Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #69, 12 pp.

Proceedings of the 35^{th} Conference on Formal Power Series and Algebraic Combinatorics (Davis)

Alternating Sign Matrices and Descending Plane Partitions: a linear number of equivalent statistics

Ilse Fischer*[1], Hans Höngesberg†[1], and Florian Schreier-Aigner‡[1]

[1]*University of Vienna, Austria*

Abstract. There is the same number of alternating sign matrices as there is of cyclically symmetric lozenge tilings of a hexagon with a central triangular hole of size 2, but finding an explicit bijection has been an open problem for about 40 years now. This is even more surprising in the view of the fact that, when restricting to vertically symmetric alternating sign matrices, their number equals the number of such lozenge tilings that also exhibit an additional reflective symmetry. To approach such bijections, we present generalizations of these results that involve a linear number of equidistributed statistics. Prior to this work, only a constant number of such statistics were known.

Keywords: alternating sign matrices, plane partitions, non-intersecting lattice paths, bijective proofs

1 Introduction

An *alternating sign matrix* (ASM) is a square matrix with entries in $\{0, \pm 1\}$ such that, in each row and each column, 1's and -1's alternate and sum to 1. All 3×3 ASMs are given next.

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \quad \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} \quad \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix} \quad \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix} \quad \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}$$

ASMs had been introduced by Robbins and Rumsey in the 1980s [15] and they conjectured that the number of $n \times n$ ASMs is $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$, which was proven after considerable effort by Zeilberger [16] (in fact, he showed a more general result that includes an additional parameter). Soon after that, Kuperberg used methods from statistical physics to give another, shorter proof [10] (of the special case).

*ilse.fischer@univie.ac.at. Ack. supp. from the Austrian Science Foundation FWF, P34931
†hans.hoengesberg@univie.ac.at. Ack. supp. from the Austrian Science Foundation FWF, P34931
‡florian.schreier-aigner@univie.ac.at. Ack. supp. from the Austrian Science Foundation FWF, P34931 and J4387

Further observations caused considerable excitement in the combinatorics community, in particular $\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$ appears also as the enumeration formula for *descending plane partitions* (DPPs) with parts no greater than n [3, 12], which are defined next. A *strict partition* is a sequence $\lambda = (\lambda_1, \ldots, \lambda_n)$ of positive integers with $\lambda_1 > \ldots > \lambda_n$, and the *shifted Young diagram* of shape λ has λ_i cells in row i, where each row is indented by one cell to the right with respect to the previous row. A DPP is a filling of a shifted Young diagram with positive integers such that rows decrease weakly and columns decrease strictly, and the first part of each row is greater than the length of its row and less than or equal to the length of the previous row. The DPPs with all entries less than or equal to 3 are given next.

$$\emptyset \quad \begin{array}{|c|}\hline 2 \\ \hline \end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 3 & 3 \\ \hline \multicolumn{1}{c|}{} & 2 \\ \cline{2-2} \end{array}$$

There has been much effort to construct explicit bijections between ASMs and DPPs, but until recently without much progress. Very recently, a bijective proof [6] of an identity that implies the equinumerosity of ASMs and DPPs was established. The bijection is quite involved, and uses signed sets and a generalization of the involution principle by Garsia and Milne [7]. It seems fair to say that a perfect understanding of these relations despite the many efforts is still missing.

The main purpose of this extended abstract is to shed light on the mysterious relation between ASMs and DPPs by introducing a pair of $n+3$ equidistributed statistics, one set of $n+3$ statistics on ASMs and the other set on DPPs. Finding such statistics is a natural approach to construct bijections. Prior to this work, a constant number of such statistics were known. The extended abstract is a summary of our two preprints [1, 5] and we refer the reader to these preprints for all the details.

The introduction of our statistics is at the cost of replacing both ASMs and DPPs by somewhat modified objects, see Sections 2 and 3, respectively, however the relation of these modified objects to ASMs and DPPs is easily established by combinatorial means. We are also able to obtain a result that is similar in vein for vertically symmetric ASMs and the corresponding DPP-objects (Section 4). As the results are also phrased in terms of *non-intersecting lattice paths* on the DPP-side, we spend the rest of the introduction by presenting the lozenge tilings that correspond to DPPs which in turn are in easy bijective correspondence with the lattice paths.

