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Polyhedral and Tropical Geometry of Flag Positroids

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Abstract. A *flag positroid* of ranks $\mathbf{r} := (r_1 < \cdots < r_k)$ on $[n]$ is a flag matroid that can be realized by a real $r_k \times n$ matrix A such that the $r_i \times r_i$ minors of A involving rows $1, 2, \ldots, r_i$ are nonnegative for all $1 \leq i \leq k$. In this abstract, we explore the polyhedral and tropical geometry of flag positroids, particularly when $\mathbf{r} := (a, a+1, \ldots, b)$ is a sequence of consecutive numbers. In this case we show that the nonnegative tropical flag variety $\mathrm{TrFl}^{\geq 0}_{\mathbf{r},n}$ equals the nonnegative flag Dressian $\mathrm{FlDr}^{\geq 0}_{\mathbf{r},n}$, and that the points $\boldsymbol{\mu} = (\mu_a, \ldots, \mu_b)$ of $\mathrm{TrFl}^{\geq 0}_{\mathbf{r},n} = \mathrm{FlDr}^{\geq 0}_{\mathbf{r},n}$ give rise to coherent subdivisions of flag positroid polytopes into (smaller) flag positroid polytopes. Our results have applications to Bruhat interval polytopes. For example, we show that a complete flag matroid polytope is a Bruhat interval polytope if and only if its (≤ 2)-dimensional faces are Bruhat interval polytopes. Our results also have applications to realizability questions. We define a *positively oriented flag matroid* to be a sequence of positively oriented matroids $(\chi_1, \ldots, \chi_k)$ which is also an oriented flag matroid. We then prove that every positively oriented flag matroid of ranks $\mathbf{r} = (a, a+1, \ldots, b)$ is realizable.

Keywords: Flag varieties, Tropical varieties, Positroids, Polytopal subdivisons, Matroid polytopes, Oriented flag matroids

1 Introduction

In recent years there has been a great deal of interest in the tropical Grassmannian [30, 19, 20, 14, 11], and subdivisions of matroid polytopes [33, 1, 15]. Further, there has been research into "positive" [29, 31, 2, 24, 25, 32, 4] and "flag" [35, 13, 12, 21, 22, 8] versions of the above objects. The aim of this paper is to illustrate the beautiful relationships between the nonnegative tropical flag variety, the nonnegative flag Dressian, and subdivisions of flag positroid polytopes, unifying and generalizing some of the existing results. We will

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particularly focus on the case of flag varieties (respectively, flag positroids) consisting of subspaces (respectively, positroids) of *consecutive* ranks. This includes both Grassmannians and complete flag varieties.

For positive integers n and d with $d < n$, we let $[n]$ denote the set $\{1, \dots, n\}$ and we let $\binom{[n]}{d}$ denote the collection of all d-element subsets of $[n]$. Given a subset $S \subseteq [n]$ we let $\mathbf{e}_S$ denote the sum of standard basis vectors $\sum_{i \in S} \mathbf{e}_i$. For a collection $\mathcal{B} \subset \binom{[n]}{d}$, we let $P(\mathcal{B}) =$ the convex hull of $\{\mathbf{e}_B : B \in \mathcal{B}\}$ in $\mathbb{R}^n$. The collection $\mathcal{B}$ is said to define a *matroid M of rank d on $[n]$* if every edge of the polytope $P(\mathcal{B})$ is parallel to $\mathbf{e}_i - \mathbf{e}_j$ for some $i \neq j \in [n]$. In this case, we call $\mathcal{B}$ the set of *bases* of M, and define the *matroid polytope $P(M)$* of M to be the polytope $P(\mathcal{B})$. When $\mathcal{B}$ indexes the nonvanishing Plücker coordinates of an element A of the Grassmannian $\mathrm{Gr}_{d,n}(\mathbb{C})$, we say that A *realizes* M, and it is well-known that $P(\mathcal{B})$ is the moment map image of the closure of the torus orbit of A in the Grassmannian [17]. We assume familiarity with the fundamentals of matroid theory as in [28] and [10].

The above definition of a matroid in terms of its polytope is due to [17]. Flag matroids are natural generalizations of matroids that admit the following polytopal definition.

Definition 1.1. [10, Corollary 1.13.5 and Theorem 1.13.6] Let $\mathbf{r} = (r_1, \dots, r_k)$ be a sequence of increasing integers in $[n]$. A *flag matroid* of ranks $\mathbf{r}$ on $[n]$ is a sequence $\boldsymbol{M} = (M_1, \dots, M_k)$ of matroids of ranks $(r_1, \dots, r_k)$ on $[n]$ such that all vertices of the polytope

$$P(\boldsymbol{M}) = P(M_1) + \cdots + P(M_k), \text{ the Minkowski sum of matroid polytopes,}$$

are equidistant from the origin. This polytope is called the *flag matroid polytope* of $\boldsymbol{M}$.

