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Fine polyhedral adjunction theory

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Abstract. Originally introduced by Fine and Reid in the study of plurigenera of toric hypersurfaces (Fine 1983, Reid et al. 1985), the Fine interior of a lattice polytope got recently into the focus of research: it is has been used for constructing canonical models in the sense of Mori Theory (Batyrev 2020). Based on the Fine interior, we propose here a modification of the original adjoint polytopes by defining the Fine adjoint polytope $P^{F(s)}$ of P as consisting of the points in P that have lattice distance at least s to all valid inequalities for P. We obtain a Fine polyhedral adjunction theory that is, in many respects, better behaved than its original analogue. Many existing results in polyhedral adjunction theory carry over, some with stronger conclusions, such as decomposing polytopes into Cayley sums, and most with simpler, more natural proofs as in the case of the finiteness of the Fine spectrum.

Keywords: Polyhedral adjunction theory, Cayley polytopes, Fine interior

1 Introduction

Let $P \subset \mathbb{R}^d$ be a rational polytope, i.e., the convex hull of a finite set of points in $\mathbb{Q}^n$. For $s > 0$ we define the Fine adjoint polytope $P^{F(s)}$ as the set of points satisfying $\langle a, x \rangle \geq b + s$ whenever $\langle a, x \rangle \geq b$ is valid for all points in P. The Fine interior of a polytope, namely the Fine adjoint polytope for $s = 1$, was first introduced by Jonathan Fine in [5], where it was referred to as the heart of a polytope. Along the lines of this idea, we take here the Fine adjoint polytope as consisting of the points of P with lattice distance at least s to *all valid inequalities* for P (cf. Definition 2.2). In this paper we argue that the resulting Fine polyhedral adjunction theory is a more natural version of the polyhedral adjunction theory introduced in [3], where only facet defining inequalities were considered. Various of the results obtained from the original polyhedral adjunction theory carry over to the Fine case. Some of them give us stronger conclusions, such as the Decomposition Theorem 4.3. Moreover, we are often able to provide simpler, more elegant proofs, as in the case of the finiteness of the Fine spectrum as mentioned in Theorem 5.4.

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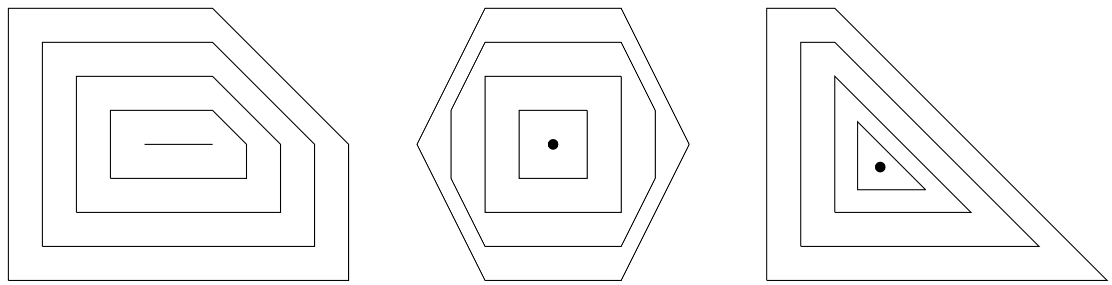

Figure 1: Examples of polytopes with their Fine adjoint polytopes.

Of particular importance is the largest value s_0 of s so that $P^{F(s)}$ is non-empty. For historical reasons, we record its reciprocal $\mu^F := 1/s_0$. A first indication that the Fine theory is better behaved is the monotonicity of this parameter:

$$P \subseteq Q \qquad \Longrightarrow \qquad \mu^F(P) \geq \mu^F(Q),$$

which does not hold for the analogous parameter μ, as defined in [3].