DPPs with parts no greater than n are known [9] to be in easy bijective correspondence with cyclically symmetric lozenge tilings of a hexagon with side lengths $n, n+2, n, n+2, n, n+2$ that have a central triangular hole of size 2, see Figure 1. Such tilings can be encoded as families of non-intersecting lattice paths (*n-DPP paths*), see Figure 1. More precisely, the *n-DPP paths* have starting points $A_i = (0, i-2)$ and end points $E_i = (i, 0)$ for $i \in S$ where $S \subseteq \{2, \ldots, n\}$ and use steps in $(1,0)$-direction and $(0,-1)$-direction (note that our convention differs from the one more frequently used

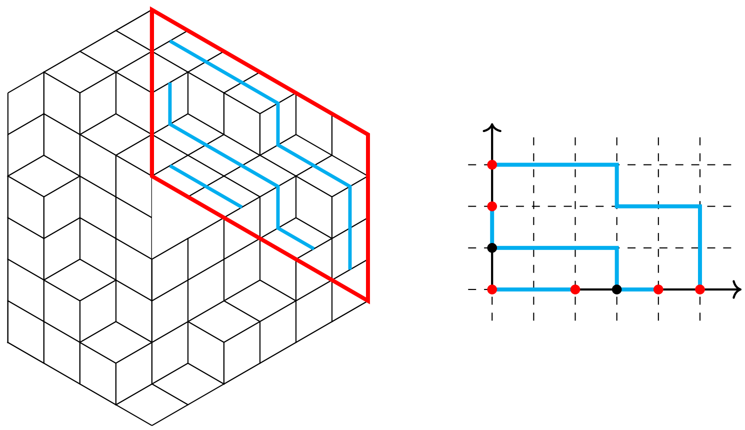

Figure 1: A cyclically symmetric lozenge tiling of a hexagon with the fundamental domain framed in red (left) and the corresponding 5-DPP paths.

by the reflection along the line $y = x$. In Section 4, we consider *vertically symmetric alternating sign matrices* (VSASMs). Due to the vertical symmetry, the central column of a VSASM has to exist and is of the form $(1, -1, 1, \ldots, -1, 1)^\top$; hence, VSASMs only exist for an odd order. Amongst the different objects that are known to be equinumerous to ASMs, cyclically symmetric lozenge tilings of a hexagon with a central triangular hole are particularly well-suited for imposing an additional condition that corresponds to the vertical symmetry of VSASMs. In fact, the axial symmetry can be literally transferred to lozenge tilings: it has been shown [14, 11] that $(2n+1) \times (2n+1)$ VSASMs are equinumerous with cyclically symmetric lozenge tilings of a hexagon with sides of alternating lengths $2n+2$ and $2n$ and with a central triangular hole of side length 2 that exhibit an additional reflective symmetry. Figure 2 illustrates an example of such a tiling.

2 Arrowed monotone triangles

A *monotone triangle* is a triangular array $(m_{i,j})_{1 \le j \le i \le n}$ of integers of the form

$$\begin{array}{ccccccccc} & & & & m_{1,1} & & & & \\ & & & m_{2,1} & & m_{2,2} & & & \\ & & \cdot^{\cdot^{\cdot}} & & & & \ddots & & \\ & m_{n-1,1} & \cdots & \cdots & \cdots & \cdots & \cdots & m_{n-1,n-1} & \\ m_{n,1} & & m_{n,2} & \cdots & \cdots & \cdots & \cdots & & m_{n,n} \end{array}$$

with weakly increasing entries along northeast- and southeast-diagonals, i.e., $m_{i+1,j} \le m_{i,j} \le m_{i+1,j+1}$, and strictly increasing rows, i.e., $m_{i,j} < m_{i,j+1}$. It is well known that

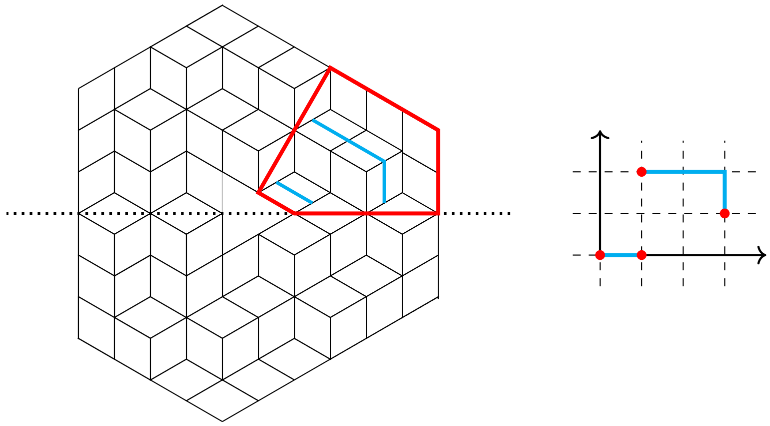

Figure 2: Cyclically symmetric lozenge tiling with an additional reflective symmetry of a hexagon with alternating side lengths $2n+2$ and $2n$ for $n=2$. The fundamental domain framed in red corresponds to a sixth of the hexagon. The axis of reflection is indicated with a dotted line. The tiling corresponds to non-intersecting lattice paths from $(i-1, 2i-2)$ to $(2i-1, i-1)$ for $1 \leq i \leq n$ with step set $\{(1,0),(0,-1)\}$.