Flag matroids are exactly the type A objects in the theory of Coxeter matroids [18, 10]. Just as a realization of a matroid is a point in a Grassmannian, a realization of a flag matroid is a point in a flag variety. More concretely, a *realization* of a flag matroid of ranks $(r_1, \dots, r_k)$ over a field is an $r_k \times n$ matrix A over that field such that for each $1 \leq i \leq k$, the first r_i rows of A is a realization of M_i. For an equivalent definition of flag matroids in terms of Plücker relations on partial flag varieties, see [21, Proposition A]. There are natural "positive" analogues of matroids, flag matroids, and their polytopes.

Definition 1.2. Let $\mathbf{r} = (r_1, \cdots, r_k)$ be a sequence of increasing integers in $[n]$. We say that a flag matroid $(M_1, \dots, M_k)$ of ranks $\mathbf{r}$ on $[n]$ is a *flag positroid* if it has a realization by a real matrix A such that the $r_i \times n$ submatrix of A formed by the first r_i rows of A has all nonnegative minors for each $1 \leq i \leq k$.

We refer to the flag matroid polytope of a flag positroid as a *flag positroid polytope*. It follows from our definition above that flag positroids are realizable.

Setting $k = 1$ in Definition 1.2 gives the well-studied notion of *positroids* and *positroid polytopes* [29, 2]. Therefore each flag positroid is a sequence of positroids.

In recent years it has been gradually understood that the tropical geometry of the Grassmannian and flag variety, and in particular, the *Dressian* and *flag Dressian*, are intimately connected to (flag) matroid polytopes and their subdivisions [33, 19, 13, 27].

The tropical geometry of the *positive* Grassmannian and of the flag variety are particularly nice: the positive tropical Grassmannian equals the positive Dressian, whose cones in turn parameterize subdivisions of the hypersimplex into positroid polytopes [31, 32, 25, 4], while the positive tropical complete flag variety equals the positive complete flag Dressian, whose cones parameterize subdivisions of the permutohedron into *Bruhat interval polytopes* [8, 22].

Our main theorem, which follows, unifies and generalizes the above results. In particular, Theorem 1.3 applies to the *nonnegative* flag variety of rank $\mathbf{r}$, where $\mathbf{r}$ consists of any consecutive integers. This includes both the Grassmannian case, when $\mathbf{r} = (d)$ and the complete flag case, when $\mathbf{r} = (1, 2 \ldots, n)$. Much of the terminology appearing in this theorem statement will be defined in Section 2.

Theorem 1.3. Suppose $\mathbf{r}$ is a sequence of consecutive integers $(a, \ldots, b)$ for some $1 \leq a \leq b \leq n$. Then, for $\boldsymbol{\mu} = (\mu_a, \ldots, \mu_b) \in \prod_{i=a}^{b} \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{i}}\right)$, the following are equivalent:

(a) $\boldsymbol{\mu} \in \mathrm{TrFl}^{\geq 0}_{\mathbf{r},n}$, the nonnegative tropicalization of the flag variety, i.e. the closure of the coordinate-wise valuation of points in $\mathrm{Fl}_{\mathbf{r},n}(\mathcal{C}_{\geq 0})$.

(b) $\boldsymbol{\mu} \in \mathrm{FlDr}^{\geq 0}_{\mathbf{r},n}$, the nonnegative flag Dressian, i.e. the "solutions" to the positive-tropical Grassmann-Plücker and incidence-Plücker relations.

(c) Every face in the coherent subdivision $\mathcal{D}_{\boldsymbol{\mu}}$ of the polytope $P(\underline{\boldsymbol{\mu}}) = P(\underline{\mu_1}) + \cdots + P(\underline{\mu_k})$ induced by $\boldsymbol{\mu}$ is a flag positroid polytope (of rank $\mathbf{r}$), where $\underline{\mu_i}$ is the support of μ_i for $1 \leq i \leq k$.

(d) Every face of dimension at most 2 in the subdivision $\mathcal{D}_{\boldsymbol{\mu}}$ of $P(\underline{\boldsymbol{\mu}})$ is a flag positroid polytope (of rank $\mathbf{r}$).

(e) The support $\underline{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ is a flag matroid, and $\boldsymbol{\mu}$ satisfies every three-term positive-tropical incidence relation (respectively, Grassmann-Plücker relation) when $a < b$ (respectively, $a = b$).

