Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #59, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

A combinatorial interpretation of the noncommutative inverse Kostka matrix

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Abstract. We provide a combinatorial formula for the expansion of immaculate noncommutative symmetric functions into complete homogeneous noncommutative symmetric functions. To do this, we introduce generalizations of Ferrers diagrams which we call GBPR diagrams. We define tunnel hooks, which play a role similar to that of the special rim hooks appearing in the Eğecioğlu-Remmel formula for the symmetric inverse Kostka matrix. We extend this interpretation to skew shapes and fully generalize to define immaculate functions indexed by integer sequences skewed by integer sequences. Finally, as an application of our combinatorial formula, we extend Campbell's results on ribbon decompositions of immaculate functions to a larger class of shapes.

Keywords: immaculate functions, inverse Kostka, NSym

1 Background and Introduction

The ring Sym of symmetric functions on a set of commuting variables consists of all polynomials invariant under the action of the symmetric group. Symmetric functions play an important role in representation theory, combinatorics, and other areas of mathematics and the physical and natural sciences. The Schur function basis for Sym, which can be defined combinatorially as the generating function for semi-standard Young tableaux or algebraically through *Bernstein creation operators*, corresponds to irreducible representations of the symmetric group. Schur function multiplication corresponds to the cohomology of the Grassmannian [6, 10, 11].

The inverse Kostka matrix is the transition matrix from the Schur basis $\{s_\lambda\}_\lambda$ of Sym to the *complete homogeneous basis* $\{h_\lambda\}_\lambda$. The *Jacobi-Trudi Formula* [10] is a determinantal formula $s_{\lambda/\nu} = \det(h_{\lambda_i - i-(\nu_j - j)})$ for decomposing Schur functions $s_{\lambda/\nu}$ into complete homogeneous symmetric functions h_λ. Eğecioğlu and Remmel's combinatorical interpretation of the inverse Kostka matrix [5] provides a method for writing Schur functions in terms of complete homogeneous symmetric functions using decompositions of the indexing shapes into *special rim hooks*.

The Hopf algebra NSym of noncommutative symmetric functions can be thought of as the free associative algebra $\mathbf{K}\langle H_1, H_2, \ldots \rangle$ generated by algebraically independent, noncommuting complete homogeneous symmetric functions H_i over a fixed commutative field $\mathbf{K}$ of characteristic zero. Set $H_a := 0$ if a is a negative integer and $H_0 := 1$. The function $H_{(\alpha_1,\alpha_2,\ldots,\alpha_\ell)} = H_{\alpha_1}H_{\alpha_2}\cdots H_{\alpha_\ell}$ maps to the complete homogeneous symmetric function h_α under the "forgetful map" from NSym to Sym.

The creation operator construction of the Schur functions can be extended to NSym to produce a new basis for NSym called the *immaculate basis* [2], corresponding to indecomposable modules of the 0-Hecke algebra [3]. We use the equivalent Jacobi-Trudi style formula [2] to define the immaculate basis and we also extend this construction to introduce the *skew immaculate functions*.

Definition 1.1. *Let $\mu, \nu \in \mathbb{Z}^k$ be sequences of integers. Define $(M_{\mu/\nu})_{i,j} = H_{(\mu_i-i)-(\nu_j-j)}$. Then $\mathfrak{S}_{\mu/\nu} = \det(M_{\mu/\nu})$, where the noncommutative determinant in* NSym *is obtained by performing the Laplace expansion row by row, starting from the top row.*

Example 1.2. *With $\mu = (4,5,2)$ and $\nu = (3,1,1)$, then*

$$\begin{aligned}\mathfrak{S}_{\mu/\nu} =& \det(M_{\mu/\nu}) = \det\begin{pmatrix} H_1 & H_4 & H_5 \\ H_1 & H_4 & H_5 \\ H_{-3} & H_0 & H_1 \end{pmatrix} \\ =& H_1H_4H_1 - H_1H_5H_0 - H_4H_1H_1 + H_4H_5H_{-3} + H_5H_1H_0 - H_5H_4H_{-3} \\ =& H_{1,4,1} - H_{1,5} - H_{4,1,1} + H_{5,1}\end{aligned}$$

since $H_0 = 1$ and $H_{-1} = 0$.

In this extended abstract, we provide a combinatorial formula for the expansion of the immaculate basis into the complete homogeneous basis for NSym using ribbon-like objects we call *tunnel hooks*, generalizing the combinatorial inverse Kostka formula to NSym. A complete discussion of these results including detailed proofs and additional examples can be found in [1].