$n \times n$ ASMs are in bijection with monotone triangles with bottom row $(1, 2, \ldots, n)$. As we see in Section 4, monotone triangles with bottom row $(0, 2, \ldots, 2n-2)$ are in bijection with $(2n+1) \times (2n+1)$ VSASMs in a similar way. Our modified ASM-objects are the following decorated monotone triangles.

An *arrowed monotone triangle* (AMT) is a monotone triangle $(m_{i,j})_{1 \leq j \leq i \leq n}$ together with a decoration of its entries by one of the symbols $\nwarrow$, $\nearrow$, $\nwarrow\!\!\nearrow$ such that the decoration of $m_{i+1,j}$ has to be $\nwarrow$ if $m_{i+1,j} = m_{i,j}$, while the decoration of $m_{i+1,j+1}$ has to be $\nearrow$ if $m_{i+1,j+1} = m_{i,j}$. In other words, an arrow indicates that the decorated entry has to be different from the entry it is pointing to. We write ${}^{\nwarrow}m_{i,j}$, $m_{i,j}{}^{\nearrow}$, ${}^{\nwarrow}m_{i,j}{}^{\nearrow}$ if the entry $m_{i,j}$ is

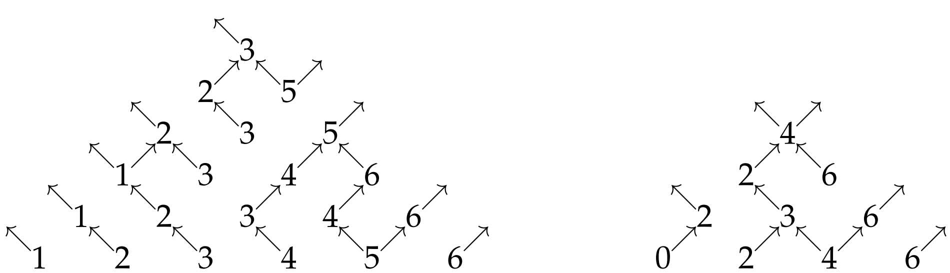

Figure 3: An arrowed monotone triangle with bottom row $(1,2,3,4,5,6)$ on the left and with bottom row $(0,2,4,6)$ on the right.

decorated with $\nwarrow$, $\nearrow$, or $\nwarrow\!\!\nearrow$, respectively. See Figure 3 for an example. We assign to an

AMT $M = (m_{i,j})_{1 \le j \le i \le n}$ the weight

$$W_M(u,v,w;\mathbf{x}) = u^{\#\nearrow} v^{\#\nwarrow} w^{\#\nwarrow\!\!\!\nearrow} \prod_{i=1}^{n} x_i^{\sum_{j=1}^{i} m_{i,j} - \sum_{j=1}^{i-1} m_{i-1,j} + \#(\nearrow \text{ in row } i) - \#(\nwarrow \text{ in row } i)}.$$

The AMT in Figure 3 (left) has weight $u^7 v^{11} w^3 x_1^2 x_2^5 x_3^2 x_4^3 x_5^3 x_6^2$ and the AMT in Figure 3 (right) has weight $u^5 v^3 w^2 x_1^4 x_2^4 x_3^2 x_4^4$. The exponents of the $n+3$ variables are the $n+3$ statistics on the ASM-side of our story. Note that there is a very satisfying analogy to the weight of Gelfand-Tsetlin patterns that corresponds to the Schur weight on semistandard Young tableaux under the classical bijection as this weight is simply $\prod_{i=1}^{n} x_i^{\sum_{j=1}^{i} a_{i,j} - \sum_{j=1}^{i-1} a_{i-1,j}}$ for a given Gelfand-Tsetlin pattern $A = (a_{i,j})_{1 \le j \le i \le n}$.

