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Generic curves and non-coprime Catalans

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Abstract. We compute the Poincaré polynomials of the compactified Jacobians for plane curve singularities with Puiseaux exponents $(nd, md, md+1)$, and relate them to the combinatorics of q,t-Catalan numbers in the non-coprime case. We also confirm a conjecture of Cherednik and Danilenko for such curves.

Keywords: Compactified Jacobians, Rational Catalan Combinatorics, Dyck Paths

1 Introduction

In this paper, we study the topology of compactified Jacobians of plane curve singularities. We focus on the case where the curve is reduced and locally irreducible (or *unibranched*), and it is known [1, 2] that in this case compactified Jacobian is irreducible as well.

Compactified Jacobians play an important role in modern geometric representation theory. First, they are closely related to Hilbert schemes of points on singular curves, singular fibers in the Hitchin fibration and affine Springer fibers. In particular, counting points in the compactified Jacobians over finite fields is related to certain orbital integrals [21, 20, 30]. Recent works [8, 19, 11] relate them to the representation theory of Coulomb branch algebras, defined by Braverman, Finkelberg and Nakajima [4]. Second, a set of conjectures of the third author, Rasmussen and Shende [26, 25] relates the homology of compactified Jacobians to the *Khovanov-Rozansky homology* of the corresponding knots and links. In particular, the conjectures imply that the homology of the compactified Jacobian is expected to be determined by the topology of the link or, in the unibranched case, by the collection of Puiseaux pairs of the singularity.

The progress in explicit computations of the homology of compactified Jacobians has been quite slow. For the quasi-homogeneous curves $C = \{x^m = y^n\}, \mathrm{GCD}(m,n) = 1$, the homology was computed in many sources, starting with Lusztig and Smelt [22]. The key observation is that in this case $\overline{JC}$ admits a paving by affine cells. These cells and their dimensions have been given a number of combinatorial interpretations in [12, 13, 15, 18], where they were related to q,t-Catalan combinatorics. In [27, 28] the third author and Yun determined the ring structure on the homology in this case.

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In a different direction, Piontkowski [29] have computed the homology of compactified Jacobians for curves with one Puiseaux pair defined by the parameterization $(x, y) = (t^n, t^m + \ldots), \mathrm{GCD}(m, n) = 1$. He showed that $\overline{JC}$ again admits an affine paving, and the combinatorics of cells depends only on (m, n) and hence agrees with the quasi-homogeneous case $(x, y) = (t^n, t^m)$. Moreover, Piontkowski computed the cell decompositions for some curves with two Puiseaux exponents, where the combinatorics becomes rather subtle. In our main theorem, we vastly generalize the results of Piontkowski and prove the following.

Theorem 1.1. *Suppose* $\mathrm{GCD}(m, n) = 1$ *and* $d \geq 1$, *consider the plane curve singularity* C *defined by the parameterization*

$$(x(t), y(t)) = (t^{nd}, t^{md} + \lambda t^{md+1} + \ldots), \quad \lambda \neq 0 \tag{1.1}$$

Then:

a) The compactified Jacobian $\overline{JC}$ *admits an affine paving where the cells are in bijection with Dyck paths* D *in an* $(nd) \times (md)$ *rectangle.*

b) The Poincaré polynomial of $\overline{JC}$ *is given by*

$$P_{\overline{JC}}(t) = \sum_{D \in \mathrm{Dyck}(nd,md)} t^{2(\delta - \mathrm{dinv}(D))}$$

where dinv *is a certain statistics on Dyck paths defined in Section 4, and* δ *is the number of boxes weakly under the diagonal in an* $(nd) \times (md)$ *rectangle.*

c) In particular, the Poincaré polynomial does not depend on λ *(as long as it is nonzero) or on the higher order terms in* (1.1).

