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Divisors, orbit harmonics, and DT invariants

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Abstract. We give a combinatorial interpretation to Donaldson–Thomas invariants of symmetric quivers by using the method of orbit harmonics. Along the way we connect Young subgroup actions on break divisors of graphs, quotients of the polynomial ring modulo power ideals of Postnikov–Ardila, and a construction of Efimov in the context of Cohomological Hall algebras.

Keywords: break divisors, Donaldson–Thomas invariants, orbit harmonics, power ideals

1 Introduction

Donaldson–Thomas (DT) invariants of quivers with potential were introduced in the seminal work of Kontsevich–Soibelman [10] as a mathematical definition of the string-theoretic concept of BPS state count. We give a high-level introduction to DT invariants with the goal of emphasizing that the algebraic formalism hides an interesting combinatorial question, which is what we address.

In [10], (quantum) DT invariants were introduced as follows: Attached to a quiver Q on k vertices is a *cohomological Hall algebra* $\mathcal{H}$ which carries a $\mathbb{Z}^k_{\geq 0}$-grading. If Q is *symmetric* then $\mathcal{H}$ is $\mathbb{Z}^k_{\geq 0} \times \mathbb{Z}$-graded. Kontsevich and Soibelman conjectured [10, Conjecture 1] that in this case $\mathcal{H}$ is a free (super-)commutative algebra generated by a $\mathbb{Z}^k_{\geq 0} \times \mathbb{Z}$-graded vector space V of the form $V^{\mathrm{prim}} \otimes \mathbb{Q}[x]$. Furthermore that, given a fixed *dimension vector* $\gamma \in \mathbb{Z}^k_{\geq 0}$, only finitely many $c_{\gamma,k} := \dim(V^{\mathrm{prim}}_{\gamma,k})$ are nonzero.

Efimov [6] proved this conjecture via an ingenious but mysterious construction for the spaces $V^{\mathrm{prim}}_{\gamma}$. The *quantum DT invariant* $\Omega_{Q,\gamma}(q)$ is the Laurent *polynomial* tracking the $c_{\gamma,k}$ as k varies over $\mathbb{Z}$. Naturally its coefficients lie in $\mathbb{Z}_{\geq 0}$. The *numerical DT-invariant* $\mathrm{DT}_{Q,\gamma}$ is the evaluation of $\Omega_{Q,\gamma}(q)$ at $q = 1$. We note that these numbers are

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called by various names in (mathematical physics) literature, e.g. LMOV invariants and BPS invariants.

At the level of Poincaré series, [10, Conjecture 1] implies a 'sum=product' flavored equality common in combinatorics and number theory. For instance, the numerical DT-invariants $\mathrm{DT}_{Q,(n)}$ for a single vertex $(m+1)$-loop quiver Q with dimension vector (n) for $n \in \mathbb{Z}_{\geq 1}$ may be defined by considering an *Euler product* factorization:

$$\sum_{n\geq 0} \frac{1}{mn+1}\binom{(m+1)n}{n+1} t^n = \prod_{n\geq 1} (1-(-1)^{mn}t^n)^{-(-1)^{mn}\, n\mathrm{DT}_{Q,(n)}}. \tag{1.1}$$

On the left-hand side is the generating function for Fuss–Catalan numbers well known to count $(m+1)$-ary trees. It is not immediate from Equation (1.1) that the exponents $\mathrm{DT}_{Q,(n)}$ are nonnegative and integral.

In what follows we give a combinatorial interpretation for $\mathrm{DT}_{Q,\gamma}$ for a symmetric quiver Q with at least one loop at each vertex and dimension vector γ. Our route is winding and hinges on realizing Efimov's construction as an instance of a construction of Dahmen–Micchelli from the 1980s.

Let X be a finite collection of nonzero vectors in $\mathbb{R}^n$. Motivated by problems in approximation theory, Dahmen–Micchelli associated a pair of vector spaces $(\mathcal{P}(X), \mathcal{D}(X))$ to X. Subsequent work [2, 9, 15] cast more light on these spaces. When the aforementioned collection of vectors is obtained from a finite connected graph G, work of Postnikov–Shapiro [15] lends a fresh perspective involving divisors on graphs. This will be our point of entry, with $\mathcal{P}(X)$ becoming the Postnikov–Shapiro slim subgraph space.

Throughout, graphs are finite connected graphs with multiple edges allowed but not self loops. Given such a graph G, we let $\mathscr{V}(G)$ (respectively $\mathscr{E}(G)$) denote its vertex set (respectively multiset of edges). An assignment $D : \mathscr{V}(G) \to \mathbb{Z}$ is called a *divisor*. Baker–Norine [3] studied this notion in depth and shed light on rich combinatorics underlying it. Another place where divisors play a role is tropical geometry, thanks to work of Mikhalkin–Zharkov [14] who introduced break divisors in this context; see [1] for more on their combinatorics. We care particularly about this class of divisors.