One of the main results of polyhedral adjunction theory is the decomposition theorem (cf. [3, 4, 7]), which gives lower bounds on the codegree of a polytope for it to have a Cayley sum structure. Here, we refer to a *Cayley sum* $P_0 \star \cdots \star P_t$ of $t+1$ polytopes as being a polytope which is constructed by positioning the $t+1$ polytopes along the vertices of a t-dimensional standard simplex and taking its convex hull (as in Definition 4.1). It was conjectured by Batyrev and Nill that there is a function $f(n)$ giving such lower bounds on the codegree of a polytope which decomposes as a Cayley sum. In particular, a conjecture posed by Dickenstein and Nill [4, Conj. 1.2] for n-dimensional polytopes where $f(n) = \frac{n+3}{2}$ has been disproven by Higashitani [8]. Instead a weaker version was proposed [3, Conj. 1.3] which states the following.

Conjecture 1.1. *If an n-dimensional lattice polytope P satisfies $\mu(P) > \frac{n+1}{2}$, then P decomposes as a Cayley sum of lattice polytopes of dimension at most $\lfloor 2(n+1-\mu(P)) \rfloor$.*

This conjecture is still open but a slightly weaker version was proven in [3, Thm. 3.4] in which $\mu(P) > \frac{n+1}{2}$ is replaced with $\mu(P) \geq \frac{n+2}{2}$. Because of the way the Fine adjoint polytopes are defined, we obtain the relation of the $\mathbb{Q}$-codegree and Fine $\mathbb{Q}$-codegree of a rational polytope P, $\mu^F(P)$, to be such that

$$\mu(P) \leq \mu^F(P).$$

It is due to this that in Theorem 4.3 we prove a Fine version of this decomposition theorem where essentially the same proof yields a stronger result.

Our second main result is related to Fujita's Spectrum Conjecture, of great interest in the context of polarized varieties, which can be translated to the case of toric varieties.

Conjecture 1.2 (Spectrum Conjecture, Fujita [6])**.** *For any $n \in \mathbb{Z}_{\geq 1}$, let S_n be the set of unnormalized spectral values of a smooth polarized n-fold. Then, for any $\varepsilon > 0$, the set $\{\mu \in S_n \mid \mu > \varepsilon\}$ is a finite set of rational numbers.*

A polyhedral version of Fujita's conjecture was proven by Paffenholz [10, Thm 3.1] even allowing certain, α-canonical, singularities (cf. Lemma 5.1 below). In this paper, we show that the analogous set $\{\mu^F \in S_n^F \mid \mu^F > \varepsilon\}$ of Fine spectral values is finite without any assumption on the singularities (cf. Theorem 5.4). As a result, our proof is simpler than Paffenholz', and it should allow for classification results in the future.

2 Redefining polyhedral adjunction theory

In what follows, unless stated otherwise, we consider $P \subseteq \mathbb{R}^n$ to be an n-dimensional rational polytope, which is described in a unique minimal way by inequalities as $P = \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \geq b_i, i = 1, \ldots, m\}$, where $b_i \in \mathbb{Q}$ and $a_i \in (\mathbb{Z}^n)^*$ are the primitive rows of a matrix A, i.e., they are not the multiple of another lattice vector, and $b \in \mathbb{Q}^m$. We will refer to P being a *rational polytope* when its vertices lie in $\mathbb{Q}^n$ and we will say that P is a *lattice polytope* when its vertices lie in $\mathbb{Z}^n$. We introduce our first definitions.

Definition 2.1. Let f be the affine functional $f(x) = \langle a, x\rangle - b$ for some $b \in \mathbb{Q}$ and $a \in (\mathbb{Z}^n)^*$. Such a functional is said to be *valid* for a polytope P if for the halfspace $\mathcal{H}_+ := \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$, we have that $P \subseteq \mathcal{H}_+$. Moreover, if there is some $p \in P$ with $f(p) = 0$, i.e., at least one point of P lies in $\mathcal{H}$, the hyperplane generated by f, we say f is a *tight* valid inequality for P.

Since P is a polyhedron, note that it can be described by a finite subset of all tight valid inequalities for P of which there is an infinite number, namely at least one for each primitive $a \in (\mathbb{Z}^n)^* \setminus \{0\}$.