We present here three useful corollaries of this result.

Corollary 1.4. Suppose $\mathbf{r}$ is a sequence of consecutive integers and let $\boldsymbol{\mu} \in \mathrm{FlDr}^{\geq 0}_{\mathbf{r},n}$. Then $P(\underline{\boldsymbol{\mu}})$ is a flag positroid polytope.

We note that in the Grassmannian case, that is, the case that $\mathbf{r} = (d)$ is a single integer, Theorem 1.3 and Corollary 1.4 describe the relationship between the nonnegative tropical Grassmannian, the nonnegative Dressian, and subdivisions of positroid polytopes (e.g. the hypersimplex, if $\boldsymbol{\mu}$ has no coordinates equal to ∞) into positroid polytopes. Moreover, when $\mathbf{r} = (1, 2, \ldots, n)$, Theorem 1.3 and Corollary 1.4 describe the relationship between the nonnegative tropical complete flag variety, the nonnegative complete flag Dressian, and subdivisions of Bruhat interval polytopes (e.g. the permutohedron, if $\boldsymbol{\mu}$ has no coordinates equal to ∞) into Bruhat interval polytopes. We illustrate this relationship in the case where $\boldsymbol{\mu}$ has no coordinates equal to ∞ in Figure 1.

We now present an application of Theorem 1.3 to flag matroid polytopes.

Corollary 1.5. For a flag matroid $\boldsymbol{M} = (M_a, M_{a+1}, \dots, M_b)$ of consecutive ranks $\mathbf{r}$, its flag matroid polytope $P(\boldsymbol{M})$ is a flag positroid polytope if and only if its (≤ 2)-dimensional faces are flag positroid polytopes (of rank $\mathbf{r}$).

In the Grassmannian case, the flag positroid polytopes of rank $\mathbf{r} = (d)$ are precisely the positroid polytopes and Corollary 1.5 appeared as [25, Theorem 3.9].

It has been shown that all *positively oriented matroids* are positroids, which is to say, realizable by a matrix with nonnegative maximal minors [3][32]. It is natural to ask when a sequence of positroids is a flag positroid, which is to say, realizable as in Definition 1.2. Note however that questions of realizability for flag matroids are rather subtle, as demonstrated in Example 2.14. By working with *oriented* flag matroids, we give an answer to this realizability question for flag positroids, in the case of consecutive ranks.

Corollary 1.6. Suppose $(M_1, \dots, M_k)$ is a sequence of positroids on $[n]$ of consecutive ranks. Then, considered as a sequence of positively oriented matroids, $(M_1, \dots, M_k)$ is a flag positroid if and only if it is an oriented flag matroid.

We define a *positively oriented flag matroid* to be a sequence of positively oriented matroids $(\chi_1, \dots, \chi_k)$ which is also an oriented flag matroid. Corollary 1.6 then says that every positively oriented flag matroid of consecutive ranks is realizable. We do not know whether Corollary 1.6 holds if $\mathbf{r}$ fails to satisfy the consecutive rank condition.

The structure of this abstract is as follows. In Section 2, we define the terms and concepts appearing in Theorem 1.3. In Section 3, we present further results related to Theorem 1.3, including an application to generalized Bruhat interval polytopes and comments on what can be said about general $\mathbf{r}$. This is an extended abstract of the full article [9].

2 Background and Definitions

2.1 Nonnegative Flag Varieties

Let $n \in \mathbb{Z}_+$ and let $\mathbf{r} = \{r_1 < \cdots < r_k\} \subseteq [n]$. For a field $\Bbbk$, let $G = \mathrm{GL}_n(\Bbbk)$, and let $\mathrm{P}_{\mathbf{r};n}(\Bbbk)$ denote the parabolic subgroup of G of block upper-triangular matrices with diagonal blocks of sizes $r_1, r_2 - r_1, \dots, r_k - r_{k-1}, n - r_k$. We define the *partial flag variety*

$$\mathrm{Fl}_{\mathbf{r};n}(\Bbbk) := \mathrm{GL}_n(\Bbbk) / \mathrm{P}_{\mathbf{r};n}(\Bbbk).$$

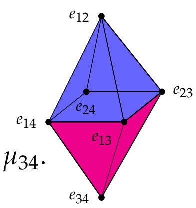
As usual, we identify $\mathrm{Fl}_{\mathbf{r};n}(\Bbbk)$ with the variety of partial flags of subspaces in $\Bbbk^n$:

$$\mathrm{Fl}_{\mathbf{r};n}(\Bbbk) = \{(V_1 \subset \cdots \subset V_k) : V_i \text{ a linear subspace of } \Bbbk^n \text{ of dimension } r_i \text{ for } i = 1, \dots, k\}$$

Let $n = 4$, $\mathbf{r} = (2)$,

$(\mu_I) \in \mathbb{R}^{\binom{[4]}{2}}$ such that

$\mu_{13} + \mu_{24} = \mu_{23} + \mu_{14} < \mu_{12} + \mu_{34}$.



