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Minimal skew semistandard Young tableaux and the Hillman–Grassl correspondence

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Abstract. Standard tableaux of skew shape are fundamental objects in enumerative and algebraic combinatorics and no product formula for the number is known. In 2014, Naruse gave a formula (NHLF) as a positive sum over excited diagrams of products of hook-lengths. Subsequently, Morales, Pak, and Panova gave several proofs and generalized it to two q-analogues. They also showed, partly algebraically, that the Hillman–Grassl map restricted to skew shapes must be a bijection. We study the problem of circumventing the algebraic part and proving the bijection completely combinatorially. For a skew shape, we define a new set of semi-standard Young tableaux, called the *minimal SSYT*, that are equinumerous with excited diagrams via a new description of the Hillman–Grassl bijection and a version of excited moves. Lastly, we relate the minimal skew SSYT with the terms of the Okounkov-Olshanski formula (OOF) for counting SYTs of skew shape. Our construction immediately implies that the summands in the NHLF are less than the summands in the OOF.

Keywords: Standard Young tableaux, Hillman–Grassl correspondence

1 Introduction

Given a partition $\lambda \vdash n$, the number f^λ of standard Young tableaux (SYT) of shape λ is an important quantity in enumerative, algebraic, and probabilistic combinatorics and is given by the following *hook-length formula* of Frame-Robinson-Thrall [2] 1954.

Theorem 1.1 (Frame-Robinson-Thrall [2]). *For λ a partition of n we have*

$$f^\lambda = \frac{n!}{\prod_{(i,j)\in[\lambda]} h(i,j)}, \tag{1.1}$$

where $[\lambda]$ is the Young diagram of λ and $h(i,j) = \lambda_i - i + \lambda'_j - j + 1$.

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Unlike for straight shapes, there is no such product formula for the number $f^{\lambda/\mu}$ of SYT of skew shape λ/μ. There is a determinantal formula for $f^{\lambda/\mu}$ like the *Jacobi–Trudi identity*, which is efficient but inherently non-positive. There is a classical positive formula for $f^{\lambda/\mu}$ involving the *Littlewood–Richardson coefficients* $c^{\lambda}_{\mu,\nu}$, which are themselves hard to compute.

Central to this paper are two other positive formulas for $f^{\lambda/\mu}$ coming from equivariant Schubert calculus or more explicitly from evaluations of *factorial Schur functions*: the Okounkov–Olshanski formula [14] from 1998 and the Naruse hook-length formula [13] from 2014. We start with the latter since it resembles (1.1).

Theorem 1.2 (Naruse [13], [10]). *For a skew shape λ/μ of size n we have*

$$f^{\lambda/\mu} = n! \sum_{S \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus S} \frac{1}{h(u)}, \tag{NHLF}$$

where $\mathcal{E}(\lambda/\mu)$ is the set of excited diagrams *of λ/μ.*

The excited diagrams of shape λ/μ, denoted by $\mathcal{E}(\lambda/\mu)$, are certain subsets of size $|\mu|$ of the Young diagram of λ obtained from the Young diagram of $[\mu]$ by recursively doing a local move β [5]. Excited diagrams are in correspondence with certain SSYT of shape μ that are *flagged*, i.e. with certain bounds on the entries in each row. The Naruse formula has been actively studied [1, 6, 9, 10].

The other positive formula by Okounkov-Olshanski is also a sum over certain SSYT of shape μ with entries at most $\ell(\lambda)$ called *Okounkov–Olshanski tableaux* $\mathcal{OOT}(\lambda/\mu)$.

Theorem 1.3 (Okounkov–Olshanskii [14]). *For a skew shape λ/μ of size n we have*

$$f^{\lambda/\mu} = \frac{n!}{\prod_{u\in[\lambda]} h(u)} \sum_{T \in \mathcal{OOT}(\lambda/\mu)} \prod_{(i,j)\in[\mu]} (\lambda_{\ell+1-T(i,j)} - i + j). \tag{OOF}$$

In [12], Morales–Zhu did a similar study of (OOF) as Morales–Pak–Panova did in [9, 10] for (NHLF). In particular, they gave in [12, Cor. 5.7] a reformulation of (OOF) in terms of the following flagged tableaux: SSYT of shape λ/μ with entries in row i at most $i-1$, whose set we denote by $\mathcal{SF}(\lambda/\mu)$.

A new formulation of Naruse's formula: A natural question, which is the start of our investigation, is to also find a reformulation of (NHLF) in terms of some SSYT of skew shape λ/μ. We define the set $\mathrm{SSYT}_{\min}(\lambda/\mu)$ of *minimal skew SSYT* obtained from the *minimum SSYT*, T_0 of shape λ/μ with entries $\{0, 1, \ldots, \lambda'_i - \mu'_i - 1\}$ by recursively doing local moves δ. Our first result is a bijection Φ between excited diagrams and tableaux $\mathrm{SSYT}_{\min}(\lambda/\mu)$ that intertwines with the respective local moves β and δ, respectively.