Definition 2.2. Let $\alpha \in (\mathbb{R}^n)^*$ we define the *distance* function associated with P as

$$d_P^F : (\mathbb{R}^n)^* \to \mathbb{R}, \quad \alpha \mapsto \min_{x\in P}\langle \alpha, x\rangle.$$

In terms of this function, for some real number $s > 0$, we may define the *Fine adjoint polytope*, which is a rational polytope, as

$$P^{F(s)} := \{x \in \mathbb{R}^n \mid \langle a, x\rangle \geq d_P^F(a) + s, \text{ for all } a \in (\mathbb{Z}^n)^* \setminus \{0\}\}.$$

We will refer to the study of such Fine adjoint polytopes as *Fine polyhedral adjunction theory*.

As previously mentioned, the Fine adjoint polytopes we have introduced are a variant of the adjoint polytopes as defined in [3, Definition 1.1]. In order to compare between these definitions, we recall the original one here.

Definition 2.3. Let P be a rational polytope of dimension n given by the inequalities $\langle a_i, \cdot \rangle \geq b_i$ for $i = 1, \ldots, m$ that define facets $F_1, \ldots, F_m$ in a minimal way. Then for $x \in \mathbb{R}^n$, the *lattice distance* from the facet F_i is given by $d_{F_i} := \langle a_i, x \rangle - b_i$ and the *lattice distance with respect to the boundary* ∂P of P is $d_P := \min_{i=1,\ldots,m} d_{F_i}(x)$. For $s > 0$, the *adjoint polytope* is defined as

$$P^{(s)} := \{x \in \mathbb{R}^n \mid d_P(x) \geq s\}.$$

Remark 2.4. In some cases, taking Fine adjoint polytopes of a polytope P is equivalent to considering the original adjoint polytopes, as is the case of the rightmost and leftmost examples in Figure 1.

In what follows, we will prove a crucial result, namely that only finitely many tight valid inequalities $f_1, \ldots, f_t$ will be relevant when computing the Fine adjoint polytopes. Moreover, from its proof we will obtain a characterization for when exactly an inequality will be relevant when computing Fine adjoint polytopes. We make this notion of a relevant inequality more precise.

Definition 2.5. Let $\mathcal{F}$ be the set of all valid inequalities for P, where an element $f \in \mathcal{F}$ is of the form $\langle a_f, x \rangle \geq b_f$. A valid inequality $f \in \mathcal{F}$ is said to be *relevant* for P if for some $s > 0$, it holds that

$$\{x \in \mathbb{R}^n \mid \langle a_f, x \rangle \geq d_P^F(a_f) + s, \forall f \in \mathcal{F}\} \neq \{x \in \mathbb{R}^n \mid \langle a_f, x \rangle \geq d_P^F(a_f) + s, \forall f \in \mathcal{F} \setminus \{f\}\}.$$

The valid inequality f is said to be *irrelevant* if it is not relevant.

The following proposition will be very useful for our results and computations below.

Proposition 2.6 ([1, Proposition 3.11])**.** *Let P be a rational polytope of dimension n. Then there exists a finite set $\mathcal{S} \subset \mathcal{F}$ of valid inequalities for P such that the set $\mathcal{S}$ contains all relevant valid inequalities for P.*

From Proposition 2.6 we obtain a useful description of the relevant valid inequalities which we state as a Corollary.

Corollary 2.7. *Let P be a rational polytope of dimension n. All $a \in (\mathbb{Z}^n)^*$ which yield valid relevant inequalities for P of the form $\langle a, \cdot \rangle \geq d_P^F(a)$ belong to* $\mathrm{conv}(a_1, \ldots, a_m)$ *where the a_i for $1 \leq i \leq m$ are the primitive inward pointing facet normals of P.*

We will now consider the polytope P to be defined as $P = \{x \in \mathbb{R}^n \mid \langle a_i, x \rangle \geq b_i, i = 1, \ldots, m\}$, where $b_i \in \mathbb{Q}$ and $a_i \in (\mathbb{Z}^n)^*$ are the primitive rows of a matrix A including all relevant valid inequalities for P.

