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Modules of the 0-Hecke algebras for genomic Schur functions

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Abstract. We construct an $H_m(0)$-module $\mathbf{G}_{\lambda;m}$ whose image under the quasisymmetric characteristic is the mth degree homogeneous component of the genomic Schur function U_λ by defining an $H_m(0)$-action on increasing gapless tableaux. We then provide a direct sum decomposition of $\mathbf{G}_{\lambda;m}$ and show that each summand of this decomposition is isomorphic to a weak Bruhat interval module.

Keywords: 0-Hecke algebra, weak Bruhat order, genomic Schur function, quasisymmetric characteristic

1 Introduction

Let $X = \mathrm{Gr}_k(\mathbb{C}^n)$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$. Since the early 2000s, several combinatorial interpretations for the *K-theoretic Littlewood-Richardson rule* have been introduced. For instance, see [3, 11, 15, 16]. In particular, Pechenik and Yong [11] gave a combinatorial interpretation by using *genomic tableaux*. Therein, they defined a symmetric function U_λ, called the *genomic Schur function*, as a generating function for genomic tableaux of shape λ for all partition λ. Further, they proved that $\{U_\lambda \mid \lambda \text{ is a partition}\}$ is a basis for the ring of symmetric functions and pointed out that genomic Schur functions are not Schur-positive in general. As an alternative positivity, Pechenik [10] showed that genomic Schur functions are fundamental positive. Specifically, for any partition λ,

$$U_\lambda = \sum_{T \in \mathrm{IGLT}(\lambda)} F_{\mathrm{comp}(T)},$$

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where IGLT(λ) is the set of *increasing gapless tableaux* of shape λ, comp(T) is the composition associated to T, and $F_{\mathrm{comp}(T)}$ is the *fundamental quasisymmetric function* associated to comp(T). For the precise definitions, see Subsection 2.4.

The 0-*Hecke algebra* $H_n(0)$ is the $\mathbb{C}$-algebra obtained from the Hecke algebra $H_n(q)$ by specializing q to 0. Duchamp, Krob, Leclerc, and Thibon [6] introduced a ring isomorphism, called *quasisymmetric characteristic*,

$$\mathrm{ch} : \mathcal{G}_0(H_\bullet(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha.$$

Here, $\mathcal{G}_0(H_\bullet(0))$ is the Grothendieck ring associated to 0-Hecke algebras and QSym is the ring of quasisymmetric functions. In view of this correspondence, there have been considerable attempts to provide a representation theoretic interpretation of noteworthy quasisymmetric functions by constructing appropriate 0-Hecke modules. For instance, see [1, 2, 4, 12, 13, 14]. Recently, Jung, Kim, Lee, and Oh [7] introduced the *weak Bruhat interval module* $\mathsf{B}(\sigma, \rho)$ to provide a unified method to study the $H_n(0)$-modules in the papers mentioned above. Here, σ and ρ are permutations in the symmetric group $\mathfrak{S}_n$.

The purpose of this paper is to provide a nice representation theoretic interpretation of genomic Schur functions. First, for each $1 \le m \le n$, we construct an $H_m(0)$-module $\mathbf{G}_{\lambda;m}$ by defining an $H_m(0)$-action on the $\mathbb{C}$-span of the set IGLT$(\lambda)_m$ of increasing gapless tableaux of shape λ with maximum entry m. And, we see that the image of $\mathbf{G}_{\lambda;m}$ under the quasisymmetric characteristic is the mth homogeneous component of U_λ. Next, we define an equivalence relation $\sim_{\lambda;m}$ on IGLT$(\lambda)_m$ and show that the $\mathbb{C}$-span of each equivalence class is closed under the $H_m(0)$-action. Thus, we obtain a direct sum decomposition

$$\mathbf{G}_{\lambda;m} = \bigoplus_{E \in \mathcal{E}_{\lambda;m}} \mathbf{G}_E,$$

where $\mathcal{E}_{\lambda;m}$ is the set of all equivalence classes with respect to $\sim_{\lambda;m}$ and $\mathbf{G}_E$ is the submodule of $\mathbf{G}_{\lambda;m}$ whose underlying space is the $\mathbb{C}$-span of E. Finally, we show that $\mathbf{G}_E$ is isomorphic to a weak Bruhat interval module for $E \in \mathcal{E}_{\lambda;m}$. To do this, we prove that there exist unique source tableau T_E and sink tableau T'_E in E. In addition, we assign a permutation $\mathsf{read}(T)$, called *standardized reading word*, to each $T \in E$. With these preparations, we show that

$$\mathbf{G}_E \cong \mathsf{B}(\mathsf{read}(T_E), \mathsf{read}(T'_E)).$$

We end with providing an avenue for future research. This paper is an extended abstract of our paper [8].

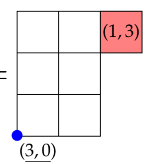
2 Preliminaries

Given any integers m and n, define $[m, n]$ to be the set $\{k \in \mathbb{Z} \mid m \le k \le n\}$ if $m \le n$ or the empty set otherwise. Throughout this section, n denotes a nonnegative integer.