Theorem 1.4. *For a skew shape λ/μ the map $\Phi : \mathcal{E}(\lambda/\mu) \to \mathrm{SSYT}_{\min}(\lambda/\mu)$ is a bijection that intertwines with the respective local moves, that is $\Phi \circ \beta = \delta \circ \Phi$.*

As a corollary, we obtain a new reformulation of (NHLF). Let $(\theta_1, \ldots, \theta_k)$ be the *Lascoux–Pragacz* decomposition of the shape λ/μ into border strips.

Theorem 1.5 (minimal SSYT version of (NHLF))**.** *For a skew shape λ/μ of size n we have*

$$f^{\lambda/\mu} = \frac{n!}{\prod_{u\in[\lambda]} h(u)} \sum_{T\in\mathrm{SSYT}_{\min}(\lambda/\mu)} \prod_{(i,j)\in[\mu]} h(i+\alpha, j+\alpha), \tag{1.2}$$

where α is the number strips θ_k such that $\overline{T}(\theta_k(j)) > \mu'_j - i$ and $\overline{T} = T - T_0$.

We give an explicit non-recursive description of the tableaux in $\mathrm{SSYT}_{\min}(\lambda/\mu)$ (Theorem 2.6) that immediately shows they are a subset of the skew flagged tableaux $\mathcal{SF}(\lambda/\mu)$ (see Lemma 3.4). As a corollary, we obtain that the Okounkov–Olshanski formula has at least as many terms as the Naruse formula and characterize the skew shapes where equality is attained. We denote the number of terms of each formula by $\mathrm{ED}(\lambda)$ and $\mathrm{OOT}(\lambda/\mu)$, respectively.

Theorem 1.6. *For a connected skew shape λ/μ with $d = \ell(\lambda)$ and $r = \max\{i \mid \mu_1 = \mu_i\}$, we have that $\mathrm{ED}(\lambda/\mu) \leq \mathrm{OOT}(\lambda/\mu)$ with equality if and only if $\lambda_d \geq \mu_r + d - r$.*

Relation with the Hillman–Grassl correspondence: The hook-length formula (1.1) has a q-analogue by Littlewood that is a special case of Stanley's hook-content formula for the generating series of SSYT of shape λ.

Theorem 1.7 (Littlewood)**.** *For a partition λ we have*

$$s_\lambda(1, q, q^2, \ldots) = q^{\sum_i (i-1)\lambda_i} \cdot \prod_{u\in\lambda} \frac{1}{1 - q^{h(u)}}. \tag{1.3}$$

This q-analogue has a bijective proof by Hillman–Grassl via the correspondence $HG(\cdot)$ between reverse plane partitions of shape λ and arrays of nonnegative integers of shape λ. This correspondence is related to the famous RSK correspondence (see [3, 10]) and has recent connections to *quiver representations* [4].

In [10], Morales–Pak–Panova gave a q-analogue of (NHLF) for the generating functions of SSYT of skew shape λ/μ.

Theorem 1.8 (Morales–Pak–Panova [10])**.** *For a skew shape λ/μ we have*

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}. \tag{1.4}$$

Interestingly, this identity corresponds to restricting the Hillman–Grassl bijection to SSYT of shape λ/μ [10]. The resulting arrays $\mathcal{A}^*(\lambda/\mu)$ have support on the complement of excited diagrams and with certain forced nonzero entries on *broken diagonals*. The proof of this connection with the Hillman–Grassl map $HG(\cdot)$ is partly algebraic and remained mysterious. Our next results elucidate on this.

First, we show that the bijection between minimal tableaux and excited diagrams coincides with the Hillman–Grassl map.

Theorem 1.9. *The map Φ is equivalent to the inverse Hillman–Grassl map HG^{-1} on the arrays $\mathcal{A}^*(\lambda/\mu)$. That is for A_D in $\mathcal{A}^*(\lambda/\mu)$ we have $HG^{-1}(A_D) = \Phi(D)$.*

As a corollary, we obtain a bijective proof of part of (1.4) by taking the *leading terms* of each summand on the right-hand-side.

Corollary 1.10. *For a shape λ/μ we have $\sum_{T\in \mathrm{SSYT}_{\min}(\lambda/\mu)} q^{|T|} = \sum_{D\in\mathcal{E}(\lambda/\mu)} q^{\sum_{(i,j)\in D}(\lambda'_j - i)}$.*

We also give a fully bijective proof of (1.4) for the case of *border strips*. The details of this last result and full version of this abstract will appear in [11].