Let $n = 3$, $\mathbf{r} = (1, 2, 3)$,

$(\mu_I) \in \mathbb{R}^{2^3}$ such that

$\mu_2 + \mu_{13} = \mu_1 + \mu_{23} < \mu_3 + \mu_{12}$.

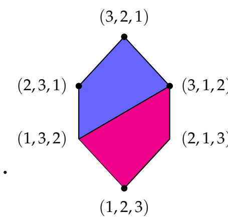


Figure 1: Coherent subdivision of the $n = 4$, $r = 2$ hypersimplex into positroid polytopes induced by a point $\boldsymbol{\mu} \in \mathrm{FlDr}^{\geq 0}_{(2),4}$ (left) and coherent subdivision of the $n = 3$ permutohedron into flag positroid polytopes induced by a point $\boldsymbol{\mu} \in \mathrm{FlDr}^{\geq 0}_{(1,2,3),3}$ (right).

We write $\mathrm{Fl}_n(\mathbb{k})$ for the complete flag variety $\mathrm{Fl}_{1,2,\ldots,n;n}(\mathbb{k})$. Note that $\mathrm{Fl}_n(\mathbb{k})$ can be identified with $\mathrm{GL}_n(\mathbb{k})/B(\mathbb{k})$, where $B(\mathbb{k})$ is the subgroup of upper-triangular matrices. There is a natural projection π from $\mathrm{Fl}_n(\mathbb{k})$ to any partial flag variety by simply forgetting some of the subspaces.

If A is an $r_k \times n$ matrix such that V_{r_i} is the span of the first r_i rows, we say that A is a *realization* of $V := (V_1 \subset \cdots \subset V_k) \in \mathrm{Fl}_{\mathbf{r};n}$. Given any realization A of V and any $1 \leq i \leq k$, we have the *Plücker coordinates* or *flag minors* $p_I(A)$ where $I \in \binom{[n]}{r_i}$; concretely, $p_I(A)$ is the determinant of the submatrix of A occupying the first r_i rows and columns I. This gives the Plücker embedding of $\mathrm{Fl}_{\mathbf{r};n}(\mathbb{k})$ into $\mathbb{P}^{\binom{[n]}{r_1}-1} \times \cdots \times \mathbb{P}^{\binom{[n]}{r_k}-1}$ taking V to $\left((p_I(A))_{I \in \binom{[n]}{r_1}}, \ldots, (p_I(A))_{I \in \binom{[n]}{r_k}} \right)$.

Definition 2.1. For integers $0 < r \leq s < n$, the *(single-exchange) Plücker relations of type* $(r, s; n)$ are polynomials in variables $\{x_I : I \in \binom{[n]}{r} \cup \binom{[n]}{s}\}$ defined as

$$\mathscr{P}_{r,s;n} = \left\{ \sum_{j \in J \setminus I} \mathrm{sign}(j, I, J) x_{I \cup j} x_{J \setminus j} \,\middle|\, I \in \binom{[n]}{r-1},\ J \in \binom{[n]}{s+1} \right\},$$

where $\mathrm{sign}(j, I, J) = (-1)^{|\{k \in J | k < j\}| + |\{i \in I | j < i\}|}$. When $r = s$, the elements of $\mathscr{P}_{r,r;n}$ are called the *Grassmann-Plücker relations* (of type $(r; n)$), and when $r < s$, the elements of $\mathscr{P}_{r,s;n}$ are called the *incidence-Plücker relations* (of type $(r, s; n)$).

As in the introduction, let $\mathbf{r} = (r_1 < \cdots < r_k)$ be a sequence of increasing integers in $[n]$. We let $\mathscr{P}_{\mathbf{r};n} = \bigcup_{\substack{r \leq s \\ r,s \in \mathbf{r}}} \mathscr{P}_{r,s;n}$, and let $\langle \mathscr{P}_{\mathbf{r};n} \rangle$ be the ideal generated by $\mathscr{P}_{\mathbf{r};n}$. It is well-known that for any field $\mathbb{k}$ the ideal $\langle \mathscr{P}_{\mathbf{r};n} \rangle$ set-theoretically carves out the partial flag variety $\mathrm{Fl}_{\mathbf{r};n}(\mathbb{k})$ embedded in $\prod_{i=1}^{k} \mathbb{P}\left(\mathbb{k}^{\binom{[n]}{r_i}}\right)$ via the Plücker embedding [16, §9].