Theorem 1.3. *The decomposition of the skew immaculate noncommutative symmetric functions $\mathfrak{S}_{\mu/\nu}$ (with $\mu \in \mathbb{Z}^k$ and ν a partition with at most k parts, possibly the empty partition) into the complete homogeneous noncommutative symmetric functions is given by the following formula.*

$$\mathfrak{S}_{\mu/\nu} = \sum_{\gamma \in \mathrm{THC}_{\mu/\nu}} \prod_{r=1}^{k} \epsilon(\mathfrak{h}(r,\tau_r))\, H_{\Delta(\mathfrak{h}(r,\tau_r))}, \tag{1.1}$$

where $\mathrm{THC}_{\mu/\nu}$ denotes the tunnel hook coverings of a diagram of shape μ/ν, and a sign $\epsilon(\mathfrak{h}(r,\tau_r))$ and integer value $\Delta(\mathfrak{h}(r,\tau_r))$ are assigned to each tunnel hook $\mathfrak{h}(r,\tau_r)$ in each THC *γ.*

Note that the product $\prod_{r=1}^{k} \epsilon(\mathfrak{h}(r,\tau_r))\, H_{\Delta(\mathfrak{h}(r,\tau_r))}$ in Eq. (1.1) is taken in order from $r = 1$ to k since the functions $H_{\Delta(\mathfrak{h}(r,\tau_r))}$ do not commute.

Loehr and Niese recently published a combinatorial interpretation of the immaculate inverse Kostka matrix [9]. Their approach uses transitive tournaments and recursively defined sums, which is quite different from our diagrammatic approach. Loehr and Niese also provide a diagrammatic method for computing the decomposition of an immaculate function into the complete homogeneous basis when the indexing shape is a partition. Our diagrammatic approach works for all indexing shapes, including all sequences of integers. Our decomposition can be determined directly by looking at the diagram and recording the values associated to the tunnel hooks.

2 GBPR diagrams and tunnel cells

Let $\mu = (\mu_1, \ldots, \mu_k)$ be a sequence and $\nu = (\nu_1, \ldots, \nu_k)$ be a partition. The *Grey-Blue-Purple-Red (GBPR) diagram* $D_{\mu/\nu}$ for the skew shape μ/ν is obtained as follows.

1. Place ν_i grey cells in row i of the diagram (for $1 \leq i \leq k$), working from bottom to top (to stay consistent with French notation).
2. For each $1 \leq i \leq k$, divide into cases to place red and blue cells into row i as follows.
   (a) If $\mu_i > 0$ and $\nu_i \leq \mu_i$, place $\mu_i - \nu_i$ blue cells in row i immediately to the right of the grey cells.
   (b) If $\mu_i > 0$ and $\mu_i < \nu_i$, place $\nu_i - \mu_i$ red cells in row i immediately to the right of the grey cells.
   (c) If $\mu_i \leq 0$, place $|\mu_i| + \nu_i$ red cells in row i immediately to the right of the grey cells.
3. Any cell in the first quadrant not colored grey, red, or blue is *purple*, but we do not typically color these in illustrations since there are infinitely many purple cells.

See Example 2.8 for an example of a GBPR diagram. Although the GBPR diagram does not necessarily contain exactly μ_i cells in row i, the value of μ_i can be determined from the number of grey, blue, and red cells. Let a_i be the number of grey cells, b_i be the number of blue cells, and c_i be the number of red cells in row i of a GBPR diagram. Then

$$a_i + b_i - c_i = \mu_i.$$

Definition 2.1. *Cells* (p, q) *and* (r, s) *are* adjacent *if and only if* $|p - r| + |q - s| = 1$. *A collection* C *of cells is* connected *if for any cells* $c, d \in C$, *there is a sequence* $c = \tau_1, \tau_2, \ldots, \tau_j = d$ *where* $\tau_i \in C$ *and* τ_i *and* τ_{i+1} *are adjacent for* $1 \leq i \leq j - 1$. *Cells* (p, q) *and* (r, s) *are* diagonally adjoining *if both* $r = p + 1$ *and* $s = q + 1$ *or both* $p = r + 1$ *and* $q = s + 1$.

At times, it will be convenient to construct diagrams for shapes obtained via prefix removal. Note that any diagram can be considered as a partial diagram by setting $r = 0$.

Definition 2.2. *Let $r \in \mathbb{Z}_{\geq 0}, \mu \in \mathbb{Z}^k$, and $\nu \in \mathbb{Z}^k_{\geq 0}$ such that $\nu_{r+1} \geq \nu_{r+2} \geq \ldots \geq \nu_k$. The* partial diagram $D^{(r)}_{\mu/\nu}$ *is obtained by constructing the GBPR diagram $D_{(\mu_{r+1},\ldots,\mu_k)/(\nu_{r+1},\ldots,\nu_k)}$ and shifting the resulting diagram up by r rows, so that the first nonempty row is in row $r+1$.*

This construction allows us to compute the GBPR diagram for row $r+1$ through row k even when the first r parts of ν are not necessarily weakly decreasing.

Example 2.3. *Let $\mu = (-3, 1, -1, 0, 3, -2)$ and $\nu = (2, 4, 1, 0, 0)$. Then $D^{(2)}_{\mu/\nu}$ is given by*

6,1	6,2	
5,1	5,2	5,3
4,1		
3,1	3,2	3,3

We now define *boundary cells*, which intuitively are the cells which lie on the boundary of ν. Boundary cells will be used to construct hooks that will play a role similar to that of *rim hooks* in the combinatorial interpretation of the symmetric inverse Kostka matrix [5].