2 Minimal semistandard tableaux of shape λ/μ

2.1 Excited diagram and broken diagonals

Given a subset D of $[\lambda]$, a cell (i,j) of D is *active* if the cells $(i+1,j),(i,j+1),(i+1,j+1)$ are not in D but are in $[\lambda]$. Given an active cell (i,j) of D, let $\beta_{(i,j)} : D \to D'$ be the map that replaces cell (i,j) in D by $(i+1,j+1)$. We call such $\beta_{(i,j)}$ an *excited move*. An *excited diagram* of λ/μ is a set of $|\mu|$ cells obtained from $[\mu] \subseteq [\lambda]$ by applying excited moves. We let $\mathcal{E}(\lambda/\mu)$ be the set of excited diagrams of λ/μ (see Figure 2a). These diagrams were introduced by Ikeda–Naruse in [5].

Each excited diagram D is associated with the *broken diagonals* $\mathrm{Br}(D)$, which are obtained from the sub-diagonals of λ/μ by applying excited moves. For $\beta_{(i,j)} : D \to D'$, then $\mathrm{Br}(D') = \mathrm{Br}(D) \setminus \{(i+1,j+1)\} \cup \{(i+1,j)\}$ (see Figure 2a).

2.2 Lascoux–Pragacz and Kreiman decomposition of λ/μ

We review two different decompositions of a Young diagram of shape λ/μ into *strips* (connected skew shapes with no 2×2 box), see Figure 1a .

The *Lascoux-Pragacz decomposition* of λ/μ is a tuple of non-intersecting lattice paths $(\theta_1,\ldots,\theta_k)$, where θ_1 is the outer border strip of λ, which starts at the southwest corner of $[\lambda/\mu]$ and ends at the northeast corner of $[\lambda/\mu]$ [8]. Then θ_2 is the outer border strip of $\lambda\setminus\theta_1$ and each θ_i is the outer border strip of $\lambda\setminus\theta_1\setminus\cdots\setminus\theta_{i-1}$. We assume each strip

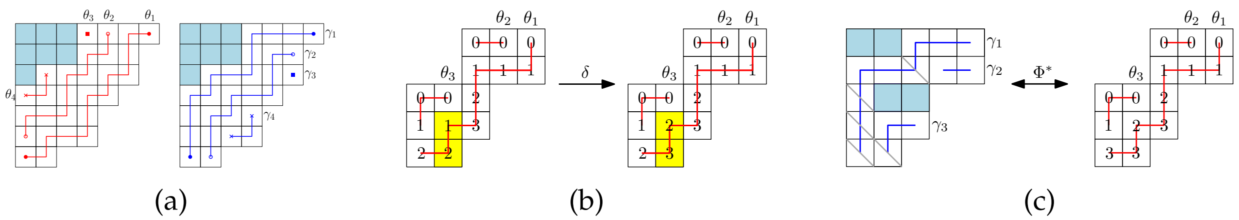


Figure 1: (a) Examples of the Lascoux–Pragacz decomposition and Kreiman decomposition of the skew shape λ/μ. (b) Excited move δ on $\mathrm{SSYT}_{\min}$ (c) Excited diagram to $\mathrm{SSYT}_{\min}$

θ_i starts at the left or bottom of $[\lambda/\mu]$ and ends at the right or top of $[\lambda/\mu]$. Given a skew shape λ/μ, and its Lascoux-Pragacz decompositions $(\theta_1, \ldots, \theta_k)$, denote $\mathsf{f}(\theta_r)$ the row number of the top most element of θ_r. We define the θ_r-height of row i, denoted as $ht_{\theta_r}(i)$ to be $i - \mathsf{f}(\theta_r)$. For example, in Figure 1a, the height of the element $(7,2)$ in θ_1 is 6 whereas the height of $(4,1)$ in θ_4 is 1.

The *Kreiman decomposition* of λ/μ is a tuple $(\gamma_1, \ldots, \gamma_k)$ of non-intersecting lattice paths with support $[\lambda/\mu]$, where each γ_i begins at the southwest corner and ends at northeast corner [7] (see Figure 1a). We denote ϵ_i the distance between γ_1 and γ_i. The support of such paths are the complement of excited diagrams [7, §5, §6]. If an excited diagram D corresponds to the tuple $(\gamma'_1, \ldots, \gamma'_k)$, we denote by $\gamma_i(D) = \gamma'_i$, the path obtained from γ_i after applying the corresponding *ladder moves* to the excited moves where $[\mu]$ is obtained from D (see Figure 2a).

Lemma 2.1. *Let $(\gamma_1, \ldots, \gamma_k)$ and $(\theta_1, \ldots, \theta_k)$ be the Kreiman and Lascoux-Pragacz decomposition of shape λ/μ. For each $i \in \{1, \ldots, k\}$, let $\gamma_i : (f_i, g_i) \to (c_i, d_i)$ and $\theta_i : (s_i, t_i) \to (u_i, v_i)$. Then $g_i - t_i = \epsilon_i$, i.e. the column difference of the starting point of γ_i and θ_i is ϵ_i.*

2.3 Minimal skew Semistandard Young Tableaux

The minimal skew SSYT are obtained from the minimum SSYT T_0 by applying a sequence of *excited moves* δ, defined as follows (see Figure 1b). Let λ/μ be a skew partition and $(\theta_1, \ldots, \theta_k)$ be its Lascoux-Pragacz decomposition. Given a strip θ_k and column j, let $\theta_k(j)$ be the jth column segment of θ_k. Let (i, j) be the top-most cell and (i', j) be the bottom-most cell of $\theta_k(j)$ (which may agree if the column has size one).