We refer to the subset of relations in $\mathscr{P}_{\mathbf{r};n}$ which are polynomial relations containing precisely three terms as the *three-term Plücker relations*. We will now often drop the $\Bbbk$ from our notation, with the understanding that $\Bbbk = \mathbb{R}$.

Definition 2.2. We say that a real matrix is *totally positive* if all of its minors are positive. We let $\mathrm{GL}_n^{>0}$ denote the subset of GL_n of totally positive matrices.

There are two natural ways to define positivity for partial flag varieties. The first notion comes from work of Lusztig [26]. The second notion uses Plücker coordinates, and was initiated in work of Postnikov [29].

Definition 2.3. We define the *(Lusztig) positive part* of $\mathrm{Fl}_{\mathbf{r};n}$, denoted by $\mathrm{Fl}_{\mathbf{r};n}^{>0}$, as the image of $\mathrm{GL}_n^{>0}$ inside $\mathrm{Fl}_{\mathbf{r};n} = \mathrm{GL}_n \,/\, \mathrm{P}_{\mathbf{r};n}$. We define the *(Lusztig) nonnegative part* of $\mathrm{Fl}_{\mathbf{r};n}$, denoted by $\mathrm{Fl}_{\mathbf{r};n}^{\geq 0}$, as the closure of $\mathrm{Fl}_{\mathbf{r};n}^{>0}$ in the Euclidean topology.

We define the *Plücker positive part* (respectively, *Plücker nonnegative part*) of $\mathrm{Fl}_{\mathbf{r};n}$ to be the subset of $\mathrm{Fl}_{\mathbf{r};n}$ where all Plücker coordinates are positive (respectively, nonnegative).[1]

It is well-known that the Lusztig positive part of $\mathrm{Fl}_{\mathbf{r};n}$ is a subset of the Plücker positive part of $\mathrm{Fl}_{\mathbf{r};n}$, and that the two notions agree in the case of the Grassmannian [34, Corollary 1.2]. The two notions also agree in the case of the complete flag variety [8, Theorem 5.21]. More generally, we have the following.

Theorem 2.4. [7, Theorem 1.1] The Lusztig positive (respectively, Lusztig nonnegative) part of $\mathrm{Fl}_{\mathbf{r};n}$ equals the Plücker positive (respectively, Plücker nonnegative) part of $\mathrm{Fl}_{\mathbf{r};n}$ if and only if the set $\mathbf{r}$ consists of consecutive integers.

See [7, Section 1.4] for more references and a nice discussion of the history. Since in this abstract we will be mainly studying the case where $\mathbf{r}$ consists of consecutive integers, we will use the two notions interchangeably when there is no ambiguity.

2.2 Tropical Geometry

Before discussing tropical varieties, we introduce the tropical hyperfield and what it means to find a solution of a tropical polynomial. For more details, see [27] and [31].

Definition 2.5. Let $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ be the set underlying the tropical hyperfield, endowed with the topology such that $-\log : \mathbb{R}_{\geq 0} \to \mathbb{T}$ is a homeomorphism. Given a point $w \in \mathbb{T}^{\binom{[n]}{r}}$, we define the *support* of w to be $\underline{w} = \{S \in \binom{[n]}{r} : w_S \neq \infty\}$. When $\underline{w}$ is the set of bases of a matroid, we identify $\underline{w}$ with that matroid. Let $\mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r}}\right)$ be the tropical projective space of $\mathbb{T}^{\binom{[n]}{r}}$, which is defined as $\left(\mathbb{T}^{\binom{[n]}{r}} \setminus \{(\infty, \ldots, \infty)\}\right) / \sim$, where $w \sim w'$ if $w = w' + (c, \ldots, c)$ for some $c \in \mathbb{R}$.

[1] The reader who is concerned about the fact that we are working with projective coordinates can replace "all Plücker coordinates are nonnegative" by "all nonzero Plücker coordinates have the same sign".

For a point $w = (w_1, \ldots, w_m) \in \mathbb{T}^m \setminus \{(\infty, \ldots, \infty)\}$, we write $\overline{w}$ for its image in $\mathbb{P}(\mathbb{T}^m)$. For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^m$, write $a \bullet w = a_1 w_1 + \cdots + a_m w_m$.