AMTs and monotone triangles are related as follows. Fix a monotone triangle M and consider the set of all AMTs which are mapped to M by forgetting the decorations. We claim that the sum of weights of these AMTs is equal to 1 when setting

$$u = v = 1, \quad w = -1, \quad x_1 = \ldots = x_n = 1. \tag{2.1}$$

Indeed, this follows from the observation that each entry of M that is different from its northwest-neighbor and its northeast-neighbor can be decorated with any of $\nwarrow, \nearrow, \nwarrow\!\!\!\nearrow$, while all the other entries can be decorated with either $\nwarrow$ or $\nearrow$. (In [1, Proposition 3.3] we also identify a weight on monotone triangles directly that is up to a simple factor the sum of weights of all arrowed monotone triangles that are obtained by decorating the monotone triangle.) Therefore, we may consider AMTs with bottom row $(1,2,\ldots,n)$ as our modified objects on the ASM-side.

For a sequence $L = (L_1, \ldots, L_n)$ of integers (not necessarily decreasing), we define the *extended Schur polynomial* $s_L(\mathbf{x})$ as $s_L(\mathbf{x}) = \frac{\det_{1 \le i,j \le n}\left(x_i^{L_j+n-j}\right)}{\prod_{1 \le i < j \le n}(x_i - x_j)}$. For an equivalent definition see [1, Equation (3.6), (5.1)]. We are now able to state our first main result.

Theorem 1 ([1, Theorem 2.2])**.** *The generating function of arrowed monotone triangles with bottom row $k_1 < \ldots < k_n$ is*

$$\prod_{i=1}^{n} (u X_i + v X_i^{-1} + w) \prod_{1 \le p < q \le n} \left(u \mathrm{E}_{k_p} + v \mathrm{E}_{k_q}^{-1} + w \mathrm{E}_{k_p} \mathrm{E}_{k_q}^{-1} \right) s_{(k_n, k_{n-1}, \ldots, k_1)}(x_1, \ldots, x_n), \tag{2.2}$$

where E_k denotes the shift operator, defined as $\mathrm{E}_k p(k) = p(k+1)$.

This is a multivariate extension of the operator formula of number of monotone triangles with bottom row [4] and one of the main ingredient in the proofs of our other results.

3 On the relation between ASMs and DPPs

We present the modified objects replacing DPPs along with the multivariate weight with $n+3$ variables that give rise to the $n+3$ statistics on the DPP-side.

Extended n-DPP paths are families of non-intersecting lattice paths with starting points $A_i' = (-i, i-2)$ and end points $E_i' = (i, -2)$ for $i \in S$ and a given subset $S \subseteq \{1, \dots, n\}$ which stay weakly below the line $y = n-2$ and have the following step sets and weights:

(i) In the region $\{(x,y) | x < 0\}$, the step set is $\{(1,0), (0,1)^2\}$ with weights $x_{p+q+2}v^{-1}$ for a $(1,0)$ step ending in (p,q), and where $(1,0)^2$ indicates that there are two types of $(0,1)$ steps, one of which has weight $x_{p+q+2}v^{-1}w$ when ending in (p,q) and the other has weight 1.

(ii) In the region $\{(x,y) | x \geq 0, y \geq -1\}$, the step set is $\{(1,0), (0,-1)\}$, with weights ux_{q+2} for a $(1,0)$ step at height q and weight 1 for a $(0,-1)$ step.

(iii) If we go below the line $y = -1$, the step set is $\{(1,-1), (0,-1)\}$, with weights w for a $(1,-1)$ step and 1 for a $(0,-1)$ step.

The weight of extended n-DPP paths is $v^{\binom{n+1}{2}}$ times the product of the weights of all steps. See Figure 4 for an example. Note that in [1, Section 4] we consider a (very minor) modification of these paths. Analogously to the situation for AMTs and ASMs, we argue

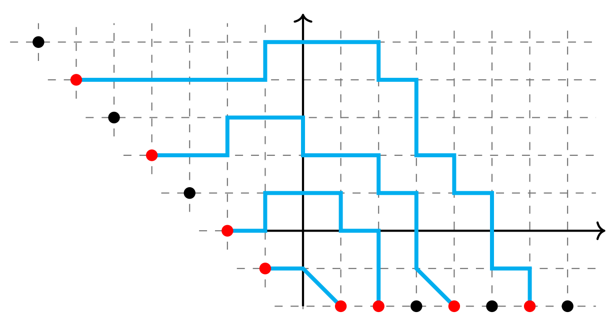

Figure 4: A family of extended 5-DPP paths with $S = \{1,2,4,6\}$ and weight $u^{11}v^5w^5x_1^5x_2^4x_3^6x_4^5x_5^2x_6^2x_7^3$ where all $(0,1)$ steps left of the y-axis are of first type.