Definition 2.2 (excited move δ)**.** *The column $\theta_k(j)$ of T in* $\mathrm{SSYT}(\lambda/\mu)$ *is* active *if*

(i) $T(i,j) < ht_{\theta_k}(i)$,

(ii) $T(i,j) < T(i,j+1)$ *and* $T(i',j) < T(i'+1,j) + 1$.

Given an active column $\theta_k(j)$ of T, the excited move δ *adds one to each entry in the column segment $\theta_k(j)$ and $T' := \delta_{(k;j)}(T)$ be the tableau obtained from T by the move δ.*

We define a bijection between $\mathcal{E}(\lambda/\mu)$ and $\mathrm{SSYT}_{\min}(\lambda/\mu)$ denoted $\Phi : \mathcal{E}(\lambda/\mu) \to \mathrm{SSYT}(\lambda/\mu)$, which commutes with the excited moves in $\mathcal{E}(\lambda/\mu)$ and $\mathrm{SSYT}_{\min}(\lambda/\mu)$.

Definition 2.3. *Given $D \in \mathcal{E}(\lambda/\mu)$, let $(\gamma_1(D), \dots, \gamma_k(D))$ and $(\theta_1, \dots, \theta_k)$ be its Kreiman and Lascoux-Pracgaz decomposition into paths and* $\mathrm{Br}(D)$ *be its set of broken diagonals. Let b_j be the number of broken diagonals on the jth column of $\gamma_i(D)$. For each $\gamma_i(D)$ and θ_i, do the following procedure starting from the left most column of each path to obtain $T := \Phi(D)$:*

1. *Denote the bottom-most element of the jth column of θ_i as (s_j, t_j). Let*

$$T(s_j, t_j) = \begin{cases} b_j & \text{if } j = 1, \\ T(s_j, t_j - 1) + b_j & \text{if } j > 1. \end{cases}$$

2. *Fill the rest of the jth column of θ_i so that each entry differs by* 1.

The proof that Φ is well defined can be found in the full version [11].

Example 2.4. *Let $D \in \mathcal{E}(\lambda/\mu)$ be the excited diagram shown in Figure 1c. We apply the correspondence to obtain $T = \Phi(D)$. The first column of $\gamma_1(D)$ has three broken diagonals, so $T(5,1) = 3$. the second column of $\gamma_1(D)$ has no broken diagonals , so $T(5,2) = 3$, then the rest of the column of θ_1 is filled with each entry differing by* 1 *to maintain column strictness. We continue the algorithm to obtain the final tableau T of skew shape λ/μ.*

For the Lascoux–Pragacz decomposition of $[\lambda/\mu]$ and the Kreiman paths $(\gamma_1, \dots, \gamma_k)$ corresponding to an excited diagram D, we denote by $\mathrm{col}_{\lambda/\mu}(\theta_i, n)$ and $\mathrm{col}_{\lambda/\mu}(\gamma_i, n)$ the column on λ/μ where the nth column of θ_i and γ_i is, respectively.

Lemma 2.5. *Given an excited diagram D in $\mathcal{E}(\lambda/\mu)$ and an active cell $u = (a, b)$ of D we have*

$$\Phi(\beta_u(D)) = \delta_{f(u)}(\Phi(D)),$$

where $f(a, b) = (i; \mu(a, b))$, i is the index of the Kreiman path $\gamma_i(D)$ modified by $\beta(a, b)$, and $\mu(a, b)$ is the column of the original cell in μ where (a, b) came from.

Proof. The excited move $\beta_{(a,b)}$ corresponds to a ladder move on $\gamma_i(D) \to \gamma_i(D')$, for some i and a broken diagonal shift from $(a + 1, b + 1)$ to $(a + 1, b)$. Let g_i and t_i be the starting column of $\gamma_i(D)$ and θ_i, respectively. Suppose $(a + 1, b)$ is in the jth column of the path $\gamma_i(D)$. Then the excited move $\beta_{(a,b)}$ shifts a broken diagonal to the j-th column of $\gamma_i(D')$, where $j = b - g_i + 1$. The j-th column of $\gamma_i(D)$ corresponds to the j-th column of θ_i, which corresponds to column $t_i + j - 1$ of shape λ/μ. By Lemma 2.1 we know that $g_i - t_i = \epsilon_i$. Thus we have that

$$\mathrm{col}_{\lambda/\mu}(\theta_i, j) = t_i + j - 1 = (g_i - \epsilon_i) + (b - g_i + 1) - 1 = b - \epsilon_i.$$