Definition 2.6. For a real homogeneous polynomial

$$f = \sum_{a \in \mathcal{A}} c_a x^a \in \mathbb{R}[x_1, \ldots, x_m], \quad \text{where } \mathcal{A} \subset \mathbb{Z}^m_{\geq 0} \text{ is finite and } c_a \in \mathbb{R} \setminus \{0\} \text{ for all } a \in \mathcal{A},$$

the *extended tropical hypersurface* $V_{\mathrm{trop}}(f)$ and the *nonnegative tropical hypersurface* $V^{\geq 0}_{\mathrm{trop}}(f)$ are subsets of the tropical projective space $\mathbb{P}(\mathbb{T}^m)$ defined by

$$V_{\mathrm{trop}}(f) = \left\{ \overline{w} \in \mathbb{P}(\mathbb{T}^m) \;\middle|\; \text{the minimum in } \min_{a \in \mathcal{A}}(a \bullet w) \text{, if finite, occurs at least twice} \right\},$$

and

$$V^{\geq 0}_{\mathrm{trop}}(f) = \left\{ \overline{w} \in \mathbb{P}(\mathbb{T}^m) \;\middle|\; \begin{array}{c} \text{the minimum in } \min_{a \in \mathcal{A}}(a \bullet w) \text{, if finite, occurs at least twice,} \\ \text{at some } a, a' \in \mathcal{A} \text{ such that } c_a \text{ and } c_{a'} \text{ have opposite signs} \end{array} \right\}.$$

We say that a point *satisfies the tropical relation* of f if it is in $V_{\mathrm{trop}}(f)$, and that it *satisfies the positive-tropical relation* of f if it is in $V^{\geq 0}_{\mathrm{trop}}(f)$.

When f is a multihomogeneous real polynomial, we define $V_{\mathrm{trop}}(f)$ and $V^{\geq 0}_{\mathrm{trop}}(f)$ similarly as subsets of a product of tropical projective spaces. We will consider tropical hypersurfaces of Plücker relations, yielding tropical analogues of partial flag varieties.

Definition 2.7. The *tropicalization* $\mathrm{TrFl}_{\mathbf{r};n}$ of $\mathrm{Fl}_{\mathbf{r};n}$, the *nonnegative tropicalization* $\mathrm{TrFl}^{\geq 0}_{\mathbf{r};n}$ of $\mathrm{Fl}_{\mathbf{r};n}$, the *flag Dressian* $\mathrm{FlDr}_{\mathbf{r};n}$, and the *nonnegative flag Dressian* $\mathrm{FlDr}^{\geq 0}_{\mathbf{r};n}$ are subsets of $\prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r_i}}\right)$ defined as

$$\mathrm{TrFl}_{\mathbf{r};n} = \bigcap_{f \in \langle \mathscr{P}_{\mathbf{r};n} \rangle} V_{\mathrm{trop}}(f) \quad \text{and} \quad \mathrm{TrFl}^{\geq 0}_{\mathbf{r};n} = \bigcap_{f \in \langle \mathscr{P}_{\mathbf{r};n} \rangle} V^{\geq 0}_{\mathrm{trop}}(f),$$

$$\mathrm{FlDr}_{\mathbf{r};n} = \bigcap_{f \in \mathscr{P}_{\mathbf{r};n}} V_{\mathrm{trop}}(f) \quad \text{and} \quad \mathrm{FlDr}^{\geq 0}_{\mathbf{r};n} = \bigcap_{f \in \mathscr{P}_{\mathbf{r};n}} V^{\geq 0}_{\mathrm{trop}}(f).$$

When $\mathbf{r} = (d)$ consists of a single integer, one obtains the (nonnegative) tropicalization of the Grassmannian and the (nonnegative) Dressian studied in [30, 31, 32, 4].

Remark 2.8. We record a useful equivalent description of the (nonnegative) tropicalization of a partial flag variety using Puiseux series.

Definition 2.9. Let $\mathcal{C} = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with coefficients in $\mathbb{C}$, with the usual valuation map $\mathrm{val} : \mathcal{C} \to \mathbb{T}$. Concretely, for $f \neq 0$, $\mathrm{val}(f)$ is the exponent of the initial term of f, and $\mathrm{val}(0) = \infty$. Let

$$\mathcal{C}_{>0} = \{f \in \mathcal{C} : \text{the initial coefficient of } f \text{ is real and positive}\} \text{ and } \mathcal{C}_{\geq 0} = \mathcal{C}_{>0} \cup \{0\}.$$

The valuation val can also be applied coordinate wise, yielding a map $\mathrm{val} : \mathcal{C}^n \to \mathbb{T}^n$,