2.1 Compositions and Diagrams

A *composition* α of a nonnegative integer n, denoted by $\alpha \models n$, is a finite ordered list of positive integers $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ satisfying $\sum_{i=1}^{k} \alpha_i = n$. We call $k =: \ell(\alpha)$ the *length* of α and $n =: |\alpha|$ the *size* of α. Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{\ell(\alpha)}) \models n$, we define $\mathrm{set}(\alpha) := \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \sum_{i=1}^{\ell(\alpha)-1} \alpha_i\} \subseteq [1, n-1]$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \models n$ satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\ell(\lambda)}$, then we say that λ is a *partition* of n and denote it by $\lambda \vdash n$. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \vdash n$, we define the *Young diagram* $\mathtt{yd}(\lambda)$ of λ by a left-justified array of n boxes where the ith row from the top has λ_i boxes for $1 \leq i \leq \ell(\lambda)$. We say that a box in $\mathtt{yd}(\lambda)$ is *in the ith row* if it is in the ith row from the top and *in the jth column* if it is in the jth column from the left. We denote by (i, j) the box in the ith row and jth column. Denoting $(i, j) \in \mathtt{yd}(\lambda)$ means that $1 \leq i \leq \ell(\lambda)$ and $1 \leq j \leq \lambda_i$. We also say that a lattice point on $\mathtt{yd}(\lambda)$ is *in the ith row* if it is in the $(i+1)$st horizontal line from the top and *in the jth column* if it is in the $(j+1)$st vertical line from the left. We denote by $(\underline{i, j})$ the lattice point in the ith row and jth column. For example, if $\lambda = (3, 2, 2)$, then

$\mathtt{yd}(\lambda) =$  ,

the box $(1, 3)$ is the box filled with red, and the lattice point $(\underline{3, 0})$ is the point marked by the blue dot. A *filling of* $\mathtt{yd}(\lambda)$ is a function $T : \mathtt{yd}(\lambda) \to \mathbb{Z}_{>0}$. Throughout this paper, we assume that

$$\begin{aligned} T((i,j)) &= \infty && \text{if } (i,j) \in (\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}) \setminus \mathtt{yd}(\lambda) \quad \text{and} \\ T((i,j)) &= -\infty && \text{if } (i,j) \in (\mathbb{Z} \times \mathbb{Z}) \setminus (\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}). \end{aligned}$$

For any filling T of $\mathtt{yd}(\lambda)$, let $\max(T) := \max\{T((i,j)) \mid (i,j) \in \mathtt{yd}(\lambda)\}$.

2.2 The 0-Hecke algebra and the quasisymmetric characteristic

To begin with, we recall that the symmetric group $\mathfrak{S}_n$ is generated by simple transpositions $s_i := (i, i+1)$ with $1 \leq i \leq n-1$. An expression for $\sigma \in \mathfrak{S}_n$ of the form $s_{i_1} s_{i_2} \cdots s_{i_p}$ that uses the minimal number of simple transpositions is called a *reduced expression* for σ. The number of simple transpositions in any reduced expression for σ, denoted by $\ell(\sigma)$, is called the *length* of σ.

The 0-*Hecke algebra* $H_n(0)$ is the $\mathbb{C}$-algebra generated by $\pi_1, \pi_2, \ldots, \pi_{n-1}$ subject to the following three relations: (1) $\pi_i^2 = \pi_i$ for $1 \leq i \leq n-1$, (2) $\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1}$ for $1 \leq i \leq n-2$, and (3) $\pi_i \pi_j = \pi_j \pi_i$ if $|i - j| \geq 2$. Pick up any reduced expression $s_{i_1} s_{i_2} \cdots s_{i_p}$ for a permutation $\sigma \in \mathfrak{S}_n$. We define the element π_σ of $H_n(0)$ by $\pi_\sigma := \pi_{i_1} \pi_{i_2} \cdots \pi_{i_p}$ It is well known that π_σ is independent of the choice of reduced expressions, and $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ is a basis for $H_n(0)$.

In [9], Norton classified all irreducible modules of the 0-Hecke algebras. It was shown that there are 2^{n-1} distinct irreducible $H_n(0)$-modules which are naturally parametrized by compositions of n. For each $\alpha \models n$, the irreducible module $\mathbf{F}_\alpha$ corresponding to α is the 1-dimensional $H_n(0)$-module spanned by a vector v_α whose $H_n(0)$-action is given by

$$\pi_i \cdot v_\alpha = \begin{cases} 0 & i \in \mathrm{set}(\alpha), \\ v_\alpha & i \notin \mathrm{set}(\alpha), \end{cases} \qquad (1 \le i \le n-1).$$

Let $\mathcal{R}(H_n(0))$ denote the $\mathbb{Z}$-span of the isomorphism classes of finite dimensional $H_n(0)$-modules. The isomorphism class corresponding to an $H_n(0)$-module M will be denoted by $[M]$. The *Grothendieck group* $\mathcal{G}_0(H_n(0))$ is the quotient of $\mathcal{R}(H_n(0))$ modulo the relations $[M] = [M'] + [M'']$ whenever there exists a short exact sequence $0 \to M' \to M \to M'' \to 0$. The irreducible $H_n(0)$-modules form a free $\mathbb{Z}$-basis for $\mathcal{G}_0(H_n(0))$. Let $\mathcal{G}_0(H_\bullet(0)) := \bigoplus_{n \ge 0} \mathcal{G}_0(H_n(0))$ be the ring equipped with the induction product. In [6], Duchamp, Krob, Leclerc, and Thibon revealed a deep connection between $\mathcal{G}_0(H_\bullet(0))$ and the ring QSym of quasisymmetric functions by providing a ring isomorphism

$$\mathrm{ch} : \mathcal{G}_0(H_\bullet(0)) \to \mathrm{QSym}, \quad [\mathbf{F}_\alpha] \mapsto F_\alpha,$$

called *quasisymmetric characteristic*. Here, F_α is the *fundamental quasisymmetric function*.