The *genus* of G is $g(G) = |\mathscr{V}(G)| - |\mathscr{E}(G)| + 1$. A divisor D is *effective* if $D(v) \geq 0$ for all $v \in \mathscr{V}(G)$. The *degree* of D is equal to $\sum_{v\in\mathscr{V}(G)} D(v)$. We say that D is a *break divisor* if its degree is $g(G)$ and furthermore, for every connected subgraph H of G the degree of D restricted to H weakly exceeds $g(H)$. This description implies that the set $\mathrm{Break}(G)$ of break divisors on G may be identified with integer lattice points of a polytope. It also implies that the automorphism group $\mathrm{Aut}(G)$ of G acts by permutations on $\mathrm{Break}(G)$.

With this elementary setup, one may ask an elementary question: *Does the* $\mathrm{Aut}(G)$ *module* $\mathrm{Break}(G)$ *carry interesting information?* We hope to convince the reader that the answer is in the affirmative by giving an enumerative application to DT theory. Given a symmetric quiver Q and dimension vector γ, we associate a connected graph $G_{Q,\gamma}$ that we call the *covering graph*. Let S_γ denote the Young subgroup determined by γ.

Unraveling Efimov's construction using $G_{Q,\gamma}$ allows us to cast his results in the context of the space $\mathcal{P}(G_{Q,\gamma})$. Here is our chief result.

Theorem 1. *The dimensions of the graded pieces of $\mathcal{P}(G_{Q,\gamma})^{S_\gamma}$ are the coefficients of the quantum DT invariant $\tilde{\Omega}_\gamma(q)$. In particular, the numerical DT invariant $\mathrm{DT}_{Q,\gamma} := \tilde{\Omega}_\gamma(1)$ equals the number of S_γ-orbits on* $\mathrm{Break}(G)$.

Theorem 1 gives, to the best of our knowledge, the first manifestly nonnegative combinatorial interpretation for $\mathrm{DT}_{Q,\gamma}$ by describing it as an orbit count. Our interpretation comes from invoking the theory of orbit harmonics (see Theorem 3). In fact, this theory will give an interpretation of quantum DT invariant as the graded dimension of a space of S_γ-invariants in a certain polynomial ring quotient (see Theorem 7).

The full version of this extended abstract with proofs appears on the arXiv [16].

2 The Postnikov–Shapiro slim subgraph space

For reasons of space, we discuss the objects in the title solely in the context they matter to us, referring the reader to [5] and references therein for a detailed treatment. Some of the jargon is matroid-theoretic, and for good reason.

Let G be a connected graph as before. We typically identify the vertex set with $[n] := \{1, \ldots, n\}$ where $n = |\mathscr{V}(G)|$. Given a nonempty proper subset $S \subset \mathscr{V}(G)$, we denote by $G[S]$ the graph induced on vertices in S. We define the *edge cut* $\partial(S)$ of G associated to S to be the set of edges with one endpoint in S and the other in $\bar{S} = \mathscr{V}(G) \setminus S$. We let $d(S) := |\partial(S)|$. We denote the set of edge cuts by $\mathscr{C}(G)$. A minimal non-empty edge cut is called a *bond*. We denote the set of bonds by $\mathscr{B}(G)$.

Suppose $H \subset \mathscr{E}(G)$ is such that the subgraph of G given by $G' := (\mathscr{V}(G), \mathscr{E}(G) \setminus H)$ is connected. We say that H (or the subgraph in G determined by it) is *slim*. Such subgraphs play a key role in two works that serve as our motivation, namely [4, 15]. We attach certain polynomials to subgraphs of G and subsequently discuss the special role played by slim subgraphs.

For each edge $e \in \mathscr{E}(G)$ joining vertices i and j where $i < j$, consider the linear form $p_e := x_i - x_j$. Given $H \subseteq \mathscr{E}(G)$, define the polynomial

$$p_H = \prod_{e \in H} p_e. \tag{2.1}$$

Consider the polynomial ring $\mathbb{Q}[p_e \mid e \in \mathscr{E}(G)]$ which must equal $\mathbb{Q}[x_i - x_j \mid i < j]$ as G is connected. In this ring, the *cocircuit ideal* $I(G)$ is defined as

$$I(G) := \langle p_H \mid H \in \mathscr{B}(G) \rangle. \tag{2.2}$$

Note that we could have defined $I(G)$ as the ideal generated by *all* p_H where $H \in \mathscr{C}(G)$. Indeed $p_{H'}$ for H' a cut is a multiple of p_H for H some bond.

Now define the *(Postnikov–Shapiro) slim subgraph space* (also known as the *central P-space*) as the $\mathbb{Q}$-vector space associated to G as follows:

$$\mathcal{P}(G) := \mathbb{Q}\left\{p_H \mid H \text{ slim subgraph of } G\right\}. \tag{2.3}$$

Example 2. Say G is the graph on three vertices (identified with [3]) with two edges between vertices 1 and 2, and two between 2 and 3. Then it is easily checked that

$$I(G) = \langle (x_1 - x_2)^2, (x_2 - x_3)^2 \rangle,$$
$$\mathcal{P}(G) = \mathbb{Q}\{1, x_1 - x_2, x_2 - x_3, (x_1 - x_2)(x_2 - x_3)\}.$$

Observe that $\mathbb{Q}[x_1 - x_2, x_1 - x_3, x_2 - x_3] = I(G) \oplus \mathcal{P}(G)$, and that $\dim(\mathcal{P}(G)) = 4$. This is no coincidence, as we explain next.