Remark 2.8. Note that in the Fine case, we have that taking Fine adjoint polytopes satisfies monotonicity with respect to inclusion of polytopes, i.e., if P and P' are two polytopes, such that $P \subseteq P'$, then we have that for any $s \geq 0$, $P^{F(s)} \subseteq P'^{F(s)}$. This holds since for any $a \in (Z^n)^*$ we have that $d_P^F(a) \geq d_{P'}^F(a)$, but this does not necessarily hold in the classical polyhedral adjunction theory case.

Now, using the Fine adjoint polytopes, we may reformulate the concept of classical $\mathbb{Q}-$codegree.

Definition 2.9. The *Fine* $\mathbb{Q}$*-codegree* of a rational polytope P is

$$\mu^F(P) := (\sup\{s > 0 \mid P^{F(s)} \neq \emptyset\})^{-1},$$

and the *Fine core* of P is

$$\operatorname{core}^F(P) := P^{F(1/\mu^F(P))}.$$

For completeness, recall that the concepts of $\mu(P)$ and $\operatorname{core}(P)$ were defined in the context of the original polyhedral adjunction theory as above, where simply the Fine adjoints $P^{F(s)}$ are the original adjoints $P^{(s)}$.

Example 2.10. In general, the core and the Fine core of a given polytope can differ and may even have different dimensions, as in the case of the polytopes in Figure 2.

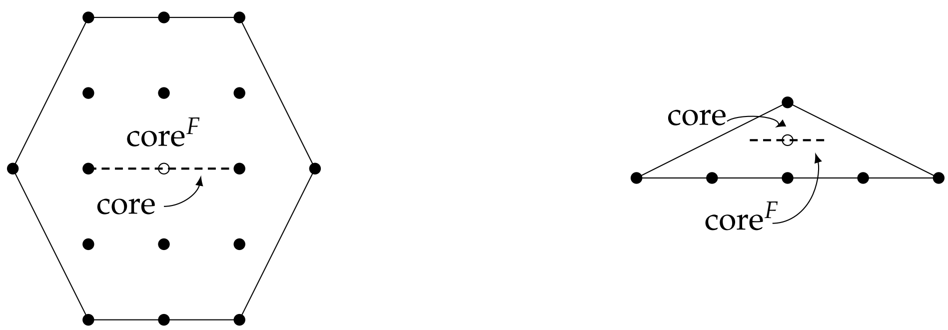

Figure 2: Examples of polytopes whose Fine and classical cores differ.

Example 2.11. Consider the polytope given as in [3, Figure 6] for the case $h = 10$ by

$$P = \operatorname{conv}\begin{bmatrix} 0 & 2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 10 & 10 \end{bmatrix}.$$

We may compute the core of P and the Fine core of P to be

$$\operatorname{core}(P) = \operatorname{conv}\begin{bmatrix} 4/3 & 4/3 \\ 4/3 & 4/3 \\ 4/3 & 2 \end{bmatrix}, \qquad \operatorname{core}^F(P) = \operatorname{conv}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 3 & 1 & 3 \end{bmatrix}.$$

Thus, in this case, as seen in Figure 3, we have that $\operatorname{core}^F(P) \not\subseteq \operatorname{core}(P)$. The core and the Fine core are even disjoint for this example.

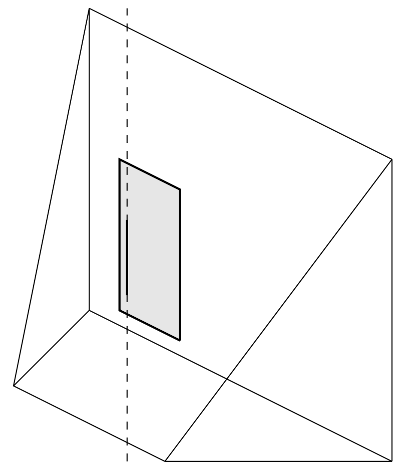

Figure 3: Polytope whose Fine core is not contained in its classical core.

We will now introduce a definition which we will need in later sections.