2.3 Weak Bruhat interval modules of the 0-Hecke algebra

Given $\sigma \in \mathfrak{S}_n$, let $\mathrm{Des}_L(\sigma) := \{i \in [1, n-1] \mid \ell(s_i\sigma) < \ell(\sigma)\}$. The *left weak Bruhat order* $\preceq_L$ on $\mathfrak{S}_n$ is the partial order on $\mathfrak{S}_n$ whose covering relation $\preceq_L^c$ is defined as follows: $\sigma \preceq_L^c s_i\sigma$ if and only if $i \notin \mathrm{Des}_L(\sigma)$. Given $\sigma, \rho \in \mathfrak{S}_n$, the *left weak Bruhat interval from σ to ρ*, denoted by $[\sigma, \rho]_L$, is the closed interval $\{\gamma \in \mathfrak{S}_n \mid \sigma \preceq_L \gamma \preceq_L \rho\}$.

Definition 2.1. ([7]) Let $\sigma, \rho \in \mathfrak{S}_n$. The *weak Bruhat interval module associated to* $[\sigma, \rho]_L$, denoted by $\mathsf{B}(\sigma, \rho)$, is the $H_n(0)$-module with the underlying space $\mathbb{C}[\sigma, \rho]_L$ and with the $H_n(0)$-action defined by

$$\pi_i \cdot \gamma := \begin{cases} \gamma & \text{if } i \in \mathrm{Des}_L(\gamma), \\ 0 & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \notin [\sigma, \rho]_L, \\ s_i\gamma & \text{if } i \notin \mathrm{Des}_L(\gamma) \text{ and } s_i\gamma \in [\sigma, \rho]_L. \end{cases}$$

2.4 Genomic Schur functions

Given $\lambda \vdash n$, an *increasing gapless tableau* of shape λ is a filling of $\mathtt{yd}(\lambda)$ such that the entries in each row strictly increase from left to right, the entries in each column strictly increase from top to bottom, and the set $T^{-1}(i)$ is nonempty for all $1 \le i \le \max(T)$. Let $\mathrm{IGLT}(\lambda)$ be the set of all increasing gapless tableaux of shape λ. Given $T \in \mathrm{IGLT}(\lambda)$ and

$1 \le i \le \max(T)$, let $\mathsf{Top}_i(T)$ (resp. $\mathsf{Bot}_i(T)$) be the highest (resp. lowest) box in T having the entry i. Let

$$(r_{\mathsf{b}}^{(i)}(T), c_{\mathsf{b}}^{(i)}(T)) := \mathsf{Bot}_i(T) \quad \text{and} \quad (r_{\mathsf{t}}^{(i)}(T), c_{\mathsf{t}}^{(i)}(T)) := \mathsf{Top}_i(T).$$

If T is clear in the context, we simply write $r_{\mathsf{b}}^{(i)}, c_{\mathsf{b}}^{(i)}, r_{\mathsf{t}}^{(i)}$, and $c_{\mathsf{t}}^{(i)}$ instead of $r_{\mathsf{b}}^{(i)}(T)$, $c_{\mathsf{b}}^{(i)}(T)$, $r_{\mathsf{t}}^{(i)}(T)$, and $c_{\mathsf{t}}^{(i)}(T)$, respectively. We call an index $i \in [1, \max(T) - 1]$ a *descent of* T if there is some instance of i strictly above some instance of $i+1$ in T. Let $\mathrm{Des}(T)$ be the set of all descents of T and let $\mathrm{comp}(T) := \mathrm{comp}(\mathrm{Des}(T))$.

Definition 2.2. ([10, 11]) For $\lambda \vdash n$, the *genomic Schur function* U_λ is defined by

$$U_\lambda := \sum_{T \in \mathrm{IGLT}(\lambda)} F_{\mathrm{comp}(T)}.$$

Given $1 \le m \le n$, we define

$$\mathrm{IGLT}(\lambda)_m := \{T \in \mathrm{IGLT}(\lambda) \mid \max(T) = m\} \quad \text{and} \quad U_{\lambda;m} := \sum_{T \in \mathrm{IGLT}(\lambda)_m} F_{\mathrm{comp}(T)}.$$

From the definition, it immediately follows that $U_{\lambda;m}$ is the mth degree homogeneous component of U_λ.

Hereafter, we assume that n is a positive integer, m is a positive integer less than or equal to n, and λ is a partition of n, unless otherwise stated.

3 0-Hecke modules from increasing gapless tableaux

3.1 An $H_m(0)$-module for $U_{\lambda;m}$

We start by introducing the necessary definitions.

Definition 3.1. Given $T \in \mathrm{IGLT}(\lambda)$ and $1 \le i \le \max(T) - 1$, we say that i is an *attacking descent* if $i \in \mathrm{Des}(T)$, and either

(a) there exists $(j,k) \in \mathtt{yd}(\lambda)$ such that $T((j,k)) = i$ and $T((j+1,k)) = i+1$, or

(b) there exists a box $B \in T^{-1}(i+1)$ placed weakly above $\mathsf{Bot}_i(T)$.