We call the curves (1.1) *generic curves*, since a generic curve with the first Puiseaux pair (nd, md) has this form. The affine paving in the statement of Theorem 1.1 is obtained by intersecting the compactified Jacobian with the Schubert cell paving of the affine Grassmannian. Thus one can define a partial order on the strata such that the boundary of an affine cell lies in the union of the cells with smaller indices in this order. Hence we conclude:

Corollary 1.2. *For generic curves, the cohomology* $H^*(\overline{JC})$ *is supported in even degrees and the weight filtration on* $H^*(\overline{JC})$ *is pure in the sense of [10, 9].*

Very recently, Kivinen and Tsai [20] used completely different methods (p-adic harmonic analysis) to count points in arbitrary compactified Jacobians over finite fields $\mathbb{F}_q$. They proved that the result is always a polynomial in q and hence recovers the weight polynomial of $\overline{JC}$. Given the above purity result, for generic curves the Poincaré polynomial of $\overline{JC}$ agrees with the weight polynomial and our result agrees with theirs.

As a corollary of Theorem 1.1, we get that the Euler characteristic of $\overline{JC}$ is given by the number of Dyck paths in $(nd) \times (md)$ rectangle. For example, for the curve

$C = (t^4, t^6 + t^7)$ the Euler characteristic $\chi(\overline{JC}) = 23$ is equal to the number of Dyck paths in a 4×6 rectangle, in agreement with [29].

Next, we address the conjectures of Cherednik, Danilenko and Philipp [6, 7], which proposed an expression for the Poincaré polynomials of compactified Jacobians in terms of certain matrix elements of certain operators in the double affine Hecke algebra, see Section 5 for more details. We are able to prove this conjecture for generic curves:

Theorem 1.3. *Consider the two-variable polynomial*

$$C_{nd,md}(q,t) = \sum_{D \in \mathrm{Dyck}(dn,dm)} q^{\mathrm{area}(D)} t^{\mathrm{dinv}(D)} \tag{1.2}$$

where $\mathrm{area}(D)$ *is the number of full boxes between a Dyck path D and the diagonal. It satisfies the following properties:*

a) It is symmetric in q and t : $C_{nd,md}(q,t) = C_{nd,md}(t,q)$

b) At $q = 1$ it specializes to the Poincaré polynomial of $\overline{JC}$ (up to a linear change of the variable).

c) It is given by the matrix element $(\gamma_{n,m}(e_d)(1), e_{nd})$ of the elliptic Hall algebra operator $\gamma_{n,m}(e_d)$.

d) It agrees with the Cherednik-Danilenko conjecture (Conjecture 5.1).

Compositional Rational Shuffle Theorem [23] implies a different, manifestly symmetric in q, t, explicit formula for $C_{nd,md}(q,t)$, which implies part (a). Part (c) also follows from the Compositional Rational Shuffle Theorem, part (d) follows from (c). Part (b) is a rephrasing of Theorem 1.1. Theorem 1.3(a) allows us to give a simple formula for the Poincaré polynomial of the compactified Jacobian:

Corollary 1.4. *The Poincaré polynomial of $\overline{JC}$ equals*

$$t^{2\delta} C_{nd,md}(1, t^{-2}) = t^{2\delta} C_{nd,md}(t^{-2}, 1) = \sum_{D \in \mathrm{Dyck}(nd,md)} t^{2(\delta - \mathrm{area}(D))}.$$

A more detailed version of the material presented in this extended abstract can be found in [14].

2 Background

2.1 Compactified Jacobians and semigroups

Let C be an irreducible (and reduced) plane curve singularity at $(0,0)$. We can parametrize C as $(x(t), y(t))$, and write the local ring of functions on C as $\mathcal{O}_C = \mathbb{C}[[x(t), y(t)]] \subset \mathbb{C}[[t]]$. Given a function $f(t) \in \mathcal{O}_C$, we can write

$$f(t) = \alpha_k t^k + \alpha_{k+1} t^{k+1} + \ldots, \ \alpha_k \neq 0$$

and define the order of $f(t)$ by $\mathrm{Ord} f(t) = k$. The compactified Jacobian $\overline{JC}$ is defined as the moduli space of rank one torsion free sheaves on C or, equivalently, $\mathcal{O}_C$-submodules $M \subset \mathbb{C}[[t]]$. Note that M is an $\mathcal{O}_C$–submodule if and only if $x(t)M \subset M$, $y(t)M \subset M$. Given such a subspace M, we define

$$\Delta_M = \{\mathrm{Ord}\, f(t) \;:\; f(t) \in M\} \subset \mathbb{Z}_{\geq 0}.$$