We know that ϵ_i is also the number of times (a, b) has been excited, which is $b - \mu(a, b)$. Therefore we have that $\mathrm{col}_{\lambda/\mu}(\theta_i, j) = \mu(a, b)$. This shows that $\Phi(\beta_u(D))$ has increments of 1 on θ_i at column $\mu(a, b)$ compared to $\Phi(D))$. This is equivalent to what the move $\delta_{(i;\mu(a,b))}$ does on $\Phi(D)$. Thus $\Phi(\beta_u(D)) = \delta_{(i;\mu(a,b))}(\Phi(D))$, as desired. □

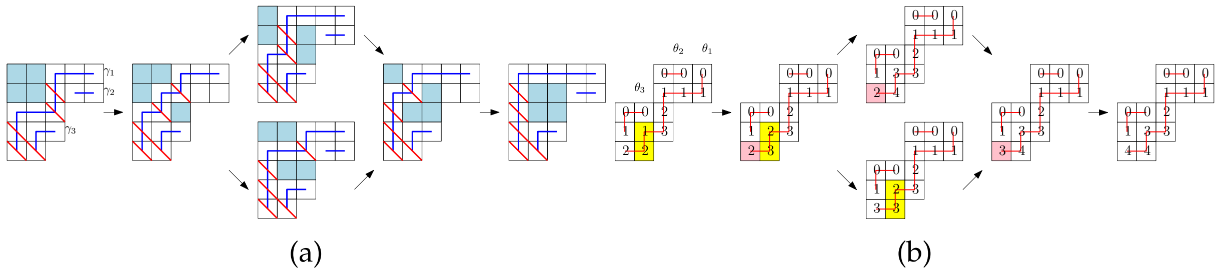


Figure 2: (a) The $\mathcal{E}(55332/22)$, their corresponding $\gamma_i(D)$ (in blue) and broken diagonals (in red). (b) The $\mathrm{SSYT}_{\min}(55332/22)$ with the active columns colored in yellow and red.

Proof sketch of Theorem 1.4. By Lemma 2.5, Φ intertwines the excited moves β and δ. One can check that Φ is injective. Note that $\Phi([\mu]) = T_0$. Given any excited diagram D in $\mathcal{E}(\lambda/\mu)$, there exists a sequence $\beta_1, \ldots, \beta_m$ of excited moves such that $D = \beta_m \circ \cdots \circ \beta_1([\mu])$. Iterating Lemma 2.5 gives that $\Phi(D) = \delta_m \circ \cdots \circ \delta_1(T_0)$.

Thus $\Phi(D)$ is in $\mathrm{SSYT}_{\min}(\lambda/\mu)$ and so $\Phi(\mathcal{E}(\lambda/\mu)) \subseteq \mathrm{SSYT}_{\min}(\lambda/\mu)$. It remains to show that $\Phi(D)$ is surjective. Given T in $\mathrm{SSYT}_{\min}(\lambda/\mu)$, there exists a sequence of $\delta_1, \ldots, \delta_k$ of excited moves such that $\delta_k \circ \cdots \circ \delta_1(T_0) = T$. Again, iterating Lemma 2.5 we have that for $D' = \beta_k \circ \cdots \circ \beta_1([\mu])$, one obtains $\Phi(D') = T$, as desired. $\square$

There is the following non-recursive description of the minimal SSYT. This new characterization will be related to the Okounkov–Olshanski tableaux in the next section. We omit the details of the proof.

Theorem 2.6. *A SSYT T of shape λ/μ is in $\mathrm{SSYT}_{\min}(\lambda/\mu)$ if and only if*

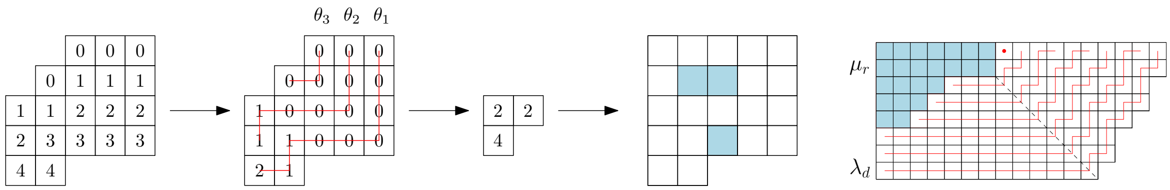
(i) For each (i_r, j_r) in θ_r, $T(i_r, j_r) \leq ht_{\theta_r}(i_r)$

(ii) For any (i_r, j_r) and $(i_r + 1, j_r)$ in θ_r, $T(i_r + 1, j_r) - T(i_r, j_r) = 1$.

In other words, the set $\mathrm{SSYT}_{\min}(\lambda/\mu)$ consists of SSYT of shape λ/μ where the values along each path θ_r are bounded by the height ht_{θ_r} and the values along entries in the same column of a strip θ_r differ by one.

