functions associated to descents of Coxeter group elements, Theorems 3.1 and 3.4 yield a method for finding a 0-Hecke module whose quasisymmetric characteristic is that function. Specifically, it suffices to determine that the set of Coxeter group elements indexing the expansion is ascent-compatible. There has been significant interest in finding 0-Hecke modules for many such functions in type A, and the type A version of these theorems was used in [14] to give a common interpretation of these modules.

Motivated by a type B analogue of Schur positivity, Mayorova and Vassilieva [10] studied several interesting subsets of $B_n$ and functions associated to them. For $X \subseteq B_n$, let

$$\mathcal{Q}(X) = \sum_{x\in X} F^B_{\mathrm{Des}(x^{-1})}.$$

Note that realizing $\mathcal{Q}(X)$ as the characteristic of an $H_n^B(0)$-module via Theorems 3.1 and 3.4 follows from ascent-compatibility of the set $X^{-1} = \{x^{-1} : x \in X\}$.

We view $B_n$ as the group of permutations $\sigma$ of $\{\pm 1, \ldots, \pm n\}$ satisfying $\sigma(-i) = -\sigma(i)$ for all $i$. Two sets $X \subseteq B_n$ whose corresponding functions $\mathcal{Q}(X)$ were studied in [10] are the *left descent classes* and the *left-unimodal* permutations. The left descent class associated to $I \subseteq [0, n-1]$ is the set $D_I^B = \{\sigma \in B_n : \mathrm{Des}(\pi) = I\}$, and the left-unimodal permutations are the elements of $L^B = \bigcup_{i=1}^n L_i^B$, where $L_i^B$ is the set of all $\sigma \in B_n$ such that $\sigma^{-1}(1) > \cdots > \sigma^{-1}(i) < \cdots < \sigma^{-1}(n)$.

**Theorem 3.6.** *The sets $(D_I^B)^{-1}$ and $(L_i^B)^{-1}$ are ascent-compatible. Therefore, both $\mathbb{C}(D_I^B)^{-1}$ and $\mathbb{C}(L_i^B)^{-1}$ are $H_n^B(0)$-modules, and we have*

$$\mathrm{ch}^B(\mathbb{C}(D_I^B)^{-1}) = \mathcal{Q}(D_I^B) \quad \textit{and} \quad \mathrm{ch}^B(\mathbb{C}(L_i^B)^{-1}) = \mathcal{Q}(L_i^B);$$

*the latter of these identities implies that $\mathrm{ch}^B(\bigoplus_{i=1}^n \mathbb{C}(L_i^B)^{-1}) = \mathcal{Q}(L^B)$.*

The sets $D_I^B$ and $L_i^B$ are also ascent-compatible, so we additionally obtain $\mathcal{Q}((D_I^B)^{-1})$ and $\mathcal{Q}((L_i^B)^{-1})$ as characteristics of $H_n^B(0)$-modules. Let us remark that each $L_i^B$ is a subset of a left descent class; more generally, any subset of a left descent class is ascent-compatible.

Another example considered in [10] is the *type B Knuth class* $C_T^B$ associated to a standard domino tableau $T$: this is the set of all $\sigma \in B_n$ that insert to $T$ under a certain type B analogue of the Robinson–Schensted correspondence (see Section 4 for the definition of a standard domino tableau). The set $(C_T^B)^{-1}$ is not in general ascent-compatible, but $C_T^B$ itself is a subset of a left descent class, thus ascent-compatible. Therefore the functions $\mathcal{Q}((C_T^B)^{-1})$ are characteristics of $H_n^B(0)$-modules.

The final example considered in [10] is the *signed arc permutations* $\mathcal{A}_n^B$. These are the elements of $B_n$ whose one-line notation is a shuffle of a word $a_1 \cdots a_p$ over the alphabet $[n]$ and a word $b_1 \cdots b_{n-p}$ over the alphabet $-[n]$ satisfying the following conditions:

- $a_{i+1} \equiv a_i + 1 \pmod{n}$ for all $1 \leq i \leq p-1$;
- $b_{j+1} \equiv b_j + 1 \pmod{n}$ for all $1 \leq j \leq n-p-1$;
- $a_1 \equiv -b_1 + 1 \pmod{n}$ if $p$ and $n-p$ are both nonzero.

The set of inverses of signed arc permutations is not ascent-compatible, but the set of signed arc permutations themselves is. Accordingly, we obtain the following theorem.