Let $\mathsf{ST}(G)$ denote the set of *spanning trees* of G, which are in bijection with the set of *bases* in the collection of vectors $\{\mathbf{e}_i - \mathbf{e}_j \mid \{i < j\} \in \mathscr{E}(G)\}$. The vectors $\mathbf{e}_1$ through $\mathbf{e}_n$ denote the standard basis vectors for $\mathbb{R}^n$. The following fundamental results then hold (see [9, Theorem 3.8] or [5, Corollary 11.23]):

1. $\mathbb{Q}[x_i - x_j \mid i < j] = \mathcal{P}(G) \oplus I(G)$,
2. $\dim \mathcal{P}(G) = |\mathsf{ST}(G)|$.

Let $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \dots, x_n]$. By throwing in the linear form $x_1 + \cdots + x_n$ into the mix, the first of these may be written as $\mathbb{Q}[\mathbf{x}_n] = \mathcal{P}(G) \oplus \hat{I}(G)$, where $\hat{I}(G)$ is now the ideal in $\mathbb{Q}[\mathbf{x}_n]$ with the same generators as $I(G)$ as well as $x_1 + \cdots + x_n$. Observe that the central P-space $\mathcal{P}(G)$ remains unchanged in both direct sum decompositions. We will henceforth abuse notation and treat $\hat{I}(G)$ and $I(G)$ as the same, with the ambient ring serving to distinguish them.

An alternative perspective on $\mathcal{P}(G)$ than the one just offered will eventually allow us to describe it by the method of orbit harmonics. $\mathcal{P}(G)$ is also the *Macaulay-inverse* to a *power ideal* determined by G. We proceed to make this precise. Like before the set of bonds $\mathscr{B}(G)$ will be important. Given nonempty $S \subseteq [n]$, let $x_S := \sum_{i \in S} x_i$. Recall that for S nonempty and proper, we denote by $d(S)$ the size of the edge cut $\partial(S)$. Consider the power ideal $\mathcal{I}(G)$ in $\mathbb{Q}[\mathbf{x}_n]$ defined by

$$\mathcal{I}(G) = \langle x_1 + \cdots + x_n,\ x_S^{d(S)} \text{ where } \emptyset \neq S \subsetneq [n] \text{ such that } \partial(S) \in \mathscr{B}(G) \rangle$$

Several nice properties of such ideals (and corresponding quotients) defined in this manner were brought forth by Ardila–Postnikov [2]. The quotient $\mathbb{Q}[\mathbf{x}_n]/\mathcal{I}(G)$ is known as the *central zonotopal algebra* attached to G.[1]

[1] We note that the ideals $\mathcal{I}(G)$ also appear in [15], but their set of generators has many redundant elements, and Postnikov–Shapiro eliminate x_n and work in a quotient of $\mathbb{Q}[\mathbf{x}_{n-1}]$.

How does $\mathcal{I}(G)$ relate to $\mathcal{P}(G)$? Let $\partial_i := \frac{\partial}{\partial x_i}$. We have ([9, Theorem 3.8 (5)], or [5, Theorem 11.25]):

$$\mathcal{P}(G) = \{f \in \mathbb{Q}[\mathbf{x}_n] \mid p(\partial_1, \dots, \partial_n)f = 0 \text{ for all } p \in \mathcal{I}(G)\}. \tag{2.4}$$

Going back to Example 2, we see for instance that the differential operator $(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2})^2$ corresponding to $(x_1 + x_2)^2 \in \mathcal{I}(G)$ annihilates the four polynomials that span $\mathcal{P}(G)$: three for degree reasons, and the fourth by a routine computation. Furthermore $\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}$ annihilates any polynomial in the differences $x_i - x_j$ for $1 \leq i < j \leq 3$.

3 Orbit harmonics and break divisors

Recall from the introduction that we denote the set of break divisors on G by $\mathrm{Break}(G)$. To motivate the connection of these divisors with the results in the preceding section, we note here the important equality $|\mathrm{Break}(G)| = |\mathsf{ST}(G)|$ [1, Theorem 4.25].