3 Reformulation of (NHLF) and comparison with (OOF)

In this section we use the bijection Φ between minimal SSYT and excited diagrams of shape λ/μ to reformulate the Naruse hook formula (NHLF) in terms of minimal SSYT. We need the following description of the inverse of Φ.



(a) (b)

Figure 3: (a) Example of bijection Φ^{-1} between a minimal SSYT and excited diagrams for the shape $\lambda/\mu = 55552/21$. (b) Illustration of skew shapes λ/μ and their Lascoux–Pragacz decomposition with $\lambda_d \geq \mu_r + d - r$ where $\mathrm{ED}(\lambda/\mu) = \mathrm{OOT}(\lambda/\mu)$.

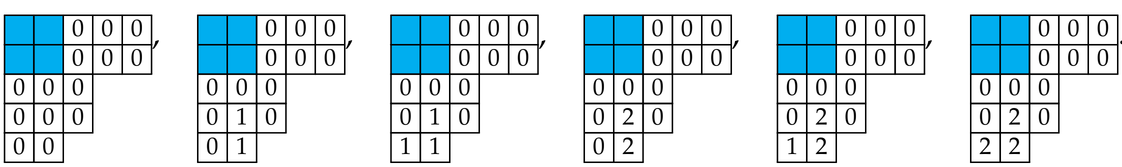
Lemma 3.1 (Inverse of Φ). *Let $D \in \mathcal{E}(\lambda/\mu)$ and recall $\overline{T} = T - T_0$, where T_0 is the minimum skew SSYT and $T = \Phi(D)$. Then*

$$D = \{(i+\alpha, j+\alpha) \mid (i,j) \in [\mu]\},$$

where α is the number of strips θ_k such that $\overline{T}(\theta_k(j)) > \mu'_j - i$.

Proof of Theorem 1.5. The result follows by combining the excited diagram formulation (NHLF) of Naruse's formula and the description of inverse of Φ from Lemma 3.1. □

Example 3.2. *For the shape $\lambda/\mu = 55332/22$, from each of its minimal tableaux in Figure 2b we find $\overline{T}$:*



Next, for each $\overline{T}$ and (i,j) in $\mu = [22]$, we find α. Putting everything together using (1.2) gives

$$f^{\lambda/\mu} = \frac{14!}{9\,8^2\,7\,6\,5^2\,4^2\,3^2\,2^4}\big(h(1,1)h(1,2)h(2,1)h(2,2) + h(1,1)h(1,2)h(2,1)h(2{+}1,2{+}1)$$
$$+ h(1,1)h(1{+}1,2{+}1)h(2,1)h(2{+}1,2{+}1) + h(1,1)h(1,2)h(2{+}1,1{+}1)h(2{+}1,2{+}1) +$$
$$h(1,1)h(1{+}1,2{+}1)h(2{+}1,1{+}1)h(2{+}1,2{+}1) + h(1{+}1,1{+}1)h(1{+}1,2{+}1)h(2{+}1,2{+}1)\big).$$

In the rest of this section we use the explicit non-recursive description of the skew minimal tableaux in $\mathrm{SSYT}_{\min}(\lambda/\mu)$ of Theorem 2.6 to relate the number $\mathrm{ED}(\lambda/\mu)$ of terms of (NHLF) and the number $\mathrm{OOT}(\lambda/\mu)$ of terms of (OOF).

Theorem 3.3. *For a skew shape λ/μ we have that $\mathrm{ED}(\lambda/\mu) \leq \mathrm{OOT}(\lambda/\mu)$.*

The proof follows immediately from the following lemma.

Lemma 3.4. *For a skew shape λ/μ we have that* $\mathrm{SSYT}_{\min}(\lambda/\mu) \subseteq \mathcal{SF}(\lambda/\mu)$.

Proof. Let T be a tableau in $\mathrm{SSYT}_{\min}(\lambda/\mu)$. By Condition (i) in Theorem 2.6, we have that for (i,j) in θ_r, $T(i,j) \leq ht_{\theta_r}(i)$. By definition of the height, we have that $ht_{\theta_r}(i) \leq i-1$. Thus T is a flagged tableau in $\mathcal{SF}(\lambda/\mu)$. □

Theorem 1.6 characterizes the skew shapes where the Naruse hook-length formula and the Okounkov–Olshanski formula have the same number of terms.

Proof sketch of Theorem 1.6. The inequality $\mathrm{ED}(\lambda/\mu) \leq \mathrm{OOT}(\lambda/\mu)$ follows from Theorem 3.3. By Lemma 3.4 we have that equality occurs if and only if all the flagged tableaux in $\mathcal{SF}(\lambda/\mu)$ are minimal. First, define the maximal tableau $T \in \mathcal{SF}(\lambda/\mu)$ as $T(i,j) = i-1$ for $(i,j) \in [\lambda/\mu]$, and let $(\theta_1,\ldots,\theta_k)$ be the Lascoux-Pragacz decomposition of λ/μ.