Definition 2.12. If we consider an element y of a k-dimensional rational polyhedral cone σ with generators $a_1, \dots, a_k$, then the *height function* associated with the cone σ is the piecewise linear function given by

$$\operatorname{height}_\sigma(y) := \max\left\{ \sum_{i=1}^{k} \lambda_i \;\middle|\; y = \sum_{i=1}^{k} \lambda_i a_i, \text{ and } \lambda_i \geq 0 \text{ for } 1 \leq i \leq k \right\}.$$

The cone σ is *α-canonical* for some $\alpha > 0$ if $\operatorname{height}_\sigma(y) \geq \alpha$ for any integral point $y \in \sigma \cap (\mathbb{Z}^n)^*$. A complete rational polyhedral fan Σ is *α-canonical* if every cone in Σ is α-canonical. Furthermore, a cone or fan is *canonical* if it is α-canonical for $\alpha = 1$.

3 Natural Projections in the Fine case

We first want to study the behaviour of the Fine $\mathbb{Q}$-codegree under projections, so we introduce the following definition.

Definition 3.1. Let $K(P)$ be the linear space parallel to $\operatorname{aff}(\operatorname{core}^F(P))$. Then the projection $\pi_P : \mathbb{R}^n \to \mathbb{R}^n / K(P)$ is called the natural projection associated with P.

We now have the following Lemma from [3] which holds with the same proof in the case of Fine Adjunction Theory.

Lemma 3.2. *Let $x \in \operatorname{relint}(\operatorname{core}^F(P))$. Let us denote by $f_1, \dots, f_t$ the relevant valid inequalities for P with $d_{f_i}(x) := \langle a_i, y \rangle - b_i = \mu^F(P)^{-1}$. Then their primitive inner normals $a_1, \dots, a_t$*

positively span the linear subspace $K(P)^\perp$. Moreover, if $\operatorname{core}^F(P) = \{x\}$, then

$$\{y \in \mathbb{R}^n \mid d_{f_i}(y) \geq 0 \text{ for all } i = 1, \ldots, t\}$$

is a rational polytope containing P.

However, we can prove the following stronger result, which does not hold in the classical polyhedral adjunction theory case. We include here the proof of the direction that was previously not valid.

Proposition 3.3. *The image $Q := \pi_P(P)$ of the natural projection of P is a rational polytope satisfying $\mu^F(Q) = \mu^F(P)$. Moreover $\operatorname{core}^F(Q)$ is the point $\pi_P(\operatorname{core}^F(P))$.*

Proof. To prove that $\mu^F(P)^{-1} \leq \mu(Q)^{-1}$, let g be a valid inequality for Q and let $p \in P$ with $\pi_P(p) = q \in \operatorname{core}^F(Q)$. Then, for $\pi_P^* g = f$, we have

$$\mu^F(Q)^{-1} = g(q) - \min_{\tilde{q} \in Q} g(\tilde{q}) = \pi^* g_P(p) - \min_{\tilde{p} \in P} \pi_P^* g(\tilde{p}) \geq \mu^F(P)^{-1}. \qquad \square$$

Remark 3.4. Note that we have described the behaviour of the $\mathbb{Q}$-codegree under the natural projection of P. However, under any projection π' of P, we have that $\mu^F(\pi'(P)) \leq \mu^F(P)$.

4 Cayley Decompositions and the Fine structure theorem

We let $P \subseteq \mathbb{R}^n$ be an n-dimensional lattice polytope and we recall the following definition.

Definition 4.1. Given lattice polytopes $P_0, \ldots, P_t \subseteq \mathbb{R}^k$, then the *Cayley sum* $P_0 * \cdots * P_t$ is the convex hull of

$$(P_0 \times 0) \cup (P_1 \times e_1) \cup \cdots \cup (P_t \times e_t) \subseteq \mathbb{R}^k \times \mathbb{R}^t$$

for $e_1, \ldots, e_t$ the standard basis of R^t.

As a means of comparison, we will now define the notion of codegree which comes up in the context of Ehrhart Theory of lattice polytopes [2].