In particular, for $M = \mathcal{O}_C$ we obtain the **semigroup** of C:

$$\Gamma = \Delta_{\mathcal{O}_C} = \{\mathrm{Ord}\, f(t) \;:\; f(t) \in \mathcal{O}_C\}.$$

If M is an $\mathcal{O}_C$-submodule then Δ_M is a Γ-module: $\Delta_M + \Gamma \subset \Delta_M$. This motivates the following

Definition 2.1. *Given a subset $\Delta \subset \mathbb{Z}_{\geq 0}$, we define the stratum in the compactified Jacobian:*

$$J_\Delta := \{M \subset \mathbb{C}[[t]] : \mathcal{O}_C M \subset M,\ \Delta_M = \Delta\}.$$

Clearly, J_Δ give a subdivision of $\overline{JC}$ and J_Δ is empty if Δ is not Γ-invariant. As a warning to the reader, J_Δ could be empty even if Δ is Γ-invariant.

2.2 Generic curves

It is well known that any curve C can be parametrized using Puiseaux expansion:

$$x = t^{nd},\ y = t^{md} + \lambda t^{md+1} + \ldots,\ \mathrm{GCD}(m,n) = 1.$$

If $d = 1$ then C has one Puiseaux pair (n,m) and its link is the (n,m) torus knot. In this paper, we will be mostly interested in the case $d > 1$.

Definition 2.2. *Assume $d > 1$. A curve C is called* generic *if $\lambda \neq 0$.*

It is easy to see that the definition of a generic curve is symmetric in n and m. Indeed, we can choose the new parameter

$$\widetilde{t} = \sqrt[md]{y(t)} = t\sqrt[md]{1 + \lambda t^{md+1} + \ldots} = t\left(1 + \frac{\lambda}{md}t + \ldots\right),$$

then

$$y = \widetilde{t}^{md},\ x = \widetilde{t}^{nd} - \frac{n\lambda}{md}\widetilde{t}^{nd+1} + \ldots.$$

If $n = 1$, then the curve has one Puiseaux pair $(d, md+1)$. Otherwise, it has two Puiseaux pairs (nd, md) and $(d, md+1)$, which completely determine the topological type of the corresponding knot as a $(d, mnd+1)$-cable of the (n,m) torus knot.

2.3 Invariant subsets

. A subset $\Delta \subset \mathbb{Z}_{\geq 0}$ is called 0-normalized if $0 \in \Delta$. We will mostly consider 0-normalized subsets, as any subset of $\mathbb{Z}_{\geq 0}$ can be shifted to a unique 0-normalized one. We call Δ cofinite if $\mathbb{Z}_{\geq 0} \setminus \Delta$ is finite. For a cofinite subset Δ and $x \geq 0$ we write

$$\mathrm{Gaps}(x) = \mathrm{Gaps}_{\Delta}(x) := [x, +\infty) \setminus \Delta.$$

Definition 2.3. *We call Δ an (nd)-invariant subset if $nd + \Delta \subset \Delta$. A number a is called an (nd)-generator of Δ if $a \in \Delta$ but $a - nd \notin \Delta$.*

It is clear that for a cofinite (nd)-invariant subset Δ there is exactly one (nd)-generator in each remainder modulo nd. We will group them according to their remainders modulo d, so that the generators $a_{j,i}$, $i = 0, \ldots, n-1$ all have remainder j modulo d. We will further reorder $a_{j,i}$ so that $a_{j,i} + m \equiv a_{j,i+1}$ modulo n. We write

$$\Delta_j = \bigcup_{i=0,\ldots,n-1} (a_{j,i} + dn\mathbb{Z}_{\geq 0}), \ \Delta = \bigcup_{j=0,\ldots,d} \Delta_j.$$

We will call the integers $a_{j,i} + md$ the combinatorial syzygys of Δ. It is clear that in each remainder modulo d there are n such syzygys.