Take any $1 \le m \le n$. For each $1 \le i \le m-1$, we define a linear operator $\boldsymbol{\pi}_i : \mathbb{C}\,\mathrm{IGLT}(\lambda)_m \to \mathbb{C}\,\mathrm{IGLT}(\lambda)_m$ by

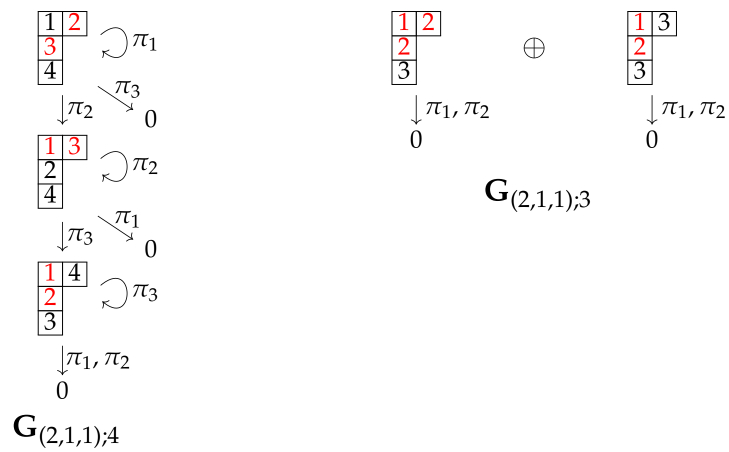
$$\boldsymbol{\pi}_i(T) := \begin{cases} T & \text{if } i \text{ is not a descent of } T, \\ 0 & \text{if } i \text{ is an attacking descent of } T, \\ s_i \cdot T & \text{if } i \text{ is a non-attacking descent of } T \end{cases} \tag{3.1}$$

for $T \in \mathrm{IGLT}(\lambda)_m$ and extending it by linearity. Here, $s_i \cdot T$ is the tableau obtained from T by replacing i and $i+1$ with $i+1$ and i, respectively. By proving $\boldsymbol{\pi}_i^2 = \boldsymbol{\pi}_i$ $(1 \le i \le m-1)$, $\boldsymbol{\pi}_i\boldsymbol{\pi}_{i+1}\boldsymbol{\pi}_i = \boldsymbol{\pi}_{i+1}\boldsymbol{\pi}_i\boldsymbol{\pi}_{i+1}$ $(1 \le i \le m-2)$, and $\boldsymbol{\pi}_i\boldsymbol{\pi}_j = \boldsymbol{\pi}_j\boldsymbol{\pi}_i$ $(1 \le i,j \le m-1$ with $|i-j| > 1)$, we obtain the following theorem. For details, see [8, Subsection 3.1].

Theorem 3.2. *For any $1 \le m \le n$, the operators $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots, \boldsymbol{\pi}_{m-1}$ satisfy the same relations as the generators $\pi_1, \pi_2, \ldots \pi_{m-1}$ for $H_m(0)$. In other words, $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots, \boldsymbol{\pi}_{m-1}$ define an $H_m(0)$-action on $\mathbb{C}\,\mathrm{IGLT}(\lambda)_m$.*

Example 3.3. (1) When $T = \begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 2&3&5&7\\\hline 4&6\\\cline{1-2}\end{array}$, we have $\boldsymbol{\pi}_3(T) = s_3 \cdot T$, $\boldsymbol{\pi}_4(T) = T$, and $\boldsymbol{\pi}_i(T) = 0$ for $i = 1, 2, 5, 6$. Here, the indices in red are used to indicate the descents of the tableau.

(2) Note that $\mathrm{IGLT}((2,1,1)) = \left\{ \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array} \right\}$. One can see that $U_{(2,1,1)} = (F_{(2,1,1)} + F_{(1,2,1)} + F_{(1,1,2)}) + 2F_{(1,1,1)}$. The following figures illustrate the $H_m(0)$-action on $\mathbf{G}_{(2,1,1);m}$ for $m = 3, 4$:



The following proposition follows immediately from (3.1) and Theorem 3.2.

Proposition 3.4. *For any $\lambda \vdash n$ and $1 \le m \le n$, $\mathrm{ch}([\mathbf{G}_{\lambda;m}]) = U_{\lambda;m}$. Consequently, $\sum_{1 \le m \le n} \mathrm{ch}([\mathbf{G}_{\lambda;m}]) = U_\lambda$.*

Remark 3.5. In [5, Theorem 4.7], Duchamp, Hivert, and Thibon described the *Ext-quiver* of $H_m(0)$. According to their result, for any $\alpha \models m$, we have $\mathrm{Ext}^1_{H_m(0)}(\mathbf{F}_\alpha, \mathbf{F}_\alpha) = 0$, equivalently, there is no indecomposable $H_m(0)$-module M such that $\mathrm{ch}([M]) = 2F_\alpha$. On the other hand, in Example 3.3, we see that $U_{(2,1,1);3} = 2F_{(1,1,1)}$. Thus, we conclude that there is no indecomposable $H_3(0)$-module M satisfying $\mathrm{ch}([M]) = U_{(2,1,1);3}$.

3.2 A direct sum decomposition of $\mathbf{G}_{\lambda;m}$ into $H_m(0)$-submodules

Let us start with necessary definitions and notation. Given $T \in \mathrm{IGLT}(\lambda)_m$, let $\mathcal{I}(T) := \{i \in [1,m] \mid |T^{-1}(i)| > 1\}$. Given $i \in \mathcal{I}(T)$, let $\Gamma_i(T)$ be the lattice path from $\underline{(r^{(i)}_{\mathtt{b}}, c^{(i)}_{\mathtt{b}} - 1)}$ to $\underline{(r^{(i)}_{\mathtt{t}} - 1, c^{(i)}_{\mathtt{t}})}$ satisfying the following two conditions:

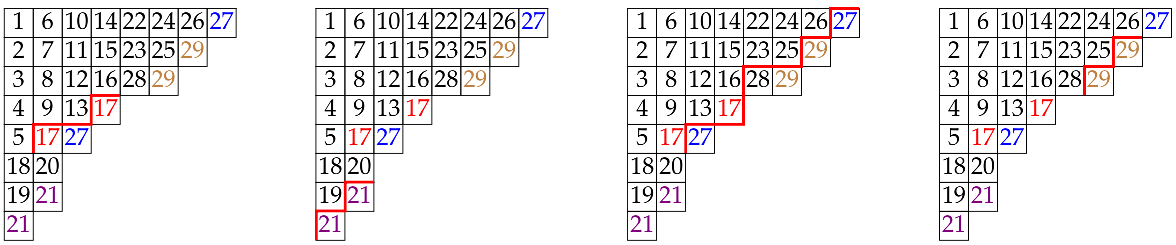
(i) if the path passes through two boxes horizontally, then the entry at the above box is smaller than i and the entry at the below box is greater than i, and

(ii) if the path passes through two boxes vertically, then the entry at the left box is smaller than i and the entry at the right box is greater than i.

Example 3.6. Let

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 6 & 10 & 14 & 22 & 24 & 26 & 27 \\ \hline 2 & 7 & 11 & 15 & 23 & 25 & 29 \\ \cline{1-7} 3 & 8 & 12 & 16 & 28 & 29 \\ \cline{1-6} 4 & 9 & 13 & 17 \\ \cline{1-4} 5 & 17 & 27 \\ \cline{1-3} 18 & 20 \\ \cline{1-2} 19 & 21 \\ \cline{1-2} 21 \\ \cline{1-1} \end{array} .$$

Note that $\mathcal{I}(T) = \{17, 21, 27, 29\}$. By following the way of defining lattice paths, we obtain the lattice paths $\Gamma_{17}(T), \Gamma_{21}(T), \Gamma_{27}(T)$, and $\Gamma_{29}(T)$ as follows:



$\Gamma_{17}(T)$ $\Gamma_{21}(T)$ $\Gamma_{27}(T)$ $\Gamma_{29}(T)$

Given a lattice path Γ, let $V(\Gamma)$ be the set of lattice points through which Γ passes. For two lattice paths Γ and Γ', we write $\Gamma = \Gamma'$ if $V(\Gamma) = V(\Gamma')$.

Definition 3.7. Let $\lambda \vdash n$ and $T_1, T_2 \in \mathrm{IGLT}(\lambda)_m$. The equivalence relation $\sim_{\lambda;m}$ on $\mathrm{IGLT}(\lambda)_m$ is defined by $T_1 \sim_{\lambda;m} T_2$ if and only if

$$\left\{ \left(\Gamma_i(T_1), T_1^{-1}(i) \right) \;\middle|\; i \in \mathcal{I}(T_1) \right\} = \left\{ \left(\Gamma_i(T_2), T_2^{-1}(i) \right) \;\middle|\; i \in \mathcal{I}(T_2) \right\}.$$

Example 3.8. Let $T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 5 & 6 \\ \hline 3 & 5 & 7 \\ \cline{1-3} \end{array}$, $T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 4 & 5 & 7 \\ \hline 3 & 5 & 6 \\ \cline{1-3} \end{array}$, $T_3 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 5 & 6 \\ \hline 3 & 4 & 7 \\ \cline{1-3} \end{array}$, and $T_4 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 2 & 3 & 5 & 6 \\ \hline 4 & 5 & 7 \\ \cline{1-3} \end{array}$.

Then, $T_1 \sim_{(4,3,2);5} T_2$, but $T_1 \not\sim_{(4,3,2);5} T_k$ for $k = 3, 4$.

Let $\mathcal{E}_{\lambda;m}$ be the set of equivalence classes of $\mathrm{IGLT}(\lambda)_m$ with respect to $\sim_{\lambda;m}$.

Theorem 3.9. *Let m and n be positive integers with $m \le n$ and let $\lambda \vdash n$. For any $1 \le i \le m-1$ and $E \in \mathcal{E}_{\lambda;m}$, $\pi_i \cdot \mathbb{C}E \subseteq \mathbb{C}E$.*

For each $E \in \mathcal{E}_{\lambda;m}$, let $\mathbf{G}_E$ be the $H_m(0)$-submodule of $\mathbf{G}_{\lambda;m}$ whose underlying space is the $\mathbb{C}$-span of E. Then, we have the following direct sum decomposition

$$\mathbf{G}_{\lambda;m} = \bigoplus_{E \in \mathcal{E}_{\lambda;m}} \mathbf{G}_E.$$

Hereafter, E denotes an equivalence class of $\mathrm{IGLT}(\lambda)_m$ with respect to $\sim_{\lambda;m}$ and T denotes a tableau contained in $\mathrm{IGLT}(\lambda)_m$ unless otherwise stated.

4 Source and sink tableaux

The goal of this section is to show that there are two distinguished tableaux, called *source* and *sink tableaux*, in each equivalence class $E \in \mathcal{E}_{\lambda;m}$. To achieve our goal, we first construct two tableaux $\mathsf{source}(T)$ and $\mathsf{sink}(T)$ for each $T \in E$. Then, we prove that $\mathsf{source}(T)$ (resp. $\mathsf{sink}(T)$) is the unique source tableau (resp. sink tableau) in E, where T is an arbitrary chosen element in E.

To begin with, we give definitions for source tableaux and sink tableaux in $\mathrm{IGLT}(\lambda)_m$.

Definition 4.1. Let $T \in \mathrm{IGLT}(\lambda)_m$.