Definition 2.4. *Let $\mu \in \mathbb{Z}^k$ and let r be a positive integer such that $1 \leq r \leq k$. Let $\nu \in \mathbb{Z}^k_{\geq 0}$ such that $(\nu_r, \nu_{r+1}, \ldots, \nu_k)$ is a partition and $r \leq p \leq k$. A cell in location (p, q) is a* boundary cell *of $D^{(r-1)}_{\mu/\nu}$ iff*

$$\nu_p + 1 \leq q \leq \nu_{p-1} + 1,$$

for $p > r$ and

$$\nu_r + 1 \leq q \leq \max\{\nu_r + 1, a_r + b_r + c_r\}$$

if $p = r$. A tunnel cell *(p, q) of $D^{(r-1)}_{\mu/\nu}$ is a boundary cell such that*

$$q = \nu_p + 1.$$

The inequality $\nu_r + 1 \leq q \leq \max\{\nu_r + 1, a_r + b_r + c_r\}$ for row r forces the cell $(r, \nu_r + 1)$, as well as all red or blue cells in row r, to be boundary cells. Also note that a cell (p, q) with $p < r$ cannot be a boundary cell of $D^{(r)}_{\mu/\nu}$. Let $\mathcal{B}^{(r)}_{\mu/\nu}$ and $\mathcal{T}^{(r)}_{\mu/\nu}$ respectively denote the collections of boundary cells and tunnel cells of $D^{(r)}_{\mu/\nu}$. Set $\mathcal{N}^{(r)}_{\mu/\nu} = \mathcal{B}^{(r)}_{\mu/\nu} - \mathcal{T}^{(r)}_{\mu/\nu}$.

Definition 2.5. *Let $\tau_r = (p, q) \in \mathcal{T}^{(r-1)}_{\mu/\nu}$ be a tunnel cell in $D^{(r-1)}_{\mu/\nu}$. The* tunnel hook *determined by τ_r is the collection $\mathfrak{h}(r, \tau_r)$ consisting of all boundary cells in rows r through p. The cell τ_r (the farthest northwest cell of $\mathfrak{h}(r, \tau_r)$) is called the terminal cell of $\mathfrak{h}(r, \tau_r)$. The* sign *of the tunnel hook $\mathfrak{h}(r, \tau_r)$, denoted by $\epsilon(\mathfrak{h}(r, \tau_r))$, equals $(-1)^{p-r}$. If cell $\hat{c} \in \mathfrak{h}(r, \tau)$ then we say that $\mathfrak{h}(r, \tau)$* covers *$\hat{c}$.*

There are many similar objects in the literature such as skew hooks, ribbons, and border strips [10], and the rim hooks [5] appearing in the combinatorial interpretation of the inverse Kostka matrix in Sym. To generalize these combinatorial objects to the NSym setting, tunnel hooks need to "tunnel" into the diagram instead of remaining on the rim.

With $\mathfrak{h}(r, \tau_r)$ a tunnel hook in the partial diagram $D^{(r)}_{\mu/\nu}$ terminating at cell $\tau = (p, q)$, set

$$\Delta(\mathfrak{h}(r, \tau_r)) = b_r - c_r + (\nu_r + 1 - q) + (p - r). \tag{2.1}$$

Note that in Equation (2.1), since ν is a partition (whose parts might be 0), the cell $(p, q) \in \mathcal{T}^{(r)}_{\mu/\nu}$ will be weakly west of $(r, \nu_r + 1)$.

The proof of the following lemma is immediate since there is only one tunnel cell in each row and moving up a row strictly increases the value of $\Delta(\mathfrak{h}(r, \tau_r))$.

Lemma 2.6. *Given a partial diagram $D^{(r)}_{\mu/\nu}$ with μ a sequence and ν a partition, for any fixed $j \in \mathbb{Z}$, there is at most one tunnel cell τ such that $j = \Delta(\mathfrak{h}(r, \tau_r))$.*

The following iterative procedure provides a method for constructing a *tunnel hook covering* (THC) of the diagram $D_{\mu/\nu}$.

Procedure 2.7. *Consider $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{Z}^k$ and a partition $\nu = (\nu_1, \nu_2, \ldots, \nu_k)$.*

1. *Construct the partial GBPR diagram $D^{(0)}_{\mu/\nu}$ of shape μ/ν. Set $\nu^{(0)} = \nu$.*
2. *Repeat the following steps, once for each value of r from 1 to k.*
   (a) *Choose a tunnel hook $\mathfrak{h}(r, \tau_r)$ in $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ and set $\alpha_r := \Delta(\mathfrak{h}(r, \tau_r))$.*
   (b) *For each $1 \le i \le k$, let $\eta^{(r)}_i$ be the number of cells in row i of $\mathfrak{h}(r, \tau_r)$ and let $\nu^{(r)}$ be the partition defined by $\nu^{(r)}_i = \nu^{(r-1)}_i + \eta^{(r)}_i$ for $1 \le i \le k$.*
   (c) *Construct the partial GBPR diagram $D^{(r)}_{\mu/\nu^{(r)}}$.*
3. *Set $\alpha = (\alpha_1, \ldots, \alpha_k)$. This will become a subscript appearing in the H-expansion of $\mathfrak{S}_\mu$.*

It is not hard to show that if $\nu^{(r)}$ is the sequence of nonnegative integers produced during Step 2b of Procedure 2.7, then $(\nu^{(r)}_{r+1}, \nu^{(r)}_{r+2}, \ldots, \nu^{(r)}_k)$ is a partition.