Definition 4.2. Let P be a rational polytope. The *codegree* of P is given by

$$\operatorname{cd}(P) := \min\{k \in \mathbb{N}_{\geq 1} \mid \operatorname{int}(kP) \cap \mathbb{Z}^n \neq \emptyset\}.$$

Let us define the value

$$d^F(P) := \begin{cases} 2(n - \lfloor \mu^F(P) \rfloor), \text{ if } \mu^F(P) \notin \mathbb{N} \\ 2(n - \mu^F(P)) + 1, \text{ if } \mu^F(P) \in \mathbb{N} \end{cases}$$

We have that $P \cong \Delta_n$ if and only if $\mathrm{cd}(P) = n + 1$. Since $\mu(P) \leq \mu^F(P) \leq \mathrm{cd}(P) \leq n + 1$ and $\mu(\Delta_n) = n + 1$, then it holds that $P \cong \Delta_n$ if and only if $\mu(P) = n + 1$, in which case also $\mu^F(P) = n + 1$.

We now come to the following strengthening of the Decomposition Theorem for Cayley sums[3, Theorem 3.4] whose proof follows the one presented in [3] slightly adapted to the Fine case.

Theorem 4.3. *Let P be an n-dimensional lattice polytope with $P \ncong \Delta_n$. If $n > d^F(P)$, then P is a Cayley sum of lattice polytopes in $\mathbb{R}^m$ with $m \leq d^F(P)$.*

Let us consider the following example where we compute the codegree in our three settings.

Example 4.4. Let $\Delta_n(a) := \mathrm{conv}(0, ae_1, e_2, \dots, e_n)$ for positive integers $a \in \mathbb{Z}_{>0}$, where for $1 \leq i \leq n$ the e_i form the standard basis of $\mathbb{R}^n$. Note that in the case where $a = 1$ this yields the standard simplex and it has been argued before that it satisfies

$$\mathrm{cd}(\Delta_n(1)) = \mu(\Delta_n(1)) = \mu^F(\Delta_n(1)) = n + 1.$$

Thus, let us consider the case where $a > 1$ and $n \geq 2$. It is easy to check that $\mathrm{cd}(\Delta_n(a)) = n$. Moreover, it has been computed in [11] that in this case the $\mathbb{Q}$-codegree is given by

$$\mu(\Delta_n(a)) = n - 1 + \frac{2}{a}.$$

Finally, since in the Fine case the inequality defined by $\sum_{i=2}^n x_i \leq 1$ is relevant, it can be computed that

$$\mu^F(\Delta_n(a)) = n.$$

Thus, we obtain that $\mu(\Delta_n(a)) < \mu^F(\Delta_n(a)) = \mathrm{cd}(\Delta_n(a))$.

From this example we see that the $\mathbb{Q}$-codegree and the Fine $\mathbb{Q}$-codegree can take different values on the same polytope P.

5 Finiteness of the Fine $\mathbb{Q}$-codegree spectrum

It has been proven already that when bounded from below by some $\varepsilon > 0$, the set of values that the $\mathbb{Q}$-codegree can take under certain conditions is finite. We will shortly review these conditions for the case of classical polyhedral adjunction theory.

Let $P \subseteq \mathbb{R}^n$ be a lattice polytope of dimension n, which we assume to be full-dimensional. We let $\mathcal{N}(P)$ denote the normal fan of P and define the following sets as in [11],

$$\mathcal{S}(n,\varepsilon) := \{P \mid P \text{ is an } n\text{-dimensional lattice polytope}, \mu(P) \geq \varepsilon\},$$

$$\mathcal{S}_\alpha^{can}(n,\varepsilon) := \{P \mid P \in \mathcal{S}(n,\varepsilon) \text{ and } \mathcal{N}(P) \text{ is } \alpha\text{-canonical}\}.$$

The theorem proven in [11] is stated as follows.