Under our usual identification of $\mathscr{V}(G)$ with $[n]$, elements of $\mathrm{Break}(G)$ become certain lattice points in $\mathbb{R}^n$. In fact, they are exactly the set of lattice points in a close cousin of a polytope attached to G. Consider the *graphical zonotope* $\mathcal{Z}_G \in \mathbb{R}^n$ determined from G by taking the Minkowski sum of line segments $[\mathbf{e}_i, \mathbf{e}_j]$ for every edge $\{i, j\} \in \mathscr{E}(G)$. Let $\Delta_{n-1,n}$ denotes the $(n-1)$-th hypersimplex obtained as the convex hull of the S_n-orbit of the point $(1^{n-1}, 0) \in \mathbb{R}^n$. This given we have [12, Proposition 2.1]:

$$\mathrm{Break}(G) = (\mathcal{Z}_G - \Delta_{n-1,n}) \cap \mathbb{Z}^n,$$

where $\mathcal{Z}_G - \Delta_{n-1,n}$ denotes the *trimmed graphical zonotope* obtained as a Minkowski difference. As mentioned in the introduction, $\mathrm{Aut}(G)$ acts on $\mathrm{Break}(G)$ by permutations. Thus we have a point set on which a linear group acts, and such a setup is ideal for the method of orbit harmonics.

The method of *orbit harmonics* gives a technique for turning an ungraded permutation representation of some linear group $\mathscr{G}$ acting on a finite point locus Y into a graded $\mathscr{G}$-module. Orbit harmonics was introduced by Kostant [13] and has seen subsequent application (for example) by Garsia–Procesi [7] in the context of Springer fibers and by Haglund–Rhoades–Shimozono [8] in the context of Macdonald-theoretic delta operators.

Consider a finite point set $Y \subset \mathbb{Q}^n$ and let $\mathsf{I}(Y) \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the ideal of polynomials which vanish on Y. The quotient of $\mathbb{Q}[\mathbf{x}_n]$ by the ideal $\mathsf{I}(Y)$ has vector space dimension $\dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x}_n]/\mathsf{I}(Y)) = |Y|$. If the set Y is $\mathscr{G}$-stable for some group $\mathscr{G} \subset \mathrm{GL}_n(\mathbb{Q})$, then we have an isomorphism of ungraded $\mathscr{G}$-modules

$$\mathbb{Q}[\mathbf{x}_n]/\mathsf{I}(Y) \cong_{\mathscr{G}} \mathbb{Q}[Y].$$

Orbit harmonics produces a quotient that will afford the structure of a *graded* $\mathscr{G}$-module. Given a nonzero polynomial $f \in \mathbb{Q}[\mathbf{x}_n]$, let $\tau(f)$ be the top degree homogeneous component of f. Define an ideal $\mathsf{T}(Y) \subseteq \mathbb{Q}[\mathbf{x}_n]$ by

$$\mathsf{T}(Y) = \langle \tau(f) \,:\, f \in \mathsf{I}(Y),\; f \neq 0 \rangle.$$

The $\mathscr{G}$-module isomorphism given above extends to a chain

$$\mathbb{Q}[\mathbf{x}_n]/\mathsf{T}(Y) \cong_{\mathscr{G}} \mathbb{Q}[\mathbf{x}_n]/\mathsf{I}(Y) \cong_{\mathscr{G}} \mathbb{Q}[Y],$$

with the added feature that the left hand side is a graded $\mathscr{G}$-module.

3.1 Applying orbit harmonics to $\mathrm{Break}(G)$

We will apply orbit harmonics to the locus $Y = \mathrm{Break}(G)$ with $\mathscr{G} = \mathrm{Aut}(G)$. Remarkably, the ideal $\mathsf{T}(Y)$ which so arises will coincide with the power ideal $\mathcal{I}(G)$. For brevity we refer to the ideals $\mathsf{I}(\mathrm{Break}(G))$ and $\mathsf{T}(\mathrm{Break}(G))$ by $\mathsf{I}(G)$ and $\mathsf{T}(G)$ respectively. We then have the following result.

Theorem 3. *We have the equality of ideals $\mathsf{T}(G) = \mathcal{I}(G)$. Consequently we have the following isomorphisms and equalities of ungraded $\mathrm{Aut}(G)$-modules:*

$$\mathbb{Q}[\mathrm{Break}(G)] \cong \mathbb{Q}[\mathbf{x}_n]/\mathsf{T}(G) = \mathbb{Q}[\mathbf{x}_n]/\mathcal{I}(G) \cong \mathcal{P}(G).$$

where the middle equality and right isomorphism are in the category of graded $\mathrm{Aut}(G)$*-modules.*

Sketch of proof. We begin by identifying a family of polynomials indexed by $\mathscr{B}(G)$ that vanish on $\mathrm{Break}(G)$. Subsequently we consider the ideal $\tilde{\mathsf{T}}(G)$ generated by top degree homogeneous summands of the polynomials in the aforementioned family. The inclusion $\tilde{\mathsf{T}}(G) \subset \mathsf{T}(G)$ on the one hand implies that

$$\dim\left(\mathbb{Q}[\mathbf{x}_n]/\tilde{\mathsf{T}}(G)\right) \geq \dim\left(\mathbb{Q}[\mathbf{x}_n]/\mathsf{T}(G)\right).$$

On the other hand, it transpires that $\tilde{\mathsf{T}}(G)$ equals the power ideal $\mathcal{I}(G)$ and thus the left hand side equals $|\mathsf{ST}(G)|$. Additionally, the right hand side equals $|\mathrm{Break}(G)|$. Given that $|\mathsf{ST}(G)| = |\mathrm{Break}(G)|$ we conclude that $\tilde{\mathsf{T}}(G) = \mathsf{T}(G)$. □

We demonstrate several key notions of this section in the following example.