(1) T is said to be a *source tableau* if there do not exist $T' \in \mathrm{IGLT}(\lambda)_m$ and $1 \le i \le m-1$ such that $\pi_i \cdot T' = T$ and $T' \neq T$.
(2) T is said to be a *sink tableau* if there do not exist $T' \in \mathrm{IGLT}(\lambda)_m$ and $1 \le i \le m-1$ such that $\pi_i \cdot T = T'$ and $T' \neq T$.

To construct the desired tableau $\mathsf{source}(T)$, we need the following preparation. Given two lattice points P and P' in the same row, we denote the horizontal line from P to P' by $\mathtt{HL}(P, P')$. For each $i \in \mathcal{I}(T)$, we define a new lattice path $\widetilde{\Gamma}_i(T)$ by extending $\Gamma_i(T)$ with the following algorithm.

Algorithm 4.2. Fix $i \in \mathcal{I}(T)$ and let $\lambda_0 := \lambda_1$.

Step 1. For each $j \in \mathcal{I}(T)$, let Γ'_j be the lattice path obtained by connecting three lattice paths $\mathtt{HL}\left(\underline{(r^{(j)}_{\mathsf{b}}, 0)}, \underline{(r^{(j)}_{\mathsf{b}}, c^{(j)}_{\mathsf{b}} - 1)}\right)$, $\Gamma_j(T)$, and $\mathtt{HL}\left(\underline{(r^{(j)}_{\mathsf{t}} - 1, c^{(j)}_{\mathsf{t}})}, \underline{(r^{(j)}_{\mathsf{t}} - 1, \lambda_{r^{(j)}_{\mathsf{t}}-1})}\right)$.

Step 2. Let $r_{\mathsf{t}} = \min\{r \mid \underline{(r, c)} \in V(\Gamma'_i)\}$ and $c_{\mathsf{t}} = \min\{c \mid \underline{(r_{\mathsf{t}}, c)} \in V(\Gamma'_i)\}$.

Step 3. If there exists $j \in \mathcal{I}(T)$ such that

$$r' < r_{\mathsf{t}} < r'' \quad \text{and} \quad c', c'' > c_{\mathsf{t}} \quad \text{for some } \underline{(r', c')}, \underline{(r'', c'')} \in V(\Gamma'_j), \tag{4.1}$$

then go to *Step 4*. Otherwise, go to *Step 5*.

Step 4. Let $j_0 = \min\{j \mid \Gamma'_j \text{ satisfies } (4.1)\}$ and $c_0 = \min\left\{c \mid \underline{(r_{\mathsf{t}}, c)} \in V(\Gamma'_{j_0})\right\}$. Then, let Γ be the lattice path satisfying that

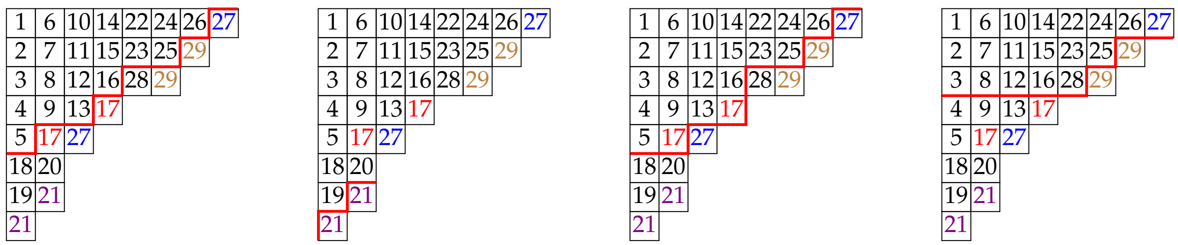
$$V(\Gamma) = V(\Gamma'_i) \setminus \{\underline{(r_{\mathsf{t}}, c)} \mid c \ge c_0\} \cup \{\underline{(r, c)} \in V(\Gamma'_{j_0}) \mid r \le r_{\mathsf{t}} \text{ and } c \ge c_0\}.$$

Set $\Gamma'_i := \Gamma$. Go to *Step 2*.

Step 5. Return $\widetilde{\Gamma}_i(T) := \Gamma'_i$ and terminate the algorithm.

If T is clear in the context, we simply write the lattice path $\widetilde{\Gamma}_i(T)$ by $\widetilde{\Gamma}_i$ for $i \in \mathcal{I}(T)$.

Example 4.3. Let us revisit Example 3.6. By applying Algorithm 4.2 to each $i \in \mathcal{I}(T)$, we obtain $\widetilde{\Gamma}_{17}$, $\widetilde{\Gamma}_{21}$, $\widetilde{\Gamma}_{27}$, and $\widetilde{\Gamma}_{29}$ as follows:



$\widetilde{\Gamma}_{17}(T)$ $\widetilde{\Gamma}_{21}(T)$ $\widetilde{\Gamma}_{27}(T)$ $\widetilde{\Gamma}_{29}(T)$

For $i \in \mathcal{I}(T)$, we set $\mathsf{p}'_i \in \{1, 2, \ldots, |\mathcal{I}(T)|\}$ satisfying the following: Let $i, j \in \mathcal{I}(T)$.

C1. If $r_{\mathsf{b}}^{(i)} < r_{\mathsf{b}}^{(j)}$, then $\mathsf{p}'_i < \mathsf{p}'_j$. And, if $r_{\mathsf{b}}^{(i)} > r_{\mathsf{b}}^{(j)}$, then $\mathsf{p}'_i > \mathsf{p}'_j$.