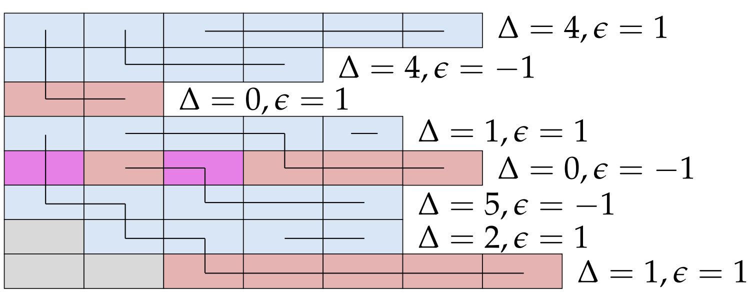
Example 2.8. *Letting $\mu/\nu = (-3, 5, 5, 0, 5, -2, 4, 6)/(2, 1)$, we construct a tunnel hook covering of $D_{\mu/\nu}$. First, we give the GBPR diagram of $D^{(0)}_{\mu/\nu}$ with cells labelled by (row,column). We then use a table to provide details for a particular tunnel hook covering. Finally, we illustrate this THC on the GBPR diagram itself.*

8,1	8,2	8,3	8,4	8,5	8,6	
7,1	7,2	7,3	7,4			
6,1	6,2					
5,1	5,2	5,3	5,4	5,5		
4,1						
3,1	3,2	3,3	3,4	3,5		
2,1	2,2	2,3	2,4	2,5		
1,1	1,2	1,3	1,4	1,5	1,6	1,7

The following table records the process of decomposing the GBPR diagram $D_{\mu/\nu}$ into tunnel hooks. Each row r indicates the situation before the r^{th} tunnel hook is placed. Here τ_r is the tunnel cell at which the tunnel hook beginning in the r^{th} row of the partial diagram terminates.

r	$(\mu_r, \mu_{r+1}, \ldots, \mu_k)$	$\nu^{(r-1)}$	τ_r	$\Delta(\mathfrak{h}(r, \tau_r))$
1	$(-3,5,5,0,5,-2,4,6)$	$(2,1,0,0,0,0,0,0)$	$(5,1)$	$-5+6=1$
2	$(5,5,0,5,-2,4,6)$	$(7,3,2,1,1,0,0,0)$	$(2,4)$	$2+0=2$
3	$(5,0,5,-2,4,6)$	$(7,5,2,1,1,0,0,0)$	$(4,2)$	$3+2=5$
4	$(0,5,-2,4,6)$	$(7,5,5,3,1,0,0,0)$	$(5,2)$	$-3+3=0$
5	$(5,-2,4,6)$	$(7,5,5,6,4,0,0,0)$	$(5,5)$	$1+0=1$
6	$(-2,4,6)$	$(7,5,5,6,5,0,0,0)$	$(8,1)$	$-2+2=0$
7	$(4,6)$	$(7,5,5,6,5,2,1,1)$	$(8,2)$	$3+1=4$
8	(6)	$(7,5,5,6,5,2,4,2)$	$(8,3)$	$4+0=4$

The following illustration of the THC of $D^{(0)}_{\mu/\nu}$ depicts all the tunnel hooks at once and omits the step of converting the colors of the cells to grey as they are covered by tunnel hooks.



3 Proof of Theorem 1.3 for non-skewed sequences

In this section, we give an overview of the proof Theorem 1.3 when the indexing shape is not a skew shape. Recall Definition 1.1 [2] states that

$$\mathfrak{S}_\mu = det(M_\mu) = \sum_{\sigma \in S_k} \epsilon(\sigma)(M_\mu)_{i,\sigma_i}.$$

Theorem 1.3 provides a combinatorial interpretation of the decomposition of immaculate functions into complete homogeneous functions using tunnel hook coverings. The main idea behind our proof is a bijection between tunnel hook coverings and permutations (Proposition 3.5). Furthermore, each tunnel hook is associated to a number equal to the subscript of the corresponding complete homogeneous function appearing in the matrix M_μ (Lemma 3.6). Finally, we show that the product of the signs of the tunnel hooks in a tunnel hook covering is equal to the sign of the corresponding permutation.