**Theorem 3.7.** *The set $\mathcal{A}_n^B$ is ascent-compatible. Therefore, $\mathbb{C}\mathcal{A}_n^B$ is an $H_n^B(0)$-module, and*

$$\mathrm{ch}^B(\mathbb{C}\mathcal{A}_n^B) = \mathcal{Q}((\mathcal{A}_n^B)^{-1}).$$

## 4 Domino tableaux and type B peak functions

A central motivation for the families of functions considered in Section 3 is their positive expansions into a type B analogue of Schur functions known as domino functions, which have a formula in terms of domino tableaux [10]. In this section, we define an action of the type B 0-Hecke algebra on standard domino tableaux, thus realizing the domino functions as quasisymmetric characteristics of 0-Hecke modules. We then consider a shifted analogue of domino tableaux introduced by Chemli [4], modify these to allow 0 entries, and introduce standard shifted domino tableaux. We show that the generating functions of the (modified) shifted domino tableaux expand positively in the type B peak functions, indexed by standard shifted domino tableaux.

Let $\lambda \vdash 2n$. A *domino tiling* of $\lambda$ is a tiling of the Young diagram of shape $\lambda$ by $1 \times 2$ and $2 \times 1$ rectangles called *dominoes*. A *standard domino tableau* of shape $\lambda$ is a bijective filling of a domino tiling of $\lambda$ with entries from $[n]$ such that entries increase from left to

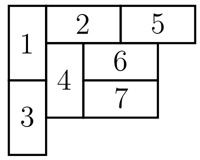


**Figure 1:** A standard domino tableau of shape $(5,4,4,1) \vdash 14$.

right along rows and from top to bottom in columns (see Figure 1). Let $\mathsf{SDT}(\lambda)$ denote the set of all standard domino tableaux of shape $\lambda$.

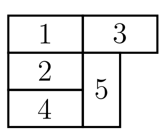
Suppose $T \in \mathsf{SDT}(\lambda)$. Let $\mathrm{dom}_i(T)$ denote the domino in $T$ filled with the entry $i$. A number $i \in [0, n-1]$ is a *descent* of $T$ if $i = 0$ and $\mathrm{dom}_1(T)$ is vertical or if $i > 0$ and $\mathrm{dom}_{i+1}(T)$ is strictly lower than $\mathrm{dom}_i(T)$. Let $\mathrm{Des}(T)$ be the set of descents of $T$. For example, the descent set of the domino tableau in Figure 1 is $\{0, 2, 5, 6\}$. The *domino function* $G_\lambda$ [10] is defined by

$$G_\lambda = \sum_{T \in \mathsf{SDT}(\lambda)} F^B_{\mathrm{Des}(T)}.$$

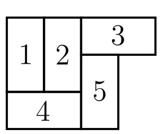
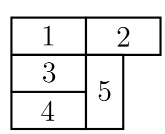
For $1 \le i \le n-1$, let $s_i(T)$ be the tableau obtained by exchanging the entries $i$ and $i+1$. If $\mathrm{dom}_1(T)$ and $\mathrm{dom}_2(T)$ tile a square in $T$, then we say these dominoes are *aligned* and let $s_0(T)$ be the tableau obtained by switching the orientation of both these dominoes (i.e., from horizontal to vertical or vice versa); otherwise, we say $s_0(T)$ is not defined (in particular, we would say $s_0(T) \notin \mathsf{SDT}(\lambda)$ in this case). We define operators $\pi_0, \pi_1, \ldots \pi_{n-1}$ on standard domino tableaux by

$$\pi_i(T) = \begin{cases} -T & \text{if } i \in \mathrm{Des}(T) \\ 0 & \text{if } i \notin \mathrm{Des}(T) \text{ and } s_i(T) \notin \mathsf{SDT}(\lambda) \\ s_i(T) & \text{if } i \notin \mathrm{Des}(T) \text{ and } s_i(T) \in \mathsf{SDT}(\lambda). \end{cases}$$

**Example 4.1.** Let $\lambda = (4,3,3)$, and suppose

$T =$  $\in \mathsf{SDT}(\lambda)$.

Then

$\pi_0(T) =$  and $\pi_2(T) =$  .

We also have $\pi_1(T) = \pi_3(T) = -T$ and $\pi_4(T) = 0$.