C2. When $r_{\mathsf{b}}^{(i)} = r_{\mathsf{b}}^{(j)}$, consider the lowest lattice point $p \in V(\widetilde{\Gamma}_i) \cap V(\widetilde{\Gamma}_j)$ such that neither $p + (\underline{-1}, 0)$ nor $p + (\underline{0}, 1)$ are contained in $V(\widetilde{\Gamma}_i) \cap V(\widetilde{\Gamma}_j)$. If $p + (\underline{-1}, 0) \in V(\widetilde{\Gamma}_i)$, then $\mathsf{p}'_i < \mathsf{p}'_j$. Otherwise, $\mathsf{p}'_i > \mathsf{p}'_j$.

Given $i, j \in \mathcal{I}(T)$, if there exist $(\underline{r'}, c'), (\underline{r''}, c'') \in V(\widetilde{\Gamma}_j)$ such that $r' < r_{\mathsf{b}}^{(i)} < r''$ and $c', c'' < c_{\mathsf{b}}^{(i)}$, then we say that $\widetilde{\Gamma}_j$ *crosses the bottom path of* $\widetilde{\Gamma}_i$. By rearranging p'_i's with the following algorithm, we define a bijection $\mathsf{p}_T : \mathcal{I}(T) \to \{1, 2, \ldots, |\mathcal{I}(T)|\}$.

Algorithm 4.4. For each $i \in \mathcal{I}(T)$, let $p_i := \mathsf{p}'_i$, where p'_i is the index defined above.

Step 1. Let $k = 1$.

Step 2. Take i_k and i_{k+1} in $\mathcal{I}(T)$ such that $p_{i_k} = k$ and $p_{i_{k+1}} = k + 1$.

Step 3. If $\widetilde{\Gamma}_{i_{k+1}}$ crosses the bottom path of $\widetilde{\Gamma}_{i_k}$, then set $p_{i_k} := k + 1$ and $p_{i_{k+1}} := k$ and go to *Step 1*. Otherwise, go to *Step 4*.

Step 4. If $k < |\mathcal{I}(T)| - 1$, then set $k = k + 1$ and go to *Step 2*. Otherwise, set $\mathsf{p}_T(i) := p_i$ for each $i \in \mathcal{I}(T)$ and go to *Step 5*.

Step 5. Return $(\mathsf{p}_T(i))_{i \in \mathcal{I}(T)}$ and terminate the algorithm.

Given $u \in [1, |\mathcal{I}(T)|\,]$, let A_u be the subdiagram of $\mathtt{yd}(\lambda)$ consisting of the boxes located above $\widetilde{\Gamma}^{(u)}$. Let

$$\mathsf{D}_u^{(1)}(T) := \mathsf{A}_u \setminus \Big(\bigcup_{1 \le v < u} \Big(\mathsf{A}_v \cup T^{-1}(\mathsf{p}_T^{-1}(v)) \Big) \Big) \quad \text{and} \quad \mathsf{D}_u^{(2)}(T) := T^{-1}(\mathsf{p}_T^{-1}(u)).$$

Now, we construct the desired tableau $\mathsf{source}(T)$ with the following algorithm.

Algorithm 4.5. Let $T \in \mathrm{IGLT}(\lambda)_m$. Set $e_0 = 0$ and $M_0 = 0$. For $1 \le u \le |\mathcal{I}(T)|$, let $e_u := |\mathsf{D}_u^{(1)}(T)| + 1$ and $M_u = \sum_{v=0}^{u} e_v$.

Step 1. Set $v := 1$.

Step 2. Fill the boxes in $\mathsf{D}_v^{(1)}(T)$ by $M_{v-1}+1, M_{v-1}+2, \ldots, M_{v-1}+e_v-1$ from left to right starting from the top.

Step 3. Fill the boxes in $\mathsf{D}_v^{(2)}(T)$ by M_v.

Step 4. If $v < |\mathcal{I}(T)|$, then set $v := v+1$ and go to *Step 2*. Otherwise, fill the remaining boxes by $M_{|\mathcal{I}(T)|}+1, M_{|\mathcal{I}(T)|}+2, \ldots, m$ from left to right starting from the top. Set $\mathsf{source}(T)$ to be the resulting filling. Return $\mathsf{source}(T)$ and terminate the algorithm.

Example 4.6. Let us revisit Example 4.3. One can easily see that $\mathsf{p}'_{17} = 2, \mathsf{p}'_{21} = 4, \mathsf{p}'_{27} = 3$, and $\mathsf{p}'_{29} = 1$. By applying Algorithm 4.4, we have

$$\mathsf{p}_T(17) = 1, \quad \mathsf{p}_T(21) = 4, \quad \mathsf{p}_T(27) = 2, \quad \text{and} \quad \mathsf{p}_T(29) = 3.$$

In addition, by applying Algorithm 4.5, we have

$T =$

1	6	10	14	22	24	26	27
2	7	11	15	23	25	29	
3	8	12	16	28	29		
4	9	13	17				
5	17	27					
18	20						
19	21						
21							

$\xrightarrow{\text{Algorithm 4.5}}$ $\mathsf{source}(T) =$

1	2	3	4	5	6	7	23
8	9	10	11	12	13	25	
14	15	16	17	24	25		
18	19	20	22				
21	22	23					
26	27						
28	29						
29							

.

Theorem 4.7. *For any $T \in E$, $\mathsf{source}(T)$ is the unique source tableau in E.*

Similarly, we construct $\mathsf{sink}(T)$ for each $T \in E$ and prove the following theorem.