next that the generating function of extended n-DPP paths is the number of n-DPP paths under the specialization (2.1). Indeed, strictly left of the y-axis $(0,1)$ steps come in pairs with weights of opposite sign; hence, left of the y-axis, we can assume to have only $(1,0)$ steps. The following is a sign-reversing involution between all such extended n-DPP paths for which not all paths end by two $(0,-1)$ steps (and hence cancels these paths): pick the right-most path ending by either a diagonal $(1,-1)$ step or by a $(1,0)$ step followed by an $(0,-1)$ step and replace in the second case the two steps by a diagonal step and vice-versa. By deleting all steps left of the y-axis and below the x-axis of the remaining extended DPP paths, we obtain the above defined DPP paths.

Our second main result is a consequence of [1, Theorem 2.6] using insights from Section 4 in that paper.

Theorem 2. *The generating function of AMTs with bottom row $(1, 2, \ldots, n)$ is equal to the generating function of extended n-DPP paths.*

It is an open problem to construct a weight-preserving bijection between AMTs with bottom row $(1, 2, \ldots, n)$ and extended n-DPP paths. One surprising observation we made in our work is the following: Suppose that one is able to find a weight preserving bijection between AMTs with bottom row $(1, 2, \ldots, n)$ and extended n-DPP paths, which is further compatible with the sign reversing involution described above and the one hinted at in the paragraph around (2.1). Then, as a direct consequence, one would obtain a longed for bijection between ASMs and DPPs. However, we show that such a compatible weight preserving bijection does not exist for $n \geq 3$.

Proposition 1. *Let $n \geq 3$. Then there exists no weight-preserving bijection between AMTs with bottom row $(1, \ldots, n)$ and extended n-DPP paths which induces a bijection between monotone triangles with bottom row $(1, \ldots, n)$ and n-DPP paths by restricting to n-DPP paths as described above and by ignoring the arrows in the corresponding AMT.*

Proof. We extend n-DPP paths to extended n-DPP paths by adding an appropriate number of $(1, 0)$ steps at the beginning of each path and by adding two $(0, -1)$ steps at the end of each path. The exponent of x_1 thereby counts the number of paths in a given family of n-DPP paths. Hence the maximum exponent of x_1 is $n - 1$ and obtained by the unique family of n-DPP paths consisting of $n - 1$ paths. For AMTs, the exponent of x_1 is $n - 1$ exactly if the top entry is n. The number of monotone triangles with top entry n is equal to the number of monotone triangles with bottom row $(1, \ldots, n - 1)$ since the top entry n forces all entries in the right-most northwest-diagonal to be equal to n. For $n \geq 3$, there are at least 2 of such monotone triangles. Hence not all of them can be reached through a weight-preserving bijection. □

There exists also a version of Theorem 2 in terms of certain plane partitions, which we discuss next. Let λ be a partition and $(a_1, \ldots, a_l | b_1, \ldots, b_l)$ its Frobenius notation, i.e., $a_i = \lambda_i - i$ and $b_i = \lambda'_i - i$, where $l = \max_i\{\lambda_i \geq i\}$. We say that λ is *near-balanced* if, for all i, either $a_i = b_i$ or $a_i = b_i + 1$. We assign the weight $\mathrm{W}(\lambda) = w^{l + \sum_{i=1}^{l}(b_i - a_i)}$. Now, a *set-valued near-balanced column strict plane partition* (SBCSPP) D of shape λ and order n is a filling of a near-balanced partition λ with non-empty subsets of $\{1, 2, \ldots, n\}$ such that strictly above the diagonal the subsets are singletons, and (i) rows decrease weakly in the sense that the maxima of the sets form a weakly decreasing sequence if read from left to right, and (ii) columns decrease strictly in the sense that for two adjacent cells in a column, all elements in the top cell are strictly greater than all elements in the bottom cell. The weight of D is as follows

$$\mathrm{W}(D) = \mathrm{W}(\lambda) \cdot u^{\text{\# cells strictly above the main diagonal}} \cdot v^{\binom{n+1}{2} - \text{\# entries on and below the main diagonal}} \cdot w^{\text{\# entries} - \text{\# cells}} \cdot \prod_{i=1}^{n} x_i^{\#i \text{ in D}}.$$

Next is an example of an SBCSPP of order 9 and weight $u^{16}v^{26}w^3x_1^5x_2^5x_3^4x_4^5x_5^4x_6^4x_7^5x_8^3$.