Proposition 2.10. [27, Theorems 3.2.3 & 6.2.15][31, Proposition 2.2] We have

$$\mathrm{TrFl}_{\mathbf{r};n} = \text{the closure of } \{\mathrm{val}(p) : \{(0,\dots,0)\} \neq p \in \mathrm{Fl}_{\mathbf{r};n}(\mathcal{C})\} \text{ in } \textstyle\prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r_i}}\right) \quad \text{and}$$

$$\mathrm{TrFl}^{\geq 0}_{\mathbf{r};n} = \text{the closure of } \{\mathrm{val}(p) : \{(0,\dots,0)\} \neq p \in \mathrm{Fl}_{\mathbf{r};n}(\mathcal{C}_{\geq 0})\} \text{ in } \textstyle\prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r_i}}\right).$$

2.3 Coherent Subdivisions

Consider a point $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in \prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r_i}}\right)$ such that its support $\underline{\boldsymbol{\mu}}$ is a flag matroid. By construction, the vertices of the flag matroid polytope $P(\underline{\boldsymbol{\mu}})$ have the form $\mathbf{e}_{B_1} + \cdots + \mathbf{e}_{B_k}$ where B_i is a basis of the matroid $\underline{\mu_i}$ for each $i = 1, \dots, k$.

Definition 2.11. We define $\mathcal{D}_{\boldsymbol{\mu}}$ to be the coherent subdivision of $P(\underline{\boldsymbol{\mu}})$ induced by assigning each vertex $\mathbf{e}_{B_1} + \cdots + \mathbf{e}_{B_k}$ of $P(\underline{\boldsymbol{\mu}})$ the weight $\mu_1(B_1) + \cdots + \mu_k(B_k)$. That is, the faces of $\mathcal{D}_{\boldsymbol{\mu}}$ correspond to the faces of the lower convex hull of the set of points

$$\{(\mathbf{e}_{B_1} + \cdots + \mathbf{e}_{B_k}, \mu_1(B_1) + \cdots + \mu_k(B_k)) \in \mathbb{R}^n \times \mathbb{R} : \mathbf{e}_{B_1} + \cdots + \mathbf{e}_{B_k} \text{ a vertex of } P(\underline{\boldsymbol{\mu}})\}.$$

In [13, Theorem A], it is shown that a point $\boldsymbol{\mu} \in \prod_{i=1}^k \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r_i}}\right)$ is in the flag Dressian $\mathrm{FlDr}_{\mathbf{r};n}$ if and only if the all faces of the subdivision $\mathcal{D}_{\boldsymbol{\mu}}$ are flag matroid polytopes.

2.4 Oriented Flag Matroids

Let $\mathbb{S} = \{-1, 0, 1\}$ be the *hyperfield of signs*, introduced in [5]. For a polynomial $f = \sum_{a \in \mathcal{A}} c_a x^a \in \mathbb{R}[x_1, \dots, x_m]$, we say that an element $\chi \in \mathbb{S}^m$ is *in the null set* of f if the set $\{\mathrm{sign}(c_a)\chi^a\}_{a \in \mathcal{A}}$ is either $\{0\}$ or contains $\{-1, 1\}$.

Definition 2.12. An *oriented matroid* of rank r on $[n]$ is a point $\chi \in \mathbb{S}^{\binom{[n]}{r}}$, called a *chirotope*, which is in the null set of each $f \in \mathscr{P}_{r,r;n}$. Similarly, an *oriented flag matroid* of ranks $\mathbf{r}$ is a point $\boldsymbol{\chi} = (\chi_1, \dots, \chi_k) \in \prod_{i=1}^k \mathbb{S}^{\binom{[n]}{r_i}}$ such that $\boldsymbol{\chi}$ is in the null set of each $f \in \mathscr{P}_{\mathbf{r};n}$.

While these definitions may seem different from those in the standard reference [6] on oriented matroids, one can show that Definition 2.12 is equivalent to [6, Definition 3.5.3].

Definition 2.13. A *positively oriented matroid* is an oriented matroid χ such that χ only takes values 0 or 1. Similarly, we define a *positively oriented flag matroid* to be an oriented flag matroid $\boldsymbol{\chi}$ such that $\boldsymbol{\chi}$ only takes values 0 or 1.

A positroid M defines a positively oriented matroid $\chi = \chi_M$ where χ takes value 1 on its bases and 0 otherwise. Every positively oriented matroid χ is realizable, i.e. has the form χ_M for some positroid M [3]. Thus, each positively oriented flag matroid is a sequence of positroids which is also an oriented flag matroid. However, we remark that imposing the oriented flag matroid condition, as in Corollary 1.6, is stronger than imposing that we have a realizable flag matroid whose consistent matroids are positroids.

