*Séminaire Lotharingien de Combinatoire* **89B** (2023)
Article #7, 12 pp.

*Proceedings of the 35$^{th}$ Conference on Formal Power Series and Algebraic Combinatorics (Davis)*

# Faces of Directed Edge Polytopes

Yasuhide Numata*[1], Yusuke Takahashi[2] , and Dai Tamaki†[3]

[1] *Department of Mathematics, Hokkaido University, Sapporo, Japan*
[2] *Graduate School of Science and Technology, Shinshu University, Matsumoto, Japan*
[3] *Department of Mathematics, Shinshu University, Matsumoto, Japan*

**Abstract.** Given a finite quiver (directed graph) without loops and multiedges, the convex hull of the column vector of the incidence matrix is called the directed edge polytope and is an interesting example of lattice polytopes. In this article, we give a complete characterization of facets of the directed edge polytope of an arbitrary finite quiver without loops and multiedges in terms of the connectivity and the existence of a rank function. Our result can be regarded as an extension of the result of Higashitani et al. [6] on facets of symmetric edge polytopes to directed edge polytopes. When the quiver in question has a rank function, we obtain a characterization of faces of arbitrary dimensions. This article is an extended abstract of [10].

**Résumé.** Pour un graphe orienté fini simple, l'enveloppe convexe du vecteur colonne de la matrice d'incidence est appelée le polytope d'arête orienté. C'est un intéressant exemple de polytopes intégraux. Dans cet article, nous donnons une caractérisation complète des facettes de polytope d'arête orienté en termes de connectivité et d'existence d'une fonction de rang. Notre résultat peut être considéré comme une extension du résultat de Higashitani et al. [6] sur les facettes de polytope d'arête symétrique. Pour un graphe orienté avec une fonction de rang, on obtient une caractérisation de faces de dimensions arbitraires. Cet article est un résumé étendu de [10].

**Keywords:** Kantorovich–Rubinstein polytopes; fundamental polytopes; symmetric edge polytopes; directed edge polytopes; characterization of faces; $f$-vectors

## 1 Introduction

For a motivation of optimal transportation problems, Vershik proposed to study the fundamental polytope constructed from a finite metric space in [11]. For a metric space $([n], d)$, the *fundamental polytope* $\mathrm{KR}([n], d)$ is defined to be the convex hull of

$$\left\{ \epsilon^d_{(i,j)} = \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{d(i,j)} \;\middle|\; i, j \in [n], i \neq j \right\},$$

*nu@math.sci.hokudai.ac.jp. The first author was partially supported by JSPS KAKENHI Grant Number JP18K03206.

†rivulus@shinshu-u.ac.jp. The third author was partially supported by JSPS KAKENHI Grant Number JP20K03579.

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. The polytope $\mathrm{KR}([n], d)$ is also called the *Kantorovich–Rubinstein polytope* [7, 8]. In the case of a tree-like metric space, Delucchi and Hoessley proved a nice formula of the $f$-vector by using the relation between tree-like metric spaces and hyperplane arrangements in [3]. The starting point of this work was the second author's attempt to extend their work to graphs with cycles.

For a finite simple graph $G$ whose vertex set is $[n]$, we can define a metric $d_G$ on $[n]$ by the minimum length of paths. The Kantorovich–Rubinstein polytope $\mathrm{KR}([n], d_G)$ has already been studied as the *symmetric edge polytope* $\mathrm{SE}(G)$, introduced by Matsui et al. [9]. If $G$ is the complete graph $K_n$, then it is also called the *root polytope* of the root system $\mathrm{A}_{n-1}$ and Cho completely determined its faces in [1].

We can also generalize the definition of symmetric edge polytope to a finite quiver (directed graph). In this article, a quiver $Q = (Q_0, Q_1)$ means a quiver without loops and multiedges. Since $Q$ has no loop and no multiedges, $Q_1$ is realized as a subset of $(Q_0 \times Q_0) \setminus \{ (i, i) \mid i \in Q_0 \}$. Let $Q$ be a quiver with $Q_0 = [n]$. We define $\varepsilon_{(i,j)} = \boldsymbol{e}_i - \boldsymbol{e}_j$ for $(i, j) \in Q_1$. Let $\varepsilon(Q_1)$ be the set of $\varepsilon_{(i,j)}$ for all $(i, j) \in Q_1$. We define the *directed edge polytope* $\mathrm{DE}(Q)$ to be the convex hull of $\varepsilon(Q_1)$. In other words, $\mathrm{DE}(Q)$ is the convex hull of column vectors of the incidence matrix of $Q$. The directed edge polytope is the convex polytope whose set of vertices is $\varepsilon(Q_1)$. See also [5]. The symmetric edge polytope $\mathrm{SE}(G)$ is nothing but $\mathrm{DE}(D(G))$, where $D(G)$ is the double of $G$, i.e. the quiver whose edge set is the collection of direct edges $(v, w)$ and $(w, v)$ for adjacent vertices $v, w$ in $G$.

The aim of this article is to give an explicit combinatorial description of all facets of $\mathrm{DE}(Q)$ for a quiver $Q$ and combinatorial descriptions of facets of $\mathrm{SE}(G) = \mathrm{KR}(G_0, d_{\mathrm{graph}})$ for a finite simple graph.