Lemma 5.1 (Paffenholz, [11, Theorem 3.1])**.** *Let $n \in \mathbb{N}$ and $\alpha, \varepsilon > 0$ be given. Then*

$$\{\mu(P) \mid P \in \mathcal{S}_\alpha^{can}(n,\varepsilon)\}$$

is finite.

Note that in this result the α-canonical assumption on the polytopes was necessary.

Example 5.2. A natural example to consider in order to see the importance of this assumption is the family of polytopes

$$\Delta_n(a) = \operatorname{conv}(0, ae_1, \ldots, e_n)$$

where the $e_1, \ldots, e_n$ are the standard basis vectors and $a \in \mathbb{Z}_{>0}$, which was previously studied in Example 4.4. For these polytopes, we have that their normal fan is $\mathbb{Q}$-Gorenstein with index a and if $a > 1$, so that

$$\mu(\Delta_n(a)) = n - 1 + \frac{2}{a},$$

but the polytopes $\Delta_n(a)$ are not α-canonical for any $\alpha > 0$ and their $\mathbb{Q}$-codegree can take an infinite number of values.

In what follows we will study a generalization of this theorem to the case of Fine adjunction theory. We will follow the proof presented in [11] and adapt it to the Fine polyhedral adjunction theory case where the remarkable difference will be the fact that we will not be assuming that the polytopes are α-canonical, hence in this new setting the theorem holds in much greater generality and with much weaker assumptions.

Definition 5.3. A vector a_i is a *Fine core normal* if for all $y \in \operatorname{core}^F(P)$,

$$\langle a_i, y\rangle = d_P^F(a_i) + \mu^F(P)^{-1}.$$

The *set of Fine core normals* will be denoted by $\mathcal{N}_{\mathrm{core}}^F(P)$. We also define the set

$$\mathcal{S}^F(n,\varepsilon) := \{P \mid P \text{ is an } n\text{-dimensional lattice polytope}, \mu^F(P) \geq \varepsilon\}.$$

We can now state our main result.

Theorem 5.4. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. Then*

$$\{\mu^F(P) \mid P \in \mathcal{S}^F(n,\varepsilon)\}$$

is finite.

The proof here will also consist of two main parts. First, we will show that up to lattice equivalence, for a fixed $n \in \mathbb{Z}_{>0}$, there are only finitely many sets of core normals for n-dimensional lattice polytopes. Then we will show that each such configuration of core normals gives rise to finitely many values for the Fine $\mathbb{Q}$-codegree above any positive threshold. Thus, if we let P be described by all relevant inequalities as

$$P = \{x \in \mathbb{R}^n \mid \langle a_i, x\rangle \geq b_i, i = 1, \dots, m\},$$

note that, up to relabelling, we can assume that $a_1, \dots, a_k$ for some $k \leq m$ is the set of valid inequalities defining the affine hull of the Fine core of P, $\operatorname{aff}(\operatorname{core}^F(P))$. That is,

$$\operatorname{aff}^F(\operatorname{core}(P)) = \{x \mid \langle a_i, x\rangle = b_i + \mu^F(P)^{-1}, 1 \leq i \leq k\}.$$

Definition 5.5. Define the set A^F_{core} to be

$$A^F_{\text{core}} := \operatorname{conv}(a_1, \dots, a_k) \subseteq (\mathbb{R}^n)^*$$

as the convex hull of the Fine core normals.

For P as defined above, the following lemmas will show that all the a_i are vertices of A^F_{core} and that the origin is a relative interior point.

Lemma 5.6 ([3, Lemma 2.2])**.** *The origin is in the relative interior of A^F_{core}.*

Moreover, the following result proven in [11] gives us precisely the vertices of A^F_{core}.

Lemma 5.7 ([11, Lemma 3.6])**.** *The vertices of A^F_{core} are $a_1, \dots, a_k$.*

We now want to show that, independently of the polytope being α-canonical or not, the origin is the only lattice point in the relative interior of A^F_{core}.

Lemma 5.8. *For A^F_{core} as above, we have that* $\operatorname{relint}(A^F_{\text{core}}) \cap (\mathbb{Z}^n)^* = \{0\}$.