3 Topology of compactified Jacobians

Throughout this section we fix a generic curve C with parametrization

$$(x(t), y(t)) = (t^{nd}, t^{md} + \lambda t^{md+1} + \ldots)$$

with $\lambda \neq 0$. We will use the notation $\mathcal{O}_C = \mathbb{C}[[x(t), y(t)]]$ for the ring of functions on C.

We also fix a cofinite (nd, md)-invariant subset $\Delta \subset \mathbb{Z}_{\geq 0}$ with (nd)-generators $a_{j,i}$ as in Section 2.3. We denote by $A = \{a_{j,i}\}$ the set of all (nd)-generators. The main goal of this section is to describe the stratum J_Δ in the compactified Jacobian $\overline{JC}$.

3.1 Equations for J_Δ

Consider an $\mathcal{O}_C$-module $M \in J_\Delta$.

Lemma 3.1. *Then for all $k \in \Delta$ there exist a unique canonical representative*

$$f_k = t^k + \sum_{l \in \mathrm{Gaps}(k)} f_{k;l-k} t^l \in M.$$

The canonical generators form a topological basis in M.

Vice versa, consider a collection of polynomials

$$\left\{g_a = t^a + \sum_{l\in \mathrm{Gaps}(a)} g_{a;l-a}t^l | a\in A\right\}.$$

It will be convenient to consider the coefficients $g_{a;l-a}$ for $l\in \mathrm{Gaps}(a)$ as parameters. For $l\neq \mathrm{Gaps}(a)$ we use conventions

$$g_{a;0}=1,\quad g_{a;l-a}=0 \text{ if } l\in\Delta\setminus\{a\}.$$

Let N be the $\mathcal{O}_C$-submodule generated by the collection, and let $\widetilde{N}$ be the $\mathbb{C}[[t^{dn}]]$ submodule generated by the same collection. Then $N\in J_\Delta$ iff $N=\widetilde{N}$, or, equivalently,

$$y(t)g_a\in \widetilde{N}\ \forall\ a\in A.$$

For every $a\in A$ let s_{a+dm} be a polynomial in t whose coefficients are polynomials in $g_{a';r}$, $a'\in A$, $r\in\mathbb{Z}_{>0}$ such that

$$y(t)g_a - s_{a+dm}\in\widetilde{N}$$

and

$$s_{a+dm}=\sum_{l\in \mathrm{Gaps}(a+dm)} s_{a+dm;l-dm-a}t^l.$$

The condition $y(t)g_a\in\widetilde{N}$ is equivalent to $s_{a+dm}=0$ for all $a\in A$. Thus J_Δ is a subset of the affine space $\mathrm{Gen}=\mathbb{C}^{G(\Delta)}$ defined by $E(\Delta)$ equations:

$$G(\Delta)=\sum_{i,j}|\mathrm{Gaps}(a_{j,i})|,\quad E(\Delta)=\sum_{i,j}|\mathrm{Gaps}(a_{j,i}+dm)|.$$

In particular, the coefficients $g_{a_{j,i};x}$ for $x\in\mathrm{Gaps}(a_{j,i})$ are natural coordinates on Gen. The equations are

$$s_{a_{j,i}+dm;x}(g)=0,\quad a_{j,i}+dm+x\notin\Delta.$$

3.2 Admissibility

Definition 3.2. *We call $x>0$ j-suspicious if $a_{j,i}+md+x\notin\Delta$ for all i, and suspicious if it is j-suspicious for at least one remainder j.*