8	8	8	7	7	6	4	1
7	7	7	6	5	5		
6	6	5	4	4	4		
5	4	3	3,2	3	2		
3	2	2,1	1				
2	1						
1							

Theorem 3 ([1, Theorem 2.6]). *The generating function of arrowed monotone triangles with bottom row $1, 2, \ldots, n$ is equal to the generating function of set-valued near-balanced column strict plane partitions with parts in $\{1, 2, \ldots, n\}$.*

Concerning the proof of Theorem 3 (and also of Theorem 4), the rough idea is to use Theorem 1 to obtain a bialternant formula for the generating function of AMTs. We then transform this generating function into a Jacobi-Trudi-type determinant, which we then interpret combinatorially using the Lindström-Gessel-Viennot theorem.

4 Reflective symmetry

We have indicated in Section 2 that there is a one-to-one correspondence between monotone triangles with bottom row $(1, 2, \ldots, n)$ and $n \times n$ ASMs. By applying similar ideas, we see that $(2n+1) \times (2n+1)$ VSASMs are in bijection with monotone triangles with bottom row $(0, 2, \ldots, 2n-2)$: First we rotate the given matrix by 90° and then map it to a monotone triangle with $2n+1$ rows of which we only need to consider the top n rows due to the axial symmetry of the original VSASM. Since the central column of the VSASM was $(1, -1, 1, -1, \ldots, -1, 1)^\top$, the top n rows constitute a monotone triangle with bottom row $(2, 4, \ldots, 2n)$. Finally, we subtract 2 from each entry.

Thanks to the correspondence between VSASMs and these monotone triangles, Theorem 1 yields an enumerational refinement of $(2n+1) \times (2n+1)$ VSASMs by $n+3$ parameters. An obvious question is whether there is a corresponding family of non-intersecting lattice paths — that might be interpreted as some kind of plane partitions or lozenge tilings afterwards — with equally distributed statistics. There are indeed several such objects but just like we needed to broaden the notion of monotone triangles in favour of AMTs in order to incorporate additional statistics, we need to consider here more intricate families of non-intersecting lattice paths than those we get from lozenge tilings; see Figure 2. However, the weights of these lattice paths are signed. In Theorem 4, we provide one such family of non-intersecting lattice paths, an example of which is depicted in Figure 5. By eliminating the sign, these paths can then be further transformed into pairs of certain plane partitions in Theorem 5.

Theorem 4 ([5, Theorem 2.2])**.** *The generating function of AMTs with bottom row $(0, 2, \dots, 2n - 2)$ is equal to the signed generating function of n lattice paths with starting points $(-1, 1)$, $(-2, 2)$,…,$(-n, n)$ and end points $(0, 1)$, $(1, 0)$,…,$(n - 1, -n + 2)$ such that:*

(i) In the region $\{(x, y)|x \leq 0\}$, the step set is $\{(1, 1),(1, 0)\}$, and steps of type $(1, 0)$ are equipped with the weight w.

(ii) In the region $\{(x, y)|x \geq 0, y \geq 1\}$, the step set is $\{(1, -1),(0, -2)\}$, and steps of type $(0, -2)$ are equipped the weight $-uv$.

(iii) In the region $\{(x, y)|x \geq 0, y \leq 1\}$, the step set is $\{(-1, 0)^2,(0, -1)\}$, and one type of horizontal steps with distance d from the line $y = 2$ is equipped with the weight ux_d and the second type is equipped with the weight vx_d^{-1}.

The paths are non-intersecting in the first and in the third region. In the second region, we distinguish between even and odd paths depending on whether they contain only even or only odd lattice points, respectively. Lattice paths of the same type are not intersecting each other, but an odd path may have an intersection with an even path.

The weight of a family of lattice paths is $\prod_{i=1}^{n} x_i^{n-1}$ multiplied by the product of the weights of all its steps where the weight of a step is 1 if it has not been specified. Let σ be the permutation so that the i-th starting point is connected to the $\sigma(i)$-th end point, then the sign of the family is $\operatorname{sgn}\sigma$.

In Section 5 of [5], we have related the lattice paths of Theorem 4 to non-intersecting lattice paths from $\{(i - 1, 2i - 2)|1 \leq i \leq n\}$ to $\{(2i - 1, i - 1)|1 \leq i \leq n\}$ with step set $\{(1, 0),(0, -1)\}$ by purely combinatorial means. The latter are in one-to-one correspondence with cyclically and vertically symmetric lozenge tilings of a hexagon with side lengths $2n + 2$, $2n$, $2n + 2$, $2n$, $2n + 2$, $2n$ and a central triangular hole of size 2 as described in Figure 2. For this purpose, we have to consider the unrefined case by setting $u = v = 1$, $w = -1$ and $x_1 = \cdots = x_n = 1$; compare with (2.1).