Example 4. Let G be the complete bipartite graph $K_{2,3}$. Let us identify the vertex set with $[5]$ with $\{1,2\}$ and $\{3,4,5\}$ being the two partite sets. Elements of $\mathrm{Break}(G)$ are certain tuples $(d_1, d_2, d_3, d_4, d_5) \in \mathbb{Z}^5_{\geq 0}$ where $\sum_{1 \leq i \leq 5} d_i = g(K_{2,3}) = 2$. Any subgraph that is a tree gives a trivial constraint. From the $S_2 \times S_3$ symmetry in G it follows that given any break divisor we can permute d_1 and d_2 (respectively d_3, d_4, and d_5) to get another break divisor. Thus, up to $S_2 \times S_3$ symmetry, the only relevant inequality is $d_1 + d_2 + d_3 + d_4 \geq 1$. In summary we obtain the following four orbit representatives for $S_2 \times S_3$-action on $\mathrm{Break}(G)$: $(2,0,0,0,0)$, $(1,1,0,0,0)$, $(1,0,1,0,0)$, $(0,0,1,1,0)$. We thus obtain $|\mathrm{Break}(G)| = 12$, which agrees with the number of spanning trees of G.

Consider the bond $\partial(S)$ determined by choosing $S = \{1\}$. This determines the linear form x_1. The smallest (respectively largest) value that x_1 attains on $\mathrm{Break}(G)$ is 0 (respectively 2). Thus the polynomial $x_1(x_1 - 1)(x_1 - 2)$ vanishes on $\mathrm{Break}(G)$. This polynomial has x_1^3 as the top degree homogeneous summand, and this polynomial is indeed in $\mathcal{I}(K_{2,3})$ as well as $\mathsf{T}(K_{2,3})$. We leave it to the reader to verify that

$$\mathsf{T}(K_{2,3}) = \langle x_1 + \cdots + x_5, x_i^3, x_j^2, (x_i + x_j)^3 \text{ where } i \in \{1,2\}, j \in \{3,4,5\} \rangle.$$

The quotient $\mathbb{Q}[\mathbf{x}_5]/\mathsf{T}(K_{2,3})$ has a monomial basis

$$\{1, x_1, x_2, x_3, x_4, x_1^2, x_2^2, x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4\}.$$

It can be shown that the elements $\{1, x_1 + x_2, x_1^2 + x_2^2, x_1x_2\}$ span the space of $S_2 \times S_3$-invariants. We thus get

$$\mathrm{Hilb}((\mathbb{Q}[\mathbf{x}_5]/\mathsf{T}(K_{2,3}))^{S_2 \times S_3}) = 1 + q + 2q^2.$$

As we shall soon see, this last polynomial is essentially a quantum DT invariant; see Example 8. This shall follow once we make a connection to Efimov's work.

We conclude this section with one final remark. In general, it follows from [15] that the quotient $\mathbb{Q}[\mathbf{x}_n]/\mathcal{I}(G)$ has a monomial basis indexed by G-parking functions. Thus, Theorem 3 provides an algebraic perspective on the fact that the set of q-reduced divisors on G (essentially G-parking functions) has the same cardinality as $\mathrm{Break}(G)$. Indeed, this same quotient arises when applying orbit harmonics to $\mathrm{Break}(G)$.

4 A construction of Efimov for quantum DT invariants

Let $A = (a_{ij})_{i,j \in [k]}$ be a symmetric matrix with nonnegative integer entries. We assume $a_{ii} \geq 1$ throughout. The matrix A determines a quiver Q on k vertices labeled 1 through k, with a_{ij} arrows from i to j for $i, j \in [k]$. Note that $a_{ii} \geq 1$ ensures at least one loop at each vertex. We further assume A is such that Q is connected. Let $\gamma = (\gamma_1, \ldots, \gamma_k) \in \mathbb{Z}_{\geq 0}^k$ be a dimension vector. We begin by describing the essential construction in [6].

Consider variables $x_{i,\alpha}$ for $1 \leq i \leq k$ and $1 \leq \alpha \leq \gamma_i$, and let σ_γ be their sum. Define the polynomial ring

$$A_\gamma = \mathbb{Q}[x_{i,\alpha} \mid 1 \leq i \leq k, 1 \leq \alpha \leq \gamma_i]. \tag{4.1}$$

Furthermore, let

$$A_\gamma^{\text{prim}} = \mathbb{Q}[x_{j,\alpha_2} - x_{i,\alpha_1} \mid 1 \le i,j \le k, 1 \le \alpha_1 \le \gamma_i, 1 \le \alpha_2 \le \gamma_j]. \tag{4.2}$$

Then we have $A_\gamma = A_\gamma^{\text{prim}} \otimes \mathbb{Q}[\sigma_\gamma]$. Define $S_\gamma := S_{\gamma_1} \times \cdots \times S_{\gamma_k}$.