Lemma 3.3. *A number x is j-suspicious if and only if $\Delta_{j+x}\subset\Delta_j+md+nd+x$.*

Definition 3.4. *We call Δ admissible, if 1 is not suspicious for Δ.*

Lemma 3.5. *Suppose that $J_\Delta\neq\emptyset$. Then Δ is admissible.*

Proof. Suppose Δ is not admissible, then there exists some j such that $a_{j,i}+md+1\notin\Delta$ for all i. This implies $a_{j,i}+1\notin\Delta$ for all i, so we have canonical generators $g_{a_{j,i}}=t^{a_{j,i}}+g_{a_{j,i};1}t^{a_{j,i}+1}+\ldots$ If $a_{j,i}+md=a_{j,i+1}+\alpha nd$ then we get

$$\begin{aligned} t^{\alpha nd}g_{a_{j,i+1}}-y(t)g_{a_{j,i}} &= t^{\alpha nd}g_{a_{j,i+1}}-(t^{md}+\lambda t^{md+1}+\ldots)g_{a_{j,i}}\\ &=(g_{a_{j,i+1};1}-g_{a_{j,i};1}-\lambda)t^{a_{j,i}+md+1}+\ldots \end{aligned}$$

hence $g_{a_{j,i+1};1}-g_{a_{j,i};1}-\lambda=0$ for all i. By adding these for all i we get $n\lambda=0$, contradiction. □

3.3 Paving by affine spaces

Recall that Gen has coordinates $g_{a_{j,i};x}$ where $a_{j,i} + x \notin \Delta$. Sometimes we will use the notation $g_{a_{j,i};x}$ for all x, assuming

$$g_{a_{j,i};x} = 0 \quad \text{if } a_{j,i} + x \in \Delta. \tag{3.1}$$

It is also convenient to consider Gen as a graded vector space $\mathrm{Gen} = \bigoplus_{x=1}^{\infty} \mathrm{Gen}_x$, where Gen_x is spanned by $g_{a_{j,i};x}$ with a fixed x. There are coordinates on Gen that are most suitable for study of equations s_k in the case Δ is admissible. Let us define

$$g^-_{j,i;x} = g_{a_{j,i};x} - g_{a_{j,i+1};x}, \quad i = 0, \ldots, n-1.$$

These functions are linearly dependent, for example $\sum_i g^-_{j,i;x} = 0$. We choose a subset of these as follows:

(1) If x is j-suspicious, we define $I(j;x) = \{0, \ldots, n-2\}$.

(2) If x is not j-suspicious, we can define $I(j;x)$ to be a set of i such that $a_{j,i} + dm + x \notin \Delta$.

Lemma 3.6. *The functions $g^-_{j,i;x}$, $i \in I(j;x)$ are linearly independent.*

Finally, if there exists at least one integer i such that $a_{j,i} + x \notin \Delta$, then we set

$$g^+_{j;x} = \sum_{i:a_{j,i}+x\notin\Delta} g_{a_{j,i};x}.$$

Thus if $I(j;x) \neq \varnothing$ then $\{g^-_{j,i;x}, g^+_{j;x}\}$, $i \in I(j;x)$ are linearly independent linear coordinates on the space of generators Gen_x. For each j, x such that $I(j;x) \neq \varnothing$ let us fix a subset $\bar{I}(j;x)$ such that $\left\{g^-_{j,i;x}, g^+_{j;x}, g_{a_{j,i'};x}\right\}$ $i \in I(j;x)$, $i' \in \bar{I}(j;x)$ is a basis of linear coordinates on Gen_x. For $I(j;x) = \varnothing$ set $\bar{I}(j;x) = \{0, 1, \ldots, n-1\}$. As we will see later the coordinates $g_{a_{j,i'};x}$, $i' \in \bar{I}(j;x)$ are not constrained by the equations for J_Δ, so we call them *free variables*. On the rest of the coordinates $g^*_{*;*}$ we introduce a partial order generated as follows. Allowing $*$ to take any independent values,

$$g^-_{*,*;x} < g^+_{*;x} < g^-_{*,*,x+1},$$

coordinates $g^-_{*,*;x}$ are ordered in any arbitrary way, and

$$g^+_{j+1;x} < g^+_{j;x}$$

(cyclic notation modulo d) for $j \neq d - x - 1$.