In order to present a class of objects with signless weights, we recall the notion of column-strictness and row-strictness regarding plane partitions. A column-strict plane partition is a filling of a Young diagram with positive integers that weakly decrease along rows and strictly decrease down columns, whereas a row-strict plane partition is

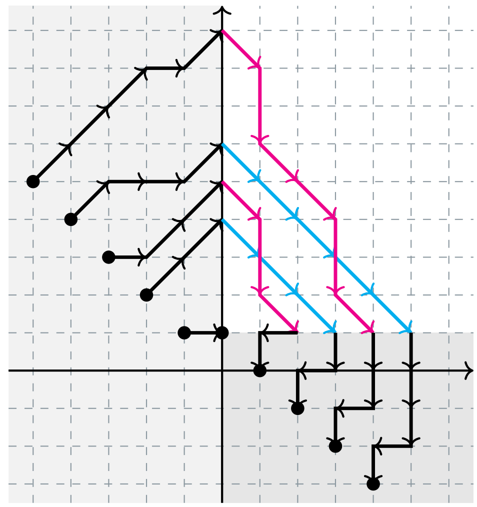

Figure 5: A family of non-intersecting lattice paths as defined in Theorem 4 for $n = 5$. In the second region, we draw even paths in red and odd paths in blue. The associated permutation is $\sigma = (1\ 3\ 2\ 5\ 4)$, and the weight is $(-uv)^3 u^4 w^4 x_1^5 x_2^5 x_3^5 x_4^5 x_5^4$ when always choosing the first type for horizontal steps in the third region.

a filling of a Young diagram with positive integers that weakly decrease down columns and strictly along rows. The following described pairs of plane partitions provide a signless generating function that coincides with the $(n+3)$-parameter refinement of VSASMs. Figure 6 shows an example of such a pair.

Theorem 5 ([5, Theorem 2.5])**.** *The generating function of AMTs with bottom row $(0, 2, \ldots, 2n-2)$ is equal to the generating function of pairs (P, Q) of plane partitions of the same shape with n rows (allowing also rows of length zero) such that P is a column-strict plane partition and Q is a row-strict plane partition, and the entries of P in the i-th row from the bottom are no greater than $2i$, while the entries of Q in the i-th row from the bottom are no greater than i. The weight of such a pair is given by the following monomial:*

$$w^{\binom{n+1}{2} - \#\text{ entries in } Q} \prod_{i=1}^{n} x_i^{n-1} (u x_i)^{\#\,(2i-1)\text{ in } P} (v x_i^{-1})^{\#\,(2i)\text{ in } P}.$$

We are able to prove Theorem 5 in two different ways: one proof is rather based on algebraic manipulations, whilst the other one is more combinatorial. In fact, we can apply two sign-reversing involutions on the lattice paths of Theorem 4 to obtain the pairs of plane partitions in Theorem 5. We essentially start with the family of lattice paths in Theorem 4, but where paths might intersect in the second and third region. The first sign-reversing involution eliminates intersections in the second region and shows that the generating function has no negative coefficients although each family of lattice paths

9	7	7	3
8	6		
5	5		
2			

4	3	2	1
4	2		
3	2		
1			

Figure 6: Pair of a column-strict plane partition (left) and a row-strict plane partition (right) of the same shape for $n = 5$. The weight of this pair is given by $w^6 x_1^4 x_2^4 x_3^4 x_4^4 x_5^4 (vx_1^{-1})^1 (ux_2)^1 (ux_3)^2 (vx_3^{-1})^1 (ux_4)^2 (vx_4^{-1})^1 (ux_5)^1$.

is equipped with a sign. The intersections in the third regions are then dealt with by a second sign-reversing involution of "Gessel-Viennot" type.