By the height condition in the definition of minimal tableaux (Condition (i) in Theorem 2.6), it follows that all border strips θ_k reach the top row. the border strips θ_k are entirely horizontal in the first μ_1 columns of λ/μ and we have $d-r$ of them starting in the first μ_1 columns. Thus $\lambda_d \geq \mu_1$. Also, since these border strips reach the top row, they must all cross the diagonal starting at (r,μ_r) and thus this diagonal extends to the last row of the shape, so $\lambda_d \geq \mu_r + d - r$, which proves the forward direction. See Figure 3b.

For the other direction, if $\lambda_d \geq \mu_r + d - r$, then it is easy to see that all border strips θ_i reach the top row and they are horizontal in the first μ_1 columns. The tableaux in $\mathcal{SF}(\lambda/\mu)$ consisting of the columns past μ_1 are of straight shape, and one can show the entries in row i are $i-1$. For the first μ_1 columns, since all border strips are horizontal, there are no further restrictions coming from the minimal tableaux. Thus $\mathrm{SSYT}_{\min}(\lambda/\mu) = \mathcal{SF}(\lambda/\mu)$, as desired. □

For shapes where $\mathrm{ED}(\lambda/\mu) = \mathrm{OOT}(\lambda/\mu)$, this number is given by the hook-content formula. We omit the proof.

Corollary 3.5. *For a connected skew shape λ/μ satisfying $\lambda_d \geq \mu_r + d - r$ where $d = \ell(\lambda)$ and $r = \max\{i \mid \mu_1 = \mu_i\}$, we have that* $\mathrm{ED}(\lambda/\mu) = \mathrm{OOT}(\lambda/\mu) = s_\mu(1^{d-r})$.

4 The Hillman–Grassl correspondence and the map Φ

In this section, we sketch the proof of Theorem 1.9.

4.1 Background on Hillman–Grassl

We denote by $HG(\cdot)$ the *Hillman–Grassl bijection* between reverse plane partitions in $\mathrm{RPP}(\lambda)$ ranked by size and integer arrays of shape λ ranked by hook weight. That is, if $HG : \pi \mapsto A$ then $|\pi| = \sum_{u\in\lambda} h(u)\cdot A_u$. We follow the definition of $HG(\cdot)$ in [15, §7.22]. This bijection also implies (1.3) as a corollary.

Definition 4.1 (Hillman–Grassl correspondence)**.** *The correspondence is obtained via a sequence of pairs $(\pi_0, A_0) := (\pi, \mathbf{0}), (\pi_1, A_1)\ldots, (\pi_k, A_k) := (\mathbf{0}, HG(\pi))$ where each π_i is a reverse plane partition and A_i is an array of nonnegative integers of shape λ obtained recursively as follows:*

(i) start at the SW most nonzero cell of π_i,

(ii) traverse a NE path: from cell (a,b), if $(\pi_i)_{a,b} = (\pi_i)_{a-1,b}$ then the path moves North, otherwise the path moves East.

(iii) Terminate when this is no longer possible.

Let $u_i = (c,d)$ where d is the column where the path starts and c is the row where it ends. Note that $|\pi_i| - |\pi_{i-1}| = h(u_i)$. We obtain the array A_{i+1} from A_i by adding one to cell u_i. We continue until π_k as only zero entries. We let $A_k = HG(\pi)$.

We extend HG to $\mathrm{SSYT}(\lambda/\mu)$ by viewing each skew SSYT T as a plane partition of shape λ with zero entries in $[\mu]$.

For any excited diagram $D \in \mathcal{E}(\lambda/\mu)$, denote by A_D the 0-1 array of shape λ with support on the broken diagonals $\mathrm{Br}(D)$ of D and let $\mathcal{A}^*(\lambda/\mu) = \{A_D \mid D \in \mathcal{E}(\lambda/\mu)\}$. Let $\mathcal{A}^*_D$ be the set of arrays A of nonnegative integers of shape λ with support contained in $[\lambda]\setminus D$ and positive entries $A_{i,j} > 0$ if $(A_D)_{i,j} = 1$.

Theorem 4.2 ([10, Thm. 7.7])**.** *The (restricted) Hillman–Grassl map HG is a bijection*

$$HG : \mathrm{SSYT}(\lambda/\mu) \to \bigcup_{D\in\mathcal{E}(\lambda/\mu)} \mathcal{A}^*_D.$$

This property combined with the correspondence between RPPs and SSYTs of straight shape yields (1.4) as a corollary (see [10, Sec. 7.1]). The proof of this result is partly algebraic and the authors of [10] asked for a fully bijective proof. We make progress on this question by (i) giving a combinatorial proof for border strips λ/μ, and (ii) characterizing what is the image of $\mathrm{SSYT}_{\min}(\lambda/\mu)$. We sketch the latter in the next section and the former is in [11].