The first lemma needed for the proof of the non-skew case of Theorem 1.3 provides foundational insight into how tunnel cells are related to the diagonals parallel to the line $y = x$. Let

$$\mathcal{L}_j := \{(p,q) : p - q + 1 = j\} = \{(j+m, 1+m) | m \in \mathbb{Z}^{\geq 0}\} \tag{3.1}$$

be the collection of cells in the j^{th} diagonal of the first quadrant of the plane, for $1 \leq j \leq k$. These diagonals (whose properties are described in the following lemma) will correspond to the entries in the permutations used when computing the determinant of the matrix M_μ.

Lemma 3.1. *Let $D^{(r-1)}_{\mu/\nu^{(r-1)}}$ be a partial GBPR diagram for $\mu \in \mathbb{Z}^k$ and $\nu^{(r-1)} \in \mathbb{Z}^k_{\geq 0}$ such that $(\nu^{(r-1)}_r, \nu^{(r-1)}_{r+1}, \ldots, \nu^{(r-1)}_k)$ is a partition. Suppose $\tau_r, t \in \mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$, $\tau_r \neq t$, and $\xi \in \mathcal{N}^{(r-1)}_{\mu/\nu^{(r-1)}}$. Furthermore, suppose $\tau_r = (p_1, q_1)$, $t = (p_2, q_2)$, and $\xi = (p_3, q_3)$. Finally, let $\mathfrak{h}(r, \tau_r)$ be a tunnel hook in the diagram $D^{(r-1)}_{\mu/\nu^{(r-1)}}$. Then*

A. $(p_1 + 1, q_1 + 1) \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$.

B. If $\mathfrak{h}(r, \tau_r)$ does not cover t then $t \in \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$.

C. If $\mathfrak{h}(r, \tau_r)$ covers t then $(p_2 + 1, q_2 + 1) \in \mathcal{T}^{(r)}_{\mu/\nu^{(r)}}$.

D. If $\mathfrak{h}(r, \tau_r)$ does not cover ξ then $\xi \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$.

E. If $\mathfrak{h}(r, \tau_r)$ covers ξ then $(p_3 + 1, q_3 + 1) \in \mathcal{N}^{(r)}_{\mu/\nu^{(r)}}$.

Lemma 3.1 implies an algorithm for identifying the cells in $\mathcal{T}^{(r-1)}_{\mu/\nu^{(r-1)}}$ available to become terminal cells at each step r in the construction of a THC. Along the way, we uncover a permutation associated with each THC. This algorithm to identify tunnel cells and produce the associated permutation is described below.

Procedure 3.2. *The following algorithm constructs a sequence of cells (which we will see are terminal cells for a tunnel hook covering) and also produces a permutation associated to each choice of cells.*

1. *Let* $\mathbb{T}_{(0)} = \{(1,1), (2,1), \ldots, (k,1)\}$.
2. *Select a cell* $\tau_1 = (p,1)$ *from* $\mathbb{T}_{(0)}$ *and set*

$$\mathbb{T}_{(1)}^{\{\tau_1\}} = \{(2,2), (3,2), \ldots, (p,2), (p+1,1), \ldots, (k,1)\};$$

   $\mathbb{T}_{(1)}^{\{\tau_1\}}$ *is the set constructed from* $\mathbb{T}_{(0)}$ *by removing* $(p,1)$ *and adding* $(1,1)$ *to each of the cells from rows lower than row* p.

3. *Let* $\sigma_1 = p - 1 + 1 = p$. *Note that* $\mathcal{L}_{\sigma_1}$ *is the diagonal containing* $(p,1)$.
4. *Repeat the following steps, once for each value of* r *from* 2 *to* k.

   (a) *Select a cell* $\tau_r = (p_r, q_r)$ *from* $\mathbb{T}_{(r-1)}^{\{\tau_1,\ldots,\tau_{r-1}\}}$.

   (b) *Construct* $\mathbb{T}_{(r)}^{\{\tau_1,\ldots,\tau_r\}}$ *from* $\mathbb{T}_{(r-1)}^{\{\tau_1,\ldots,\tau_{r-1}\}}$ *by removing* τ_r *and adding* $(1,1)$ *to each of the cells from rows lower than row* p_r.

   (c) *Set* $\sigma_r = p_r - q_r + 1$. *Note that* $\mathcal{L}_{\sigma_r}$ *is the diagonal containing* (p_r, q_r).

5. *Set* $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$.

The following lemma shows that Procedure 3.2 provides a method for selecting the terminal cells of a tunnel hook covering of any non-skew shape μ. Although the cells chosen do not depend on the diagram μ (provided μ is non-skew), the value Δ of each tunnel hook selected will vary based on the diagram μ.

Lemma 3.3. *Let* μ *be an arbitrary integer sequence with* k *parts and* ν *be the empty partition. Let* $\mu/\nu^{(r-1)}$ *be the shape obtained after* $r-1$ *iterations of the second step in Procedure 2.7 with terminal cells* $\{\tau_1, \tau_2, \ldots \tau_{r-1}\}$. *Then the cells contained in* $\mathbb{T}_{(r-1)}^{\{\tau_1,\ldots,\tau_{r-1}\}}$ *are precisely the tunnel cells in* $\mathcal{T}_{\mu/\nu^{(r-1)}}^{(r-1)}$ *for* $1 \le r-1 \le k$. *Furthermore, the sequence* $\sigma = (\sigma_1, \ldots, \sigma_k)$ *produced in step (5) of the procedure is a permutation in* S_k.