In addition to providing a signless interpretation in terms of plane partitions, there is another benefit of Theorem 5: It yields an expansion of the generating function of AMTs with bottom row $(0, 2, \ldots, 2n-2)$ into symplectic characters since column-strict plane partitions as they appear in Theorem 5 are in bijective correspondence with symplectic tableaux as described by Koike and Terada [8]. The coefficients in this expansion are given by *totally symmetric self-complementary plane partitions*. This can be seen as follows: Given a row-strict plane partition with n rows (including rows of length zero) and entries in the i-th row from the bottom not being greater than i, we replace each entry π by $n+1-\pi$ and conjugate the resulting array such that we obtain a semistandard Young tableau with entries from 1 to n such that the entries in the i-th column are no smaller than i. By applying the standard procedure of mapping semistandard Young tableaux into Gelfand-Tsetlin patterns and adding an additional diagonal and adding 1 to each entry, we obtain so-called *Magog triangles* — triangular arrays of positive integers with weakly increasing $\nearrow$- and $\searrow$-diagonals such that no entry in the ith $\nearrow$-diagonal from the left exceeds i for every i — which are known to be in bijection with totally symmetric self-complementary plane partitions in a $(2n+2) \times (2n+2) \times (2n+2)$ box [13]. We give the Magog triangle that corresponds to the row-strict plane partition in Figure 6 below. Note that a similar result is known for the ordinary case: the generating function of AMTs with bottom row $(1, 2, \ldots, n)$ exhibits an expansion into Schur functions whose coefficients are given by weights of *totally symmetric plane partitions* [2].

4	3	2	1
4	2		
3	2		
1			

$$\Rightarrow \quad \begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 1 & & 3 & & & \\ & & 1 & & 1 & & 2 & & 4 & & \\ & 1 & & 1 & & 2 & & 4 & & 4 & \\ 1 & & 1 & & 2 & & 2 & & 4 & & 5 \end{array}$$

References

[1] F. Aigner and I. Fischer. "The relation between alternating sign matrices and descending plane partitions: n+3 pairs of equivalent statistics". 2021. arXiv:2106.11568.

[2] F. Aigner, I. Fischer, M. Konvalinka, P. Nadeau, and V. Tewari. "Alternating sign matrices and totally symmetric plane partitions". *Sém. Lothar. Combin.* **84B** (2020), # 77.

[3] G. Andrews. "Plane partitions. III. The weak Macdonald conjecture". *Invent. Math.* **53**.3 (1979), pp. 193–225. DOI.

[4] I. Fischer. "The number of monotone triangles with prescribed bottom row". *Adv. in Appl. Math.* **37**.2 (2006), pp. 249–267. DOI.

[5] I. Fischer and Höngesberg. "Alternating sign matrices with reflective symmetry and plane partitions: $n+3$ pairs of equivalent statistics". 2022. arXiv:2207.04469.

[6] I. Fischer and M. Konvalinka. "The mysterious story of square ice, piles of cubes, and bijections". *Proc. Natl. Acad. Sci. USA* **117**.38 (2020), pp. 23460–23466. DOI.

[7] A. M. Garsia and S. C. Milne. "Method for constructing bijections for classical partition identities". *Proc. Nat. Acad. Sci. U.S.A.* **78**.4, part 1 (1981), pp. 2026–2028. DOI.

[8] K. Koike and I. Terada. "Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank". *Adv. Math.* **79**.1 (1990), pp. 104–135. DOI.

[9] C. Krattenthaler. "Descending plane partitions and rhombus tilings of a hexagon with a triangular hole". *European J. Combin.* **27**.7 (2006), pp. 1138–1146. DOI.

[10] G. Kuperberg. "Another proof of the alternating-sign matrix conjecture". *Internat. Math. Res. Notices* 3 (1996), pp. 139–150. DOI.

[11] G. Kuperberg. "Symmetry classes of alternating-sign matrices under one roof". *Ann. of Math. (2)* **156**.3 (2002), pp. 835–866. DOI.

[12] W. Mills, D. P. Robbins, and H. Rumsey Jr. "Alternating sign matrices and descending plane partitions". *J. Comb. Theory, Ser. A.* **34**.3 (1983), pp. 340–359. DOI.

[13] W. Mills, D. P. Robbins, and H. Rumsey Jr. "Self-complementary totally symmetric plane partitions". *J. Comb. Theory, Ser. A.* **42**.2 (1986), pp. 277–292. DOI.

[14] W. H. Mills, D. P. Robbins, and H. Rumsey Jr. "Enumeration of a symmetry class of plane partitions". *Discrete Math.* **67**.1 (1987), pp. 43–55. DOI.

[15] D. P. Robbins and H. Rumsey Jr. "Determinants and alternating sign matrices". *Adv. in Math.* **62**.2 (1986), pp. 169–184. DOI.

[16] D. Zeilberger. "Proof of the alternating sign matrix conjecture". *Electron. J. Combin.* **3**.2 (1996). The Foata Festschrift, Research Paper 13, 84 pp. (electronic). Link.