Define J_γ to be the smallest S_γ-stable A_γ^{prim}-submodule of A_γ^{prim} such that the following holds: for any decomposition of $\gamma = \delta + \bar{\delta}$ where both δ and $\bar{\delta}$ are nonzero we have that

$$f_{\delta,\bar{\delta}} = \prod_{i \neq j \in [k]} \prod_{\alpha_1=1}^{\delta_i} \prod_{\alpha_2=\delta_j+1}^{\gamma_j} (x_{j,\alpha_2} - x_{i,\alpha_1})^{a_{ij}} \prod_{i \in [k]} \prod_{\alpha_1=1}^{\delta_i} \prod_{\alpha_2=\delta_i+1}^{\gamma_i} (x_{i,\alpha_2} - x_{i,\alpha_1})^{a_{ii}-1} \in J_\gamma. \tag{4.3}$$

We shall later reinterpret $f_{\delta,\bar{\delta}}$ in terms of cuts (as the indexing hints) in a graph constructed from (A,γ).

Letting $\mathcal{H}_\gamma^{\text{prim}} := (A_\gamma^{\text{prim}})^{S_\gamma}$, consider the decomposition [6, p. 1139]

$$\mathcal{H}_\gamma^{\text{prim}} = V_\gamma^{\text{prim}} \oplus J_\gamma^{S_\gamma}. \tag{4.4}$$

The quantum DT invariants of the quiver Q with dimension vector γ (assuming trivial potential and stability) arise as dimensions of the graded pieces of the graded vector space V_γ^{prim} as explained in [6, Section 4]. We describe the $\mathbb{Z}$-grading employed.

Given dimension vectors γ and δ, the *Euler form* $\chi_Q(\gamma,\delta)$ is defined as

$$\chi_Q(\gamma,\delta) := \sum_{1 \le i \le k} \gamma_i \delta_i - \sum_{1 \le i,j \le k} a_{ij} \gamma_i \delta_j. \tag{4.5}$$

Homogeneous polynomials $f \in \mathcal{H}_\gamma$ of degree d get assigned the grading $2d + \chi_Q(\gamma,\gamma)$. We let $V_{\gamma,m}^{\text{prim}}$ be the space of elements in V_γ^{prim} with grading m. Following [6, Section 4], and as in the introduction, set $c_{\gamma,m} := \dim(V_{\gamma,m}^{\text{prim}})$, and consider the polynomial in $\mathbb{Z}_{\ge 0}[q^{\pm\frac{1}{2}}]$:

$$\Omega_\gamma(q) = \sum_{m \in \mathbb{Z}} c_{\gamma,m} q^{m/2}. \tag{4.6}$$

These $\Omega_\gamma(q)$ are the *quantum DT-invariants* of the quiver Q.

We get a necessary condition for when $V_{\gamma,m}^{\text{prim}}$ is nonzero in [6, Theorem 1.2] (also presented as [6, Theorem 3.10]). It involves a statistic $N_\gamma(Q)$ [6, Section 1] whose precise definition is unnecessary for the purposes of this abstract.

Theorem 5 ([6, Theorem 3.10]). *If* $V_{\gamma,m}^{\text{prim}} \neq 0$, *then* $\gamma \neq 0$, $m \equiv \chi_Q(\gamma,\gamma) \mod 2$, *and* $\chi_Q(\gamma,\gamma) \le m < \chi_Q(\gamma,\gamma) + 2N_\gamma(Q)$.

In view of Theorem 5 we may rewrite $\Omega_\gamma(q)$ as

$$\Omega_\gamma(q) = q^{\frac{1}{2}\chi_Q(\gamma,\gamma)} \sum_{0 \le m \le N_\gamma(Q)-1} c_{\gamma,2m+\chi_Q(\gamma,\gamma)} q^m. \tag{4.7}$$

Denote the sum on the right by $\tilde{\Omega}_\gamma(q)$. It lies in $\mathbb{Z}_{\ge 0}[q]$ and its degree is bounded by $N_\gamma(Q) - 1$. We abuse notation and refer to $\tilde{\Omega}_\gamma(q)$ as the quantum DT invariant as well.[2]

4.1 The covering graph construction

The reader might find the preceding construction both ingenious and mysterious. It transpires that the space $V^{\text{prim}}_{\gamma,m}$ is the space of S_γ-invariants for a slim subgraph space determined naturally from the quiver and the dimension vector. We proceed to describe this construction.

Our point of departure from Efimov is to consider an analogue of (4.4) where we do not take S_γ-invariants. We will construct an S_γ-stable space W^{prim}_γ so that

$$A^{\text{prim}}_\gamma = W^{\text{prim}}_\gamma \oplus J_\gamma. \tag{4.8}$$

Our next construction is crucial to this end.