Lemma 3.3 proves that Procedure 3.2 is equivalent to the THC construction procedure, since selecting a tunnel hook starting in row r can be done by simply selecting its terminal cell. Every possible terminal cell for a tunnel hook starting in row r (for the diagram $D_{\mu/\nu^{(r-1)}}^{(r-1)}$) is included in $\mathbb{T}_{(r-1)}^{\{\tau_1,\ldots,\tau_{r-1}\}}$.

Example 3.4. *Let* σ *be a permutation of length* $k = 10$ *and consider step* $i = 6$ *in a tunnel hook covering construction of* μ. *Assume the first* 5 *tunnel hooks have been constructed with terminal cells* $\tau_1, \ldots, \tau_5$ *and* $\sigma_i \in \{2, 3, 6, 7, 9\}$ *for* $1 \le i \le 5$. *Then the tunnel cells in* $\mathbb{T}_{(5)}^{\{\tau_1,\ldots,\tau_5\}}$ *are in diagonals* $\mathcal{L}_1$, $\mathcal{L}_4$, $\mathcal{L}_5$, $\mathcal{L}_8$, *and* $\mathcal{L}_{10}$, *and, moreover,*

$$\mathbb{T}_{(5)}^{\{\tau_1,\ldots,\tau_5\}} = \{(6,6), (7,4), (8,4), (9,2), (10,1)\}.$$

If, for example, the cell $(8,4)$ is selected as the terminal cell τ_6 for the next tunnel hook, then the new collection of tunnel cells will become

$$\mathbb{T}_{(6)}^{\{\tau_1,\ldots,\tau_6\}} = \{(7,7),(8,5),(9,2),(10,1)\},$$

since $(1,1)$ is added to the first two entries of $\mathbb{T}_{(5)}^{\{\tau_1,\ldots,\tau_5\}}$ and $(8,4)$ is removed.

The following proposition is a direct consequence of Procedure 3.2 and Lemma 3.3.

Proposition 3.5. *Let $\mu = (\mu_1, \ldots, \mu_k)$ be a sequence. There is a bijection between tunnel hook coverings of the GBPR diagram for μ and permutations $\sigma \in S_k$.*

Additionally, given a permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ in one-line notation, consider the subscripts of the entries of the submatrix of $(M_\mu)_{i,j} = H_{\mu_i - i + j}$ obtained by removing row i and column σ_i for $1 \leq i \leq r-1$. These subscripts equal the Δ values of the tunnel hooks in the partial diagram obtained after $r-1$ tunnel hooks (corresponding to $\sigma_1, \sigma_2, \ldots, \sigma_{r-1}$) have been constructed.

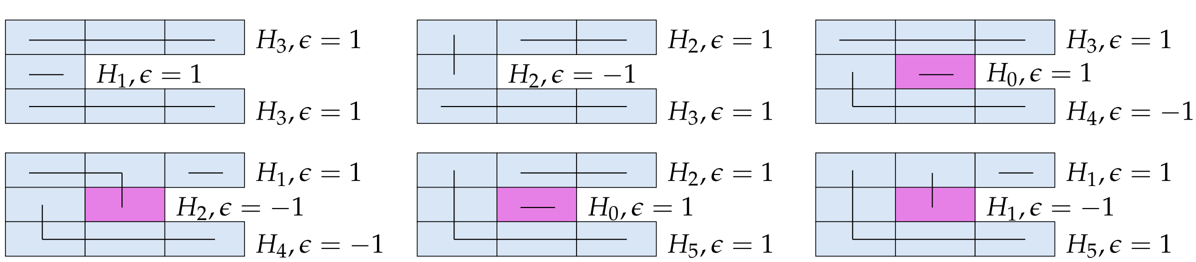
Lemma 3.6. *Let μ be a sequence of length k and let M_μ be defined as above. Assume the first $r-1$ tunnel hooks have been constructed so that their terminal cells do not lie in $\mathcal{L}_j$. Then $(M_\mu)_{r,j} = H_{\Delta(\mathfrak{h}(r,\tau_r))}$, where τ_r is the unique cell in the diagonal $\mathcal{L}_j$ that is also in $\mathbb{T}_{(r-1)}^{\{\tau_1,\ldots,\tau_{r-1}\}}$.*

We now have all the pieces we need to complete the proof of Theorem 1.3.