Lemma 3.7. *For x such that $a_{ji} + dm + x \notin \Delta$ we have*

$$s_{a_{j,i}+dm;x} = g^-_{j,i;x} + \textit{polynomial in free variables and variables} < g^-_{j,i;x} \tag{3.2}$$

and if x is j-suspicious then

$$\sum_{i=0}^{n-1} s_{a_{j,i}+dm;x} = \lambda g^+_{j;x-1} + \textit{polynomial in free variables and variables} < g^+_{j;x-1} \tag{3.3}$$

Proposition 3.8. *If Δ is admissible then*

$$J_\Delta = \mathbb{C}^{\dim(\Delta)}, \quad \dim(\Delta) = G(\Delta) - E(\Delta).$$

Proof. Recall that J_Δ is defined by the equations $s_{a_{j,i}+dm;x} = 0$ for j, i, x such that $a_{j,i} + dm + x \notin \Delta$. We can modify this system of equations as follows. Whenever x is j-suspicious, replace $s_{a_{j,n-1}+dm;x} = 0$ by $\sum_{i=0}^{n-1} s_{a_{j,i}+dm;x} = 0$. Clearly, the new system of equations is equivalent to the old one. Furthermore, according to Lemma 3.7, the new system of equations expresses some of the elements of a basis of the space Gen in terms of the smaller variables with respect to the order $<$. Therefore, one can use the equations to eliminate these variables one by one. Since $\dim \mathrm{Gen} = G(\Delta)$ and there are $E(\Delta)$ equations, we obtain the required result. □

4 Combinatorics

4.1 More on invariant subsets

Let Δ be an (nd, md)-invariant subset. We call b an (md)-cogenerator for Δ if $b \notin \Delta$ but $b + md \in \Delta$.

Lemma 4.1. *Let Δ be an admissible cofinite (nd, md)-invariant subset. The dimension of J_Δ equals to the number of pairs (a, b) such that a is an (nd)-generator of Δ, b is an (md)-cogenerator and $a < b$.*

Let Θ be an n, m-invariant subset in $\mathbb{Z}$. As before, n, m are relatively prime.

Definition 4.2. *The* skeleton *S of Θ is the union of the n-generators and m-cogenerators of Θ.*

4.2 Equivalence classes of dn, dm-invariant subsets

Let us remind the definitions of the equivalence classes of invariant subsets from [16]. Let $\Theta = \left\{\Theta_0^0, \ldots, \Theta_{d-1}^0\right\}$ be a collection of 0-normalized (n, m)-invariant subsets. For every $(x_1, \ldots, x_{d-1}) \in \mathbb{R}_{\geq 0}^{d-1}$ consider

$$\Delta = \Delta(x_1, \ldots, x_{d-1}) := \bigcup_{k=0}^{d-1} (d\Theta_k^0 + x_k),$$

where $x_0 = 0$. If all shift parameters $x_0, \ldots, x_{d-1}$ are integers with different remainders modulo d, then Δ is a cofinite 0-normalized (nd, md)-invariant subset.