Given Q as before, we construct an undirected graph $G_{Q,\gamma}$: Consider a set of vertices $v_{i,\alpha}$ for $1 \le i \le k$ and $1 \le \alpha \le \gamma_i$. For $i \in [k]$, the restriction of $G_{Q,\gamma}$ to the vertices $v_{i,1}, \ldots, v_{i,\gamma_i}$ is the clique on γ_i vertices with $a_{ii} - 1$ edges between any two distinct vertices. In particular, if $a_{ii} = 1$, then we have a collection of γ_i totally disconnected vertices. For $i \ne j \in [k]$ we draw a_{ij} edges between any vertex v_{i,α_1} and v_{j,α_2} for $1 \le \alpha_1 \le \gamma_i$ and $1 \le \alpha_2 \le \gamma_j$. This determines $G_{Q,\gamma}$. Note that S_γ is a subgroup of $\mathrm{Aut}(G_{Q,\gamma})$.

We will assume throughout that $G_{Q,\gamma}$ is connected. For the remainder of this section, we fix our symmetric quiver Q and dimension vector γ, and we will drop them from notation. In particular, unless otherwise noted, let $G := G_{Q,\gamma}$.

We reinterpret the important element $f_{\delta,\bar{\delta}} \in J_\gamma$ from Equation (4.3) as the polynomial p_H for some cut in G.

Lemma 6. *Consider a decomposition $\gamma = \delta + \bar{\delta}$ where $\delta, \bar{\delta} \ne \gamma$. The following hold.*

1. *$f_{\delta,\bar{\delta}} = p_H$ for some $H \in \mathscr{C}(G)$.*
2. *For $\emptyset \ne S \subsetneq \mathscr{V}(G)$ such that $\partial(S) \in \mathscr{B}(G)$, we have that $p_{\partial(S)} \in J_\gamma$. More specifically, there exists a decomposition $\gamma = \delta + \bar{\delta}$ such that $\sigma(f_{\delta,\bar{\delta}}) = p_{\partial(S)}$ for some $\sigma \in S_\gamma$.*

The preceding lemma simply states that Efimov's ideal J_γ is precisely the ideal generated by p_H for $H \in \mathscr{B}(G)$, i.e. $J_\gamma = I(G)$ where the latter is the cocircuit ideal from Equation (2.2). We thus obtain the following theorem that 'combinatorializes' Efimov's construction and describes the DT invariants in terms of $\mathcal{P}(G)$.

[2] As the careful reader may have realized, we already do so in the introduction.

Theorem 7. *The following decomposition holds:* $A_\gamma^{\text{prim}} = \mathcal{P}(G) \oplus J_\gamma$. *Consequently we have the equality*

$$\text{Hilb}(\mathcal{P}(G)^{S_\gamma}) = \tilde{\Omega}_\gamma(q).$$

From the chain of isomorphisms in Theorem 3 we have $\text{DT}_{Q,\gamma} = \tilde{\Omega}_\gamma(1) = \dim(\mathbb{Q}[\text{Break}(G)]^{S_\gamma})$, *i.e. the number of* S_γ*-orbits on* $\text{Break}(G)$.

To demonstrate our main result, we revisit Example 4, this time starting with a quiver and a dimension vector.

Example 8. Consider the quiver with two vertices, each with a self-loop, and two edges between the vertices, one in each direction. Pick $\gamma = (2,3)$. The covering graph construction then yields $G_{Q,\gamma}$ as the bipartite graph $K_{2,3}$. We reindex our variables $x_1 = x_{11}$, $x_2 = x_{12}$, $x_3 = x_{21}$, $x_4 = x_{22}$, and $x_5 = x_{32}$. We let $S_2 \times S_3$ act on $\mathbb{Q}[x_1, \ldots, x_5]$ by letting S_2 (respectively S_3) act on x_1, x_2 (respectively x_3, x_4, x_5).

$\mathcal{P}(G)$ is spanned by p_H for slim subgraphs H which may be obtained as the S_γ-orbit of elements in $\{1, x_1 - x_3, (x_1 - x_3)(x_1 - x_4), (x_1 - x_3)(x_2 - x_4)\}$. The space $\mathcal{P}(G)^{S_\gamma}$ has basis elements

$$\{1, \sum_{\sigma \in S_\gamma} \sigma \cdot (x_1 - x_3), \sum_{\sigma \in S_\gamma} \sigma \cdot (x_1 - x_3)(x_1 - x_4), \sum_{\sigma \in S_\gamma} \sigma \cdot (x_1 - x_3)(x_2 - x_4))\}.$$

Explicitly, other than the constant polynomial 1, up to normalization these equal

$$\begin{aligned}
&3(x_1 + x_2) - 2(x_3 + x_4 + x_5) \\
&3(x_1^2 + x_2^2) - 2(x_1 + x_2)(x_3 + x_4 + x_5) + 2(x_3x_4 + x_3x_5 + x_4x_5) \\
&3x_1x_2 - (x_1 + x_2)(x_3 + x_4 + x_5) + (x_3x_4 + x_3x_5 + x_4x_5).
\end{aligned}$$

We thus infer that

$$\tilde{\Omega}_\gamma(q) = 1 + q + 2q^2,$$

and therefore that $\tilde{\Omega}_\gamma(1) = 4$. Going back to Example 4, this agrees with the number of $S_2 \times S_3$-orbits on $\text{Break}(K_{2,3})$. We also encountered the polynomial $1 + q + 2q^2$ in that same example. Back then we obtained it as $\text{Hilb}((\mathbb{Q}[\mathbf{x}_5]/\mathcal{I}(K_{2,3}))^{S_2 \times S_3})$.