**Theorem 4.2.** *The operators $\pi_0, \pi_1, \ldots, \pi_{n-1}$ define an $H_n^B(0)$-action on the space spanned by the standard domino tableaux of shape $\lambda$. Moreover, the type B quasisymmetric characteristic of the resulting $H_n^B(0)$-module is precisely the domino function $G_\lambda$.*

We now consider the shifted domino tableaux [4]. The definition of these tableaux requires the notion of the *2-quotient* of a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, which is the pair of partitions $(\mu, \nu)$ obtained as follows. Let $\lambda^\star$ be the partition obtained by adding $k - i$ to the part $\lambda_i$ for each $1 \leq i \leq k$. Now let $w$ be obtained from $\lambda^\star$ by replacing the odd parts of $\lambda^\star$ with $1, 3, 5, \ldots$ and the even parts of $\lambda^\star$ with $0, 2, 4, \ldots$, from right to left. Then $\mu$ (respectively, $\nu$) is the partition found by subtracting the even (respectively, odd) entries of $w$ from the even (respectively, odd) entries of $\lambda^\star$, dividing the result by 2, and removing any 0 entries.

**Example 4.3.** If $\lambda = (7, 7, 6, 5, 1)$, then $\lambda^\star = (11, 10, 8, 6, 1)$, $w = (3, 4, 2, 0, 1)$, $\mu = (3, 3, 3)$, and $\nu = (4)$.

Let $\lambda$ be a partition whose 2-quotient $(\mu = (\mu_1, \ldots, \mu_p), \nu = (\nu_1, \ldots, \nu_q))$ satisfies $\mu_p \geq p$ and $\nu_q \geq q$. A domino tiling of $\lambda$ is a *shifted tiling* of $\lambda$ if there is no vertical domino $d$ lying on the main diagonal such that all dominoes left of $d$ and adjacent to $d$ are strictly below the main diagonal. Given a shifted tiling of $\lambda$, Chemli [4] defined a *semistandard shifted domino tableau* of shape $\lambda$ to be a filling of the dominoes in the tiling lying weakly above the main diagonal with entries from $1' < 1 < 2' < 2 < \cdots$ such that

1. entries weakly increase from left to right along rows and from top to bottom in columns
2. each row contains at most one $i'$, and each column contains at most one $i$.

Under this definition, the weight generating function of semistandard shifted domino tableaux of shape $\lambda$ is equal to the product of the Schur $Q$-functions indexed by $\mu$ and $\nu$ [4].

We modify Chemli's definition of semistandard shifted domino tableaux to introduce *zeroed semistandard shifted domino tableaux*. In these tableaux, we allow 0 entries (but not $0'$), where $0 < 1' < 1 < \cdots$; we also require that all dominoes with a 0 entry are horizontal. See the left of Figure 2. Let $\mathsf{SSShDT}(\lambda)$ denote the set of zeroed semistandard shifted domino tableaux. For $T \in \mathsf{SSShDT}(\lambda)$, let the *weight* $\mathrm{wt}(T)$ of $T$ be defined by $\mathrm{wt}(T) = (\mathrm{wt}_0(T), \mathrm{wt}_1(T), \ldots)$, where $\mathrm{wt}_i(T)$ is the number of entries in $T$ equal to $i$ or $i'$ (so $\mathrm{wt}_0(T)$ is just the number of 0 entries in $T$). Let $x^{\mathrm{wt}(T)} = \prod_{i \geq 0} x_i^{\mathrm{wt}_i(T)}$. We define the *shifted domino function* $H_\lambda$ by

$$H_\lambda = \sum_{T \in \mathsf{SSShDT}(\lambda)} x^{\mathrm{wt}(T)}.$$

**Figure 2:** A semistandard shifted domino tableau $T \in \mathsf{SSShDT}(\lambda)$ (left) and a standard shifted domino tableau $U \in \mathsf{SShDT}(\lambda)$ (right), where $\lambda = (7,7,6,5,1)$. We have $\mathrm{wt}(T) = (1,4,0,1,2,2)$ and $\mathrm{Des}(U) = \{1,5,7,8\}$.

The zeroed variant of semistandard shifted domino tableaux is analogous to a zeroed variant of a semistandard version of (nonshifted) domino tableaux introduced in [10], whose weight generating function is $G_\lambda$. In this way, the shifted domino functions may be regarded as a type B analogue of the Schur $Q$-functions, similarly to how domino functions provide a type B analogue of Schur functions.