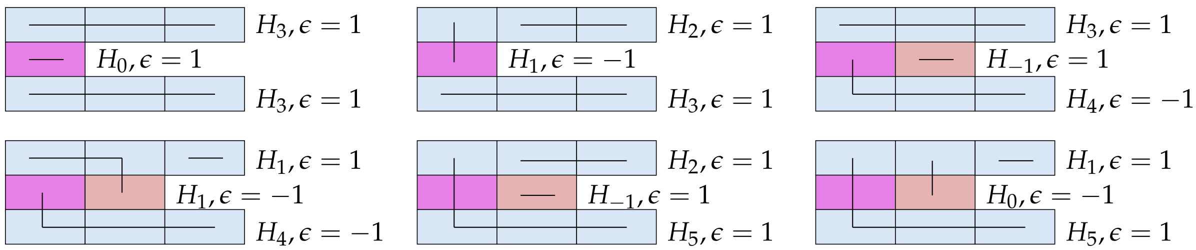
Sketch of Theorem 1.3 proof (non-skew case). Recall that $\mathfrak{S}_\mu = det(M_\mu) = \sum_{\sigma \in S_k} \epsilon(\sigma)(M_\mu)_{i,\sigma_i}$. Proposition 3.5 gives a bijection between permutations and tunnel hook coverings. In Lemma 3.6 we show that the subscripts on the entries of the matrix M_μ equal the Δ values for the corresponding tunnel hooks. All that remains is to show that $\epsilon(\sigma)$ equals the product of the signs of the tunnel hooks. This is done using a connection between the Lehmer code [8], the sign of a permutation, and the number of rows covered by a tunnel hook. □

The following three examples demonstrate how to construct the tunnel hook coverings for three different indexing shapes. Notice that the 6 configurations of terminal cells are the same in all three examples, but the shapes of the actual tunnel hooks (as well as the resulting Δ values) differ based on the initial diagram μ.

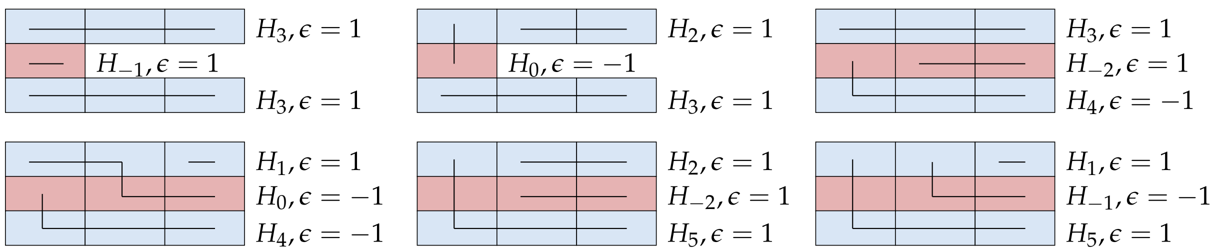
Example 3.7. $\mathfrak{S}_{(3,1,3)} = H_{(3,1,3)} - H_{(3,2,2)} - H_{(4,0,3)} + H_{(4,2,1)} + H_{(5,0,2)} - H_{(5,1,1)}$



Example 3.8. $\mathfrak{S}_{(3,0,3)} = H_{(3,0,3)} - H_{(3,1,2)} + H_{(4,1,1)} - H_{(5,0,1)}$



Example 3.9. $\mathfrak{S}_{(3,-1,3)} = -H_{(3,0,2)} + H_{(4,0,1)}$.



We now briefly discuss what happens when an immaculate is indexed by a sequence skewed by partition. First note that the partial GBPR diagrams produced during the construction of a tunnel hook covering are GBPR diagrams of skew shapes. Therefore finding the tunnel hook coverings of a shape skewed by a partition amounts to choosing an appropriate initial sequence and collection of initial tunnel hooks and then completing the remaining tunnel hook coverings. See [1] for details.

4 Immaculate functions indexed by sequences skewed by non-partition shapes

We now extend Theorem 1.3 to find the H-decomposition of $\mathfrak{S}_{\mu/\nu}$ for any pair of integer sequences $\mu, \nu \in \mathbb{Z}^k$. If all parts of ν are nonnegative, set $\hat{\nu} = (\nu_1, \nu_2, \ldots, \nu_{p-1}, \nu_{p+1} - 1, \nu_p + 1, \nu_{p+2}, \ldots, \nu_k)$ and $(M_{\mu/\hat{\nu}})_{i,j} = H_{(\mu_i - i) - (\hat{\nu}_j - j)}$, so that $(M_{\mu/\nu})_{i,j} = (M_{\mu/\hat{\nu}})_{i,j}$ for $j \notin \{p, p+1\}$. Furthermore, since $\hat{\nu}_{p+1} = \nu_p + 1$ and $\hat{\nu}_p = \nu_{p+1} - 1$, we have $\mathfrak{S}_{\mu/\nu} = -\mathfrak{S}_{\mu/\hat{\nu}}$, since we obtain $M_{\mu/\hat{\nu}}$ by swapping columns p and $p+1$ of $M_{\mu/\nu}$. If the entries of ν are not weakly decreasing, apply the straightening operator $\hat{\nu}$ (adjusting the sign each time) until the result is a partition (in which case Theorem 1.3 applies) or a sequence containing a one-step increase (in which case the skew immaculate function is zero). If any terms of ν are negative, let ν_j be the smallest part of ν. Add $-\nu_j$ to every part of μ and every part of ν; call the resulting sequences $\mathrm{aug}_{\nu_j}(\mu)$ and $\mathrm{aug}_{\nu_j}(\nu)$ respectively. Then $M_{\mathrm{aug}_{\nu_j}(\mu)/\,\mathrm{aug}_{\nu_j}(\nu)} = M_{\mu/\nu}$, so $\mathfrak{S}_{\mu/\nu} = \mathfrak{S}_{\mathrm{aug}_{\nu_j}(\mu)/\,\mathrm{aug}_{\nu_j}(\nu)}$ and we can apply the above techniques to $\mathfrak{S}_{\mathrm{aug}_{\nu_j}(\mu)/\,\mathrm{aug}_{\nu_j}(\nu)}$ to find the decomposition of $\mathfrak{S}_{\mu/\nu}$ into the complete homogeneous noncommutative symmetric functions.