5 Two applications

We close with two representation-theoretic results of independent interest, with motivation coming from [4]. Like before, we set $G := G_{Q,\gamma}$.

The top degree of $\mathcal{P}(G)$ as an S_γ-module. The graph G (with $\mathscr{V}(G) = [n]$) determines a hyperplane arrangement $\mathcal{A}_G$ in $\mathbb{C}^n$ by considering the union of hyperplanes $x_i - x_j =$

0 for edges $\{i,j\} \in E(G)$. The *de Rham cohomology ring* $H^*(\mathbb{C}^n \setminus \mathcal{A}_G)$ over $\mathbb{C}$ of the complement $\mathbb{C}^n \setminus \mathcal{A}_G$ is the exterior algebra over the generators $\mathrm{d}\log(\beta) = \frac{d\beta}{\beta}$ where β ranges over the linear forms that cut out $\mathcal{A}_G$. Consider the collection of vectors $X_G = \{\mathbf{e}_i - \mathbf{e_j} \mid \{i < j\} \in \mathscr{E}(G)\}$, and endow it with a total order. The top degree graded piece of $H^*(\mathbb{C}^n \setminus \mathcal{A}_G)$, denoted by $H^*(\mathbb{C}^n \setminus \mathcal{A}_G)_{\text{top}}$, has a basis indexed by so-called *unbroken bases*. Such bases also form a basis for the top degree of $\mathcal{P}(G)$, denoted by $\mathcal{P}(G)_{\text{top}}$. This equality in dimension of the two spaces is a shadow of an S_γ-isomorphism between two vector spaces. The following result generalizes [4, Theorem 5].

Proposition 9. *We have the following S_γ-isomorphism:*

$$\mathcal{P}(G)_{\text{top}} \cong H^*(\mathbb{C}^n \setminus \mathcal{A}_G)_{\text{top}} \otimes \varepsilon_{\gamma_1}^{a_{11}-1} \otimes \cdots \otimes \varepsilon_{\gamma_k}^{a_{kk}-1}.$$

Here $a_{ii} - 1$ is one less than the number of loops at vertex i in Q, and ε_{γ_i} denotes the sign representation *of S_{γ_i}.*

The $(m+1)$-loop quiver. Let m be a positive integer. We revisit the case of the $(m+1)$-loop quiver Q with dimension vector $\gamma = (n)$ where $n \geq 1$. The covering graph $G := G_{Q,\gamma}$ is the *complete multipartite graph K_n^m*, i.e. the graph on n vertices with m edges between any two distinct vertices. $\mathrm{Aut}(G)$ is clearly the symmetric group S_n. Theorem 3 then gives us (isomorphic) spaces with dimension $m^{n-1}n^{n-2}$.

Berget–Rhoades [4] consider the S_n-modules $\mathcal{P}(K_n^m)$ and establish several interesting results. While the graded Frobenius characteristics of these modules remains elusive, we are able to shed light on related matters. In fact these modules encode a large class of DT invariants; see [16, Section 5.4] for more.

Theorem 3 allows us to connect the space $\mathcal{P}(K_n^m)$ to $\mathbb{Q}[\mathrm{Break}(K_n^m)]$ and, among other things, resolve [11, Conjecture 3.3] and recover one of the main results in [4].

Corollary 10. *The following hold.*

1. *We have the isomorphism $\mathcal{P}(K_n^m) \cong_{S_n} \mathbb{Q}[\mathrm{Break}(K_n^m)]$ of ungraded representations. In particular, the* Frobenius characteristic $\mathrm{Frob}(\mathcal{P}(K_n^m))$ *is h-positive.*

2. *The graded multiplicity of the trivial representation of S_n in $\mathcal{P}(K_n^m)$ is given by the quantum DT invariant $\tilde{\Omega}_{(n)}(q)$. At $q = 1$ we obtain*

$$\dim(\mathcal{P}(K_n^m)^{S_n}) = \mathrm{DT}_{Q,(n)} = \frac{1}{mn^2} \sum_{d|n} (-1)^{mn + \frac{mn}{d}} \mu(d) \binom{\frac{(m+1)n}{d} - 1}{\frac{n}{d}}.$$

3. *On the level of graded Frobenius characteristics, we have*

$$\mathrm{grFrob}(\mathcal{P}(K_n) \downarrow^{S_n}_{S_{n-1}}) = \mathrm{grFrob}(\mathrm{PF}_{n-1}),$$

where PF_{n-1} is the parking function representation of S_{n-1}, with basis indexed by parking functions and grading given by the sum of entries in a parking function.

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