Proof. We prove this by contradiction. Hence, assume that there is some vector $a \in (\mathbb{Z}^n)^* \setminus \{0\}$ contained in the relative interior of A^F_{core}. As $0 \in \operatorname{relint}(A^F_{\text{core}})$, the point a is contained in the cone spanned by the vertices of some facet F of A^F_{core}, and defines the valid inequality $\langle a, x\rangle \geq b$ for P. If we let $a_1, \dots, a_k$ be the vertices of A^F_{core}, we can find $\lambda_1, \dots, \lambda_k \geq 0$ with $\lambda_i = 0$ if $a_i \notin F$ such that $a = \sum_{i=1}^k \lambda_i a_i$ and satisfying

$\sum_{i=1}^k \lambda_i < 1$. This last inequality follows from the fact that a is in the relative interior of A^F_{core}. Let $x_{\text{core}} \in \operatorname{relint}(\operatorname{core}^F(P))$. By definition, we have that $\langle a, x_{\text{core}}\rangle - b \geq (\mu^F(P))^{-1}$. Now, considering the sum over all valid inequalities associated to the core normals a_i for $1 \leq i \leq k$, for such y we obtain

$$\langle a, x_{\text{core}}\rangle - b = \sum_{i=1}^{k} \lambda_i(\langle a_i, x_{\text{core}}\rangle - b_i) = \sum_{i=1}^{k} \lambda_i (\mu^F(P))^{-1} < (\mu^F(P))^{-1}$$

where the last inequality follows from the fact that $\sum_{i=1}^k \lambda_i < 1$, but this contradicts the previous relation. □

The last result we need in this first part of the proof is the following one by Lagarias and Ziegler. We say two lattice polytopes P and Q are *lattice equivalent* if there is an affine lattice isomorphism mapping P onto Q.

Theorem 5.9 (Lagarias, Ziegler [9, Theorem 1])**.** *Let integers $n, l \geq 1$ be given. There are, up to lattice equivalence, only finitely many different lattice polytopes of dimension d with exactly l interior points in the lattice $\mathbb{Z}^n$.*

Since we have proven that for A^F_{core} the only lattice point in its relative interior is $\{0\}$, combining this result with Theorem 5.9 we have shown that for a fixed $n \in \mathbb{Z}_{>0}$, only finitely many sets define the Fine core normals of an n-dimensional lattice polytope P. We record this as the following result.

Lemma 5.10. *Let $n \in \mathbb{Z}_{>0}$ be fixed and let P be an n-dimensional lattice polytope. Then only finitely many sets define the Fine core normals of P.*

In what follows, we will continue with the second step of the proof. We make use of the following lemma proven in [11] where we do not require the α-canonicity of P.

Lemma 5.11 ([11, Lemma 3.10])**.** *Fix some $\varepsilon > 0$ and some $n \in \mathbb{Z}_{>0}$. Let P be an n-dimensional lattice polytope with set of Fine core normals $\mathcal{A}$. Then the set*

$$\{\mu^F(P) \mid P \text{ is } n\text{-dimensional with set of Fine core normals } \mathcal{A}, \mu^F(P) \geq \varepsilon\}$$

is finite.

Proof of Theorem 5.4. Combining this last lemma together with the previous one we obtain the following. First of all, by Lemma 5.10 we have that up to lattice equivalence, there are only finitely many sets of Fine core normals for some n-dimensional lattice polytope. Finally, by Lemma 5.11, the set of values μ^F is finite for n-dimensional lattice polytopes with a fixed set $\mathcal{A}$ of Fine core normals. □

In the Fine case, a stronger version of the theorem regarding the finiteness of the $\mathbb{Q}$-codegree spectrum holds, dropping the α-canonicity assumption. Hence, considering all valid inequalities is a condition that highly restricts the shape and properties of the polytope A^F_{core}, i.e., the convex hull of the Fine core normals of a polytope P, since all such polytopes contain just one interior lattice point, namely the origin. Due to this we are able to prove the result in greater generality for the Fine $\mathbb{Q}$-codegree spectrum.

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