Theorem 4.8. *For any $T \in E$, $\mathsf{sink}(T)$ is the unique sink tableau in E.*

We denote by T_E and T'_E the unique source tableau and sink tableau in E, respectively.

5 A weak Bruhat interval module description of $\mathbf{G}_E$

Throughout this section, we let $\mathrm{Des}(T_E) = \{d_1 < d_2 < \cdots < d_k\}$, $d_0 := 0$, and $d_{k+1} := m$. For each $1 \le j \le k+1$, let $\mathtt{H}_j := T_E^{-1}([d_{j-1}+1, d_j])$.

Example 5.1. When

$T_E =$

1	2	3	4	5	11	12	14	15
6	7	8	10	11	14			
9	10							
13	14							

,

we have that $\mathrm{Des}(T_E) = \{5, 8, 12\}$. In this case, $\mathtt{H}_j$ $(1 \le j \le 4)$ are given as follows:

$\mathtt{H}_1$	$\mathtt{H}_1$	$\mathtt{H}_1$	$\mathtt{H}_1$	$\mathtt{H}_1$	$\mathtt{H}_3$	$\mathtt{H}_3$	$\mathtt{H}_4$	$\mathtt{H}_4$
$\mathtt{H}_2$	$\mathtt{H}_2$	$\mathtt{H}_2$	$\mathtt{H}_3$	$\mathtt{H}_3$	$\mathtt{H}_4$			
$\mathtt{H}_3$	$\mathtt{H}_3$							
$\mathtt{H}_4$	$\mathtt{H}_4$							

For each $1 \leq j \leq k$, let $\mathrm{w}^{(j)}(T)$ be the word obtained by reading the entries of T contained in $\mathtt{H}_j$ from right to left. Note that if an integer i appears multiple times in $\mathrm{w}^{(j)}(T)$, then the integer i's are placed consecutively. We define $\overline{\mathrm{w}}^{(j)}(T)$ as the word obtained from $\mathrm{w}^{(j)}(T)$ by erasing all i's except one i for each i that appears in $\mathrm{w}^{(j)}(T)$.

Definition 5.2. For $T \in E$, *the standardized reading word* $\mathsf{read}(T)$ *of* T is defined to be the word $\overline{\mathrm{w}}^{(1)}(T)\overline{\mathrm{w}}^{(2)}(T)\cdots\overline{\mathrm{w}}^{(k+1)}(T)$ obtained by concatenating $\overline{\mathrm{w}}^{(j)}(T)$ for $1 \leq j \leq k+1$.

Example 5.3. We revisit Example 5.1. One can see that

$$\mathsf{read}(T_E) = 5\ 4\ 3\ 2\ 1\ 8\ 7\ 6\ 12\ 11\ 10\ 9\ 15\ 14\ 13 \in \mathfrak{S}_{15}.$$

Theorem 5.4. *For each $E \in \mathcal{E}_{\lambda;m}$, $\mathbf{G}_E \cong \mathsf{B}(\mathsf{read}(T_E), \mathsf{read}(T'_E))$ as $H_m(0)$-modules.*

6 Further avenue

In [10], Pechenik proved that for all $\lambda = (\lambda_1, \lambda_2) \vdash n$,

$$U_\lambda = \sum_{l_\lambda \leq m \leq n} \ \sum_{\mu \in \mathsf{Par}(\lambda;m)} s_\mu, \tag{6.1}$$

where $l_\lambda := \max\{\lambda_1, \lambda_2 + 1\}$, $\mathsf{Par}(\lambda; n) := \{(\lambda_1, \lambda_2)\}$, and

$$\mathsf{Par}(\lambda; m) := \begin{cases} \left\{(\lambda_1 - k_m, \lambda_1 - k_m, 1^{k_m})\right\} & \text{if } \lambda_1 = \lambda_2, \\ \left\{(\lambda_1 - k_m, \lambda_2 - k_m, 1^{k_m}), (\lambda_1 - k_m, \lambda_2 - k_m + 1, 1^{k_m - 1})\right\} & \text{if } \lambda_1 > \lambda_2 \end{cases}$$

for all $l \leq m < n$. Here, $k_m := n - m$ and $s_\mu := 0$ if μ is not a partition. And, for each $\lambda \vdash m$, Searles [12] introduced the $H_m(0)$-module X_λ such that $\mathrm{ch}([X_\lambda]) = s_\lambda$. The study of representation theoretic interpretation for (6.1) will be pursued in the near future by using $\mathbf{G}_{\lambda;m}$ and X_μ. In this direction, we leave the following conjecture.

Conjecture 6.1. *Let $\lambda = (\lambda_1, \lambda_2) \vdash n$. For each $l_\lambda \leq m \leq n$, there exists a partition $\{\mathcal{E}_\mu \mid \mu \in \mathsf{Par}(\lambda; m)\}$ of $\mathcal{E}_{\lambda;m}$ satisfying the following: For each $\mu \in \mathsf{Par}(\lambda; m)$,*

(1) $\sum_{E \in \mathcal{E}_\mu} \mathrm{ch}([\mathbf{G}_E]) = s_\mu$,

(2) *there exist a total order $\prec_\mu$ on $\mathcal{E}_\mu = \{E_1 \prec_\mu \cdots \prec_\mu E_{|\mathcal{E}_\mu|}\}$ and a filtration $M_0 = \{0\} \subseteq M_1 \subseteq \cdots \subseteq M_{|\mathcal{E}_\mu|} = X_\mu$ such that $\mathbf{G}_{E_i} \cong M_i / M_{i-1}$ for all $1 \leq i \leq |\mathcal{E}_\mu|$.*

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