For each Θ_k^0 let S_k^0 be its skeleton. Consider the space $\mathbb{R}_{\geq 0}^l$ of all possible shifts $x_1, \ldots, x_{d-1}$. Consider the subset $\Sigma_\Theta \subset \mathbb{R}_{\geq 0}^{d-1}$ consisting of all shifts $x_1, \ldots, x_{d-1}$ for which there exists i and j such that

$$dS_i^0 + x_i \cap dS_j^0 + x_j \neq \emptyset.$$

Clearly, Σ_Θ is a hyperplane arrangement. We say that two (nd, md)-invariant subsets are equivalent if the corresponding shifts belong to the same connected component of the complement to Σ_Θ. One can show that every connected component of the complement to Σ_Θ contains at least one point corresponding to an (dn, dm)-invariant subset. To summarize, in order to get all the equivalence classes of the cofinite (dn, dm)-invariant subsets, one should consider all possible d-tuples of 0-normalized (n, m)-invariant subsets $\Theta_0^0, \ldots, \Theta_{d-1}^0$, and for each such d-tuple consider the set of connected components of the complement to Σ_Θ in the space of shifts. Furthermore, one should consider these connected components up to symmetry: if two of the subsets are equal $\Theta_i^0 = \Theta_j^0$ then switching the corresponding shift coordinates x_i and x_j interchanges the connected components corresponding to the same equivalence class of (dn, dm)-invariant subsets.

4.3 Admissible representatives

Let Δ be a cofinite 0-normalized (dn, dm)-invariant subset. In the spirit of the above notations, let $\Delta = \bigcup_k d\Theta_k^0 + x_k + k$, where $(x_1, \ldots, x_{d-1}) \in d\mathbb{Z}_{\geq 0}^{d-1} \subset \mathbb{R}_{\geq 0}^{d-1}$, $x_0 = 0$, and $\Theta_0^0, \ldots, \Theta_{d-1}^0$ are 0-normalized (n, m)-invariant subsets. Denote $\Theta_k := \Theta_k^0 + \frac{x_k}{d}$ for all $k \in \{0, \ldots, d-1\}$.

In these notations we get that Δ is admissible if for every $k \in \{0, \ldots, d-1\}$ one has

$$d\Theta_{k+1} + (k+1) \not\subseteq d\Theta_k + k + dn + dm + 1,$$

or, equivalently,

$$\Theta_{k+1} \not\subseteq \Theta_k + n + m.$$

The following is our key combinatorial result:

Theorem 4.3. *Every equivalence class contains a unique admissible representative.*

Let $Inv(dm, dn)$ be the set of cofinite 0-normalized (dm, dn)-invariant subsets. The following Theorem is the main result of [16]:

Theorem 4.4 ([16])**.** *There exists a bijection* $\mathcal{D} : \mathrm{Inv}(dm, dn)/\sim \to \mathrm{Dyck}(dm, dn)$*, where* $\sim$ *is the equivalence relation defined above, and* $\mathrm{Dyck}(dm, dn)$ *is the set of* (dm, dn)*-Dyck paths. Furthermore,* $\dim \Delta = \delta - \mathrm{dinv}(\mathcal{D}(\Delta))$ *for all* $\Delta \in \mathrm{Inv}(nd, md)$*.*

4.4 Proof of Theorem 1.1

We combine all of the above results to prove Theorem 1.1. The compactified Jacobian $\overline{JC}$ is stratified into locally closed subsets J_Δ and by Proposition 3.8 they are isomorphic to affine spaces of dimension $\dim(\Delta)$ if Δ is admissible and empty otherwise. Therefore the Poincaré polynomial has the form

$$P(t) = \sum_{\Delta \text{ admissible}} t^{2\dim(\Delta)}.$$

Next, we consider the infinite set $\mathrm{Inv}(nd, md)$ of all (nd, md)-invariant subsets in $\mathbb{Z}_{\geq 0}$ and the equivalence relation on it. By Theorem 4.3 in each equivalence class there is a unique admissible representative. Furthermore, by Lemma 4.1 the "combinatorial dimension" $\dim(\Delta)$ depends only on the order of generators and cogenerators, and hence is constant on each equivalence class. Therefore we can write

$$P(t) = \sum_{\Delta \in \mathrm{Inv}(nd,md)/\sim} t^{2\dim(\Delta)}.$$

Finally, by Theorem 4.4 there is a bijection between the equivalence classes and Dyck paths in $(nd) \times (md)$ rectangle, and the statistic $\dim(\Delta)$ on the former corresponds to the statistic $\mathrm{codinv} = \delta - \mathrm{dinv}$ on the latter, so

$$P(t) = \sum_{D \in \mathrm{Dyck}(nd,md)} t^{2(\delta - \mathrm{dinv}(d))}.$$

5 Rational Shuffle Theorem and generic curves

5.1 Elliptic Hall algebra

We briefly recall some notations for the elliptic Hall algebra [5], and refer the reader to [3, 17, 23, 24] for more precise statements and details.