Just as the Schur $Q$-functions expand positively in the peak functions, we claim the shifted domino function expands positively in the type B peak functions. To this end, we define a *standard shifted domino tableau* of shape $\lambda$ to be a bijective filling of the dominoes weakly above the main diagonal in a shifted tiling of $\lambda$ with entries in $[m]$ (where $m$ is the number of dominoes weakly above the main diagonal) such that entries strictly increase from left to right along rows and from top to bottom in columns. See the right of Figure 2. Let $\mathsf{SShDT}(\lambda)$ denote the set of standard shifted domino tableaux of shape $\lambda$.

Let $U \in \mathsf{SShDT}(\lambda)$, and suppose $U$ uses the entries $1,\ldots,m$. Write $\mathrm{dom}_i(U)$ for the domino filled with the entry $i$ in $U$. An integer $i \in [0, m-1]$ is a *descent* of $U$ if $i = 0$ and $\mathrm{dom}_1(U)$ is vertical or if $i > 0$ and $\mathrm{dom}_{i+1}(U)$ is strictly lower than $\mathrm{dom}_i(U)$. Let $\mathrm{Des}(U)$ be the set of descents of $U$. Recall the definition of the type B peak function $\Delta^B(I)$ from Section 2.1.

**Theorem 4.4.** *We have*

$$H_\lambda = \sum_{U \in \mathsf{SShDT}(\lambda)} \Delta^B(\mathrm{Des}(U)).$$

We remark that the same operators used in Theorem 4.2 also define a $H_n^B(0)$-action on the space spanned by $\mathsf{SShDT}(\lambda)$.

# 5 A type B analogue of the 0-Hecke–Clifford algebra

The *0-Hecke–Clifford algebra* $HCl_n(0)$ is the algebra with generators $\pi_1,\ldots,\pi_{n-1}$ and $c_1,\ldots,c_n$ such that the $\pi_i$ generate $H_n(0)$, the $c_j$ satisfy the Clifford relations $c_j^2 = -1$

and $c_i c_j = c_j c_i$ for $i \neq j$, and the $\pi_i$ and $c_j$ satisfy the additional relations

$$\begin{aligned} \pi_i c_j &= c_j \pi_i \quad \text{for } j \neq i, i+1; \\ \pi_i c_i &= c_{i+1} \pi_i; \\ (\pi_i + 1) c_{i+1} &= c_i (\pi_i + 1). \end{aligned}$$

We construct an analogue $HCl_n^B(0)$ of this algebra using the type B 0-Hecke algebra $H_n^B(0)$. This algebra has generators $\pi_0, \pi_1, \ldots, \pi_{n-1}$ and $c_1, \ldots, c_n$ such that the $\pi_i$ generate $H_n^B(0)$, the $c_j$ satisfy the same Clifford relations $c_j^2 = -1$ and $c_i c_j = c_j c_i$ for $i \neq j$, and the $\pi_i$ and $c_j$ satisfy the additional relations

$$\begin{aligned} \pi_i c_j &= c_j \pi_i \quad \text{for } j \neq i, i+1; \\ \pi_i c_{i+1} &= c_{i+1} \pi_i \quad \text{for } 1 \leq i \leq n-1; \\ (\pi_i + 1) c_i &= c_{i+1} (\pi_i + 1); \\ \pi_0 c_1 &= \sqrt{-1} \pi_0. \end{aligned}$$

Recall that we write $S_I^B$ for the simple $H_n^B(0)$-module corresponding to the set $\{s_i : i \in I\}$ of simple generators of $B_n$. Let $M_I^B$ denote the $HCl_n^B(0)$-module induced from $S_I^B$, and consider the restriction $\mathrm{Res}_{H_n^B(0)} M_I^B$ of $M_I^B$ to $H_n^B(0)$. The following theorem, whose type A analogue was proven in [3], provides a representation-theoretic interpretation of Petersen's type B peak functions $\Delta^B(I)$ [12].

**Theorem 5.1.** *Let $I \subseteq [0, n-1]$ and let $I^c = [0, n-1] \setminus I$. Then*

$$\mathrm{ch}^B(\mathrm{Res}_{H_n^B(0)} M_{I^c}^B) = \Delta^B(I).$$

*Moreover, two induced modules $M_I^B$ and $M_J^B$ are isomorphic if and only if*

$$\mathrm{Peak}^B(I^c) = \mathrm{Peak}^B(J^c) \quad \textit{and} \quad 0 \notin I \triangle J.$$

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