Example 4.1. *Let $\mu = (2,-5,0,1)$ and $\nu = (2,-3,1,6)$. To find $\mathfrak{S}_{\mu/\nu}$, first add 3 to every part of ν and every part of μ to get $\mathrm{aug}_3(\mu)/\,\mathrm{aug}_3(\nu) = (5,-2,3,4)/(5,0,4,9)$. Next, since $\mathrm{aug}_3(\nu)$ is not a partition, apply the straightening operator to get*

$$\mathfrak{S}_{\mu/\nu} = -\,\mathfrak{S}_{(5,-2,3,4)/(5,0,8,5)} = \mathfrak{S}_{(5,-2,3,4)/(5,7,1,5)} = -\,\mathfrak{S}_{(5,-2,3,4)/(6,6,1,5)} = \mathfrak{S}_{(5,-2,3,4)/(6,6,4,2)}\,.$$

Corollary 4.2. *For $1 \leq m \leq k$, we have*

$$\mathfrak{S}_{\mu} = \sum_{\pi \in A_{k,m}} \epsilon(\pi) H_{\mu_1-1+\pi_1} \cdots H_{\mu_{m-1}-(m-1)+\pi_{m-1}}\, \mathfrak{S}_{\mu/\nu^{(m)}},$$

where $\nu^{(m)}$ is the partition obtained from the construction of the tunnel hooks corresponding to π in the diagram D_μ and $A_{k,m}$ is the set of all ordered m-element subsets of a k-element set.

For example, the skew immaculate $\mathfrak{S}_{(5,-1,3,4)/(3,1,0,0)}$ can be obtained by selecting the terms appearing in $\mathfrak{S}_{(3,3,3,5,-1,3,4)}$ whose first three terms are $H_7H_4H_2$. Applying the forgetful map to this expansion produces the expansion of a Schur function in terms of skew Schur functions with complete homogeneous symmetric functions as coefficients.

5 Ribbon decompositions of immaculate functions

Ribbons are a Schur-like basis for NSym related to Schur functions by the forgetful map; there are a number of excellent sources for background on ribbons in Sym and NSym [7, 10]. The ribbon basis expands positively into the immaculate basis via standard immaculate tableaux [2], but the expansion of the immaculate basis into the ribbon basis is only known for certain special cases. Campbell [4] provides formulas for the ribbon expansion of immaculate functions indexed by rectangles and immaculate functions indexed by products of two rectangles satisfying certain size conditions as follows.

Theorem 5.1. *[4] The ribbon expansion of an immaculate function indexed by either a rectangle $\alpha = m^{\ell(\alpha)}$ or the product $\alpha = (a^b, c^d)$ of rectangles satisfying $b \leq c$ and $b \leq a$ is given by*

$$\mathfrak{S}_{\alpha} = \sum_{\sigma \in S_{\ell(\alpha)}} \epsilon(\sigma) R_{(\alpha_1-1+\sigma_1, \alpha_2-2+\sigma_2, \ldots, \alpha_{\ell(\alpha)}-\ell(\alpha)+\sigma_{\ell(\alpha)})}.$$

It is not true in general that the ribbon decomposition has the same indexing set as the homogeneous decomposition. One open question is to classify the compositions for which the indexing compositions are the same. We use tunnel hooks to extend Campbell's results to a much larger class of compositions.

Theorem 5.2. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition for which there exists j with $1 \leq j \leq k$ such that $\alpha_i \geq i$ for $1 \leq i \leq j$, $\alpha_{j+1} \geq j$, and $\alpha_\ell = \alpha_{j+1}$ for $j+2 \leq \ell \leq k$. Then*

$$\mathfrak{S}_\alpha = \sum_{\sigma \in S_k} \epsilon(\sigma) R_{(\alpha_1 - 1 + \sigma_1, \alpha_2 - 2 + \sigma_2, \ldots, \alpha_k - k + \sigma_k)},$$

with the convention that R_α vanishes if α contains any nonpositive parts.

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