The elliptic Hall algebra $\mathcal{E}$ is generated by elements $P_{kn,km}$ for all possible (kn, km). The universal cover of the group $\mathrm{SL}(2,\mathbb{Z})$ acts on $\mathcal{E}$ by automorphisms. For coprime m and n we denote by $\gamma_{n,m}$ an element of $\mathrm{SL}(2,\mathbb{Z})$ such that $\gamma_{n,m}(1,0) = (n,m)$. Then one gets $P_{kn,km} = \gamma_{n,m}(P_{k,0})$.

Let Λ be the ring of symmetric functions in infinitely many variables. The elliptic Hall algebra $\mathcal{E}$ acts on Λ, and the multiplication operators by power sum symmetric functions p_k correspond (up to a scalar) to the generators $P_{k,0}$ of $\mathcal{E}$. Furthermore, $\mathcal{E}$ is graded, and the grading is compatible with the grading on Λ. The generator $P_{kn,km}$ has degree kn.

5.2 Cherednik-Danilenko conjecture

Consider a Puiseaux expansion of plane curve singularity:

$$y = b_1 x^{\frac{m_1}{r_1}} + b_2 x^{\frac{m_2}{r_1 r_2}} + b_3 x^{\frac{m_3}{r_1 r_2 r_3}} + \ldots, \qquad b_i \neq 0.$$

Here we assume $\mathrm{GCD}(m_i, r_1 \cdots r_i) = 1$. The exponents are related to characteristic pairs (r_i, s_i) by the equations $m_1 = s_1, m_i = s_i + r_i m_{i-1}$ $(i > 1)$. Given a sequence of characteristic pairs $(r_1, s_1), \ldots, (r_\ell, s_\ell)$, the authors of [6] define a sequence of symmetric functions $f_{\ell+1}, f_\ell, \ldots, f_1$ by setting $f_{\ell+1} = p_1$ and $f_k = \gamma_{r_\ell, s_\ell}(f_{k+1})(1)$, where f_{k+1} is viewed as a multiplication operator on Λ, and thus an element of $\mathcal{E}$.

Conjecture 5.1 ([6]). *Let f_1 be the symmetric function of degree $r_1 \cdots r_\ell$ obtained by the above procedure. The specialization of $(f_1, e_{r_1 \cdots r_\ell})$ at $t = 1$ agrees with the Poincaré polynomial of the compactified Jacobian of an algebraic curve with characteristic pairs $(r_1, s_1), \ldots, (r_\ell, s_\ell)$.*

In the case of generic curves (1.1) we have

$$y = x^{\frac{m}{n}} + \lambda x^{\frac{md+1}{nd}} + \ldots,\ m_1 = m,\ r_1 = n,\ m_2 = md+1,\ r_2 = d.$$

This means that $s_1 = m$ and $s_2 = 1$. To follow the above procedure, we first need to compute the operator $\gamma_{r_2,s_2}(p_1)$ and the corresponding symmetric function f_2. By [17, Corollary 6.5] we have $f_2 = \gamma_{d,1}(p_1)(1) = P_{d,1}(1) = e_d$. Therefore the next symmetric function is $f_1 = \gamma_{n,m}(e_d)(1)$. Then $(f_1, e_{nd}) = C_{nd,md}(q,t)$ follows from Rational Shuffle Theorem [23]. We conclude that for generic curves Conjecture 5.1 is true and follows from Theorem 1.1.

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