Example 2.14. We give an example of a realizable flag matroid that has positroids as its constituent matroids but is not a flag positroid. This example also appeared in [22, Example 5] and [7, Example 6]. Let (M, M') be matroids of ranks 1 and 2 on [3] whose sets of bases are $\{1, 3\}$ and $\{12, 13, 23\}$, respectively. Both are positroids. We can realize (M, M') as a flag matroid using the matrix

$$\begin{bmatrix} a & 0 & b \\ c & d & e \end{bmatrix},$$

where the minors $a, b, ad, -bd, ae - bc$ are nonzero. In order to realize (M, M') as a flag positroid, we need all these minors to be strictly positive, which is impossible. Since (M, M') does *not* form an oriented flag matroid, this is consistent with Corollary 1.6.

3 Further Results

We begin by presenting two results that are crucial to the proof Theorem 1.3, and which are also interesting in their own rights.

Theorem 3.1. Suppose $\boldsymbol{\mu} = (\mu_1, \mu_2) \in \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r}}\right) \times \mathbb{P}\left(\mathbb{T}^{\binom{[n]}{r+1}}\right)$ satisfies every three-term positive-tropical incidence relation. If the support $\underline{\boldsymbol{\mu}}$ is a flag matroid, then $\boldsymbol{\mu} \in \mathrm{FlDr}^{\geq 0}_{r,r+1;n}$.

We also expect that our proof of Theorem 3.1 in [9] adapts well to arbitrary perfect hyperfields, in the language of [5]. The following result gives a characterization of positively oriented flag matroids (and equivalently, by Corollary 1.6, of flag positroids).

Theorem 3.2. The set of positively oriented flag matroids of ranks $\mathbf{r}$ can be identified with the set of points of the nonnegative flag Dressian $\mathrm{FlDr}^{\geq 0}_{\mathbf{r};n}$ whose coordinates are all either 0 or ∞.

We continue with a further corollary of Theorem 1.3. A *generalized Bruhat interval polytope* [35, Definition 7.8 and Lemma 7.9] can be defined as the moment map image of the closure of the torus orbit of a point A in the nonnegative part $(G/P)^{\geq 0}$ (in the sense of Lusztig) of a flag variety G/P. When $\mathbf{r}$ is a sequence of consecutive integers, it then follows from [7] that generalized Bruhat interval polytopes for $\mathrm{Fl}^{\geq 0}_{\mathbf{r};n}$ are precisely the flag positroid polytopes of ranks $\mathbf{r}$. In the complete flag case, a generalized Bruhat interval polytope is just a *Bruhat interval polytope* [23], that is, the convex hull of the permutation vectors $(z(1), \ldots, z(n))$ for all permutations z lying in some Bruhat interval $[u, v]$. Thus, we can now restate Corollary 1.5 as follows.

Corollary 3.3. For a flag matroid on $[n]$ of consecutive ranks $\mathbf{r}$, its flag matroid polytope is a generalized Bruhat interval polytope if and only if its (≤ 2)-dimensional faces are generalized Bruhat interval polytopes. In particular, for a complete flag matroid on $[n]$, its flag matroid polytope is a Bruhat interval polytope if and only if its (≤ 2)-dimensional faces are Bruhat interval polytopes.

As a final point, we comment on what happens when $\mathbf{r}$ does not consist of consecutive integers. A key fact we use about flag positroids of consecutive ranks is that they can be extended to complete flag positroids. This is no longer necessarily true if $\mathbf{r}$ does not consist of consecutive integers.

Example 3.4. Let the sets of bases of M and M' be $\{1, 2, 3, 4\}$ and $\{123, 234\}$, respectively. These form a flag positroid (M, M') on $[4]$ of ranks $(1, 3)$. After row reduction, a realization of this flag matroid can be written as

$$\begin{bmatrix} 1 & a & b & c \\ 0 & x & y & 0 \\ 0 & z & w & 0 \end{bmatrix}$$

where $a, b, c > 0$ and $xw - yz > 0$. The minors of the matrix formed by the first two rows include $x, y, -cx, -cy$, which cannot be all nonnegative since $c > 0$ and not both of x and y are zero. Thus, there is no flag positroid (M, M_2, M') with rank of M_2 equal to 2.

Passing through complete flag positroids is a key step in our proof of Theorem 1.3 and, in fact, most of Theorem 1.3 no longer holds if $\mathbf{r}$ does not consist of consecutive integers. However, we still have the following:

Lemma 3.5. For any $\mathbf{r}$, the modification of (c) of Theorem 1.3 to:

(c′) Every face in the coherent subdivision $\mathcal{D}_{\mu}$ of $P(\underline{\mu})$ is the flag matroid polytope of a positively oriented flag matroid.

is equivalent to (b) of Theorem 1.3.

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