4.2 Hillman–Grassl on minimal skew SSYT

In this section we give the necessary lemmas to prove Theorem 1.9. Fix a skew shape λ/μ with Lascoux–Pragacz decomposition $(\theta_1, \ldots, \theta_k)$.

Let $A_{\gamma_i(D)}$ be the a subarray of A_D of shape $\lambda \setminus \cup_{j=1}^{i-1}\gamma_j(D)$ with support on the broken diagonal of $\gamma_i(D)$. We define the sum $A_{\gamma_i(D)} + A_{\gamma_{i+1}(D)}$ as an array of shape $A_{\gamma_i(D)}$ with supports on the broken diagonals of $\gamma_i(D)$ and $\gamma_{i+1}(D)$. In other words, it is the entry-wise sum of the arrays where alignment of $A_{\gamma_i(D)}$ and $A_{\gamma_{i+1}(D)}$ corresponds to the original A_D (see Figure 4a).

Similarly, let T_{θ_i} be a SSYT of shape $\lambda \setminus \cup_{j=1}^{i-1}\theta_j(D)$ with tableau entries in θ_i. We define the $T_{\theta_i} \cup T_{\theta_{i+1}}$ to be the tableaux of shape T_{θ_i} with entries of both θ_i and θ_{i+1}.

The first lemma is a variation of Theorem 1.9 restricted to the individual γ_i and θ_i (see Figure 4a).

Lemma 4.3. *Let $D \in \mathcal{E}(\lambda/\mu)$ and $T = \Phi(D) \in \mathrm{SSYT}_{\min}(\lambda/\mu)$. Then for $i = 1, \ldots, k$ we have that $HG(T_{\theta_i}) = A_{\gamma_i(D)}$.*

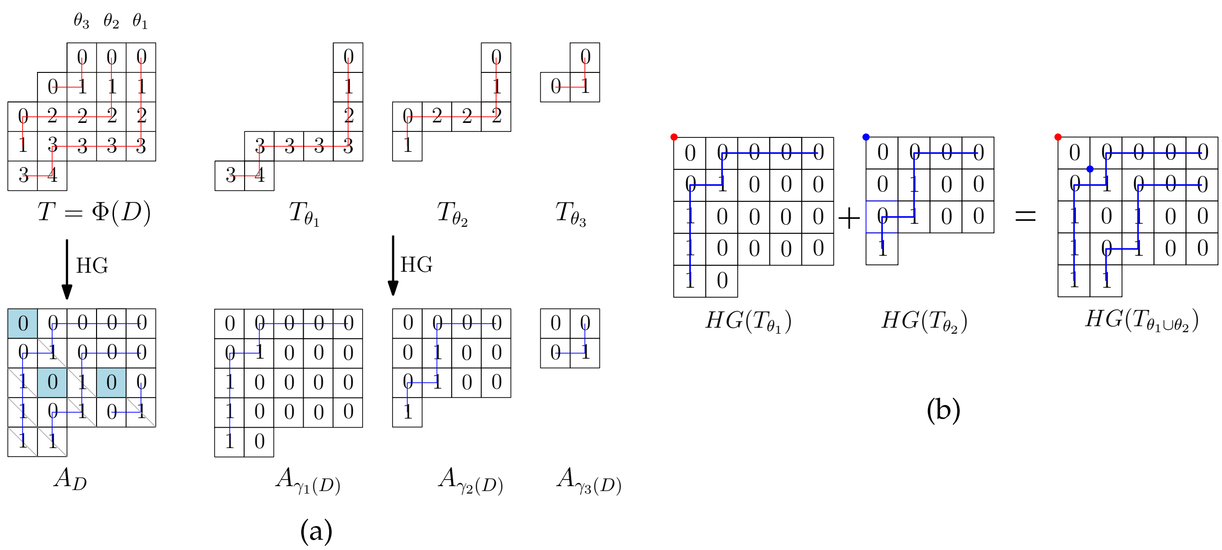


Figure 4: (a) The Hillman–Grassl map of T_{θ_1}, T_{θ_2}, and T_{θ_3} separately (b) Additivity of the Hillman–Grassl correspondence on T_{θ_i}.

In order to show Theorem 1.9, we need the following additivity results (see Section 4.2). We define the sum of arrays $HG(T_{\theta_i})$ and $HG(T_{\theta_{i+1}})$ similarly to the sum of $A_{\gamma_i(D)}$ and $A_{\gamma_{i+1}(D)}$. We omit the technical proof by induction.

Lemma 4.4. *We have that $\sum_{i=1}^{k} HG(T_{\theta_i}) = HG(T_{\theta_1 \cup \cdots \cup \theta_k})$.*

Proof sketch of Theorem 1.9. We have that $T_{\theta_1 \cup \cdots \cup \theta_k} = \Phi(D)$. We obtain the desired identity $HG(\Phi(D)) = \sum_{i=1}^{k} A_{\gamma_i(D)} = A_D$ by Lemma 4.3 and Lemma 4.4. □

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