In general, the directed edge polytope $\mathrm{DE}(Q)$ with $Q_0 = [n]$ is a convex polytope in the vector space $\mathbb{R}^n$. Since the set of vertices of $\mathrm{DE}(Q)$ is given by $\varepsilon(Q_1)$, we can describe any face of $\mathrm{DE}(Q)$ in the form $\mathrm{DE}(R)$ for a subquiver $R$ with $R_0 = Q_0$. Let us call such a subquiver a *lluf subquiver*.

Our problem is to determine the lluf subquiver $R$ of $Q$ such that $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, i.e. $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$ and $\dim \mathrm{DE}(R) = \dim \mathrm{DE}(Q) - 1$. The existence of a rank function plays a key role in both problems. We call $\rho\colon Q_0 \to \mathbb{R}$ a *rank function* of $Q$ if $\rho(v) + 1 = \rho(w)$ for each edge $(v, w) \in Q_1$. As in Remark 3.3, such a function makes the vertex set $Q_0$ into a graded poset. To consider the dimension, we also need to consider a connected component $C$ of a quiver $Q$, i.e. a maximal subquiver such that any two distinct vertices in $C$ are connected by an undirected walk in $C$, where an undirected walk is a sequence $(v_0, v_1, \dots, v_l)$ of vertices in $Q$ such that $(v_t, v_{t+1}) \in Q_1$ or $(v_{t+1}, v_t) \in Q_1$ for all $t$. We define the *coconnectivity* $c(Q)$ of $Q$ by $c(Q) = |Q_0| - |\pi_0(Q)|$, where $\pi_0(Q)$ stands for the set of connected components of $Q$. We can describe the dimension of $\mathrm{DE}(Q)$ by the coconnectivity as follows:

**Theorem 1.1.** *For a finite quiver $Q$ without loops and multiedges, we have*

$$\dim(\mathrm{DE}(Q)) = \begin{cases} c(Q) - 1 & (\textit{if } Q \textit{ has a rank function}) \\ c(Q) & (\textit{otherwise}). \end{cases}$$

To determine faces, the contraction is also a key notion. We say that a luff subquiver $R$ of $Q$ is *component-wise full* if $C_1 = \{ (v, w) \in Q_1 \mid v, w \in C_0 \}$ for each connected component $C$ of $R$. For a component-wise full subquiver $R$ of a quiver $Q$, we define an equivalence relation $\sim$ on $Q_0$ by

$$v \sim w \iff v \text{ and } w \text{ are in the same connected component of } R\,,$$

and the equivalence class of $v \in Q_0$ is denoted by $[v]$. We define the contraction $Q/R$ to be the quiver such that $(Q/R)_0 = Q_0/_\sim$ and $(Q/R)_1 = \{ ([v], [w]) \mid (v, w) \in Q_1 \setminus R_1 \}$. Roughly speaking, $Q/R$ is the quiver obtained from $Q$ by collapsing each connected component of $R$ to a point. We can characterize facets as follows:

**Theorem 1.2.** *Let $Q$ be a finite quiver without loops and multiedges. For a lluf subquiver $R$ of $Q$ with $\dim(\mathrm{DE}(R)) = \dim(\mathrm{DE}(Q)) - 1$, $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$ if and only if one of the following conditions holds:*

1. *$c(R) = c(Q) - 1$, $R$ is a component-wise full subquiver of $Q$, and the contraction $Q/R$ is acyclic.*
2. *$c(R) = c(Q)$ and there exists a rank function $\rho$ of $R$ such that*

$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0$$

*for any $(v, w), (v', w') \in Q_1 \setminus R_1$.*

By iterating the process of taking facets, we can obtain lower dimensional faces. Thus we obtain a characterization of faces of $\mathrm{DE}(Q)$ for a quiver $Q$ with a rank function.

**Theorem 1.3.** *Suppose $Q$ has a rank function. For a proper subquiver $R$ of $Q$, the polytope $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$, if and only if $R$ is a component-wise full subquiver of $Q$ and $Q/R$ is acyclic.*

In the case of $D(G)$ for a simple graph $G$, it does not have a rank function and the condition (2) in Theorem 1.2 applies. Hence we have the following:

**Corollary 1.4.** *Let $G$ be a finite simple graph. For a lluf subquiver $R$ of $D(G)$ with $\dim(\mathrm{DE}(R)) = \dim(\mathrm{SE}(G)) - 1$, the following are equivalent:*

1. *$\mathrm{DE}(R)$ is a facet of $\mathrm{SE}(G)$.*

2. *$c(R) = c(D(G))$ and there exists a rank function $\rho$ of $R$ such that*

$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0$$

*for any $(v,w),(v',w') \in D(G)_1 \setminus R_1$.*

3. *$c(R) = c(D(G))$ and there exists a map $\rho$ from the set of vertices of $G$ to $\mathbb{R}$ such that*

$$\rho(v) - \rho(w) = \begin{cases} 1 & ((v,w) \in R_1) \\ -1 & ((w,v) \in R_1) \\ 0 & (\textit{otherwise}) \end{cases}$$

*for $(v,w) \in D(G)_1$.*

*Remark* 1.5. This is essentially equivalent to a characterization of facets of symmetric edge polytopes in Higashitani–Jochemko–Michałek [6] for a connected graph $G$. See [10, Remark 5.5] for the details.

*Remark* 1.6. In the case of the Kantorovich–Rubinstein polytope $\mathrm{KR}([n], d)$ for a general metric $d$, Gordon and Petrov [4] defined a quiver $Q(F)$ with vertex set $[n]$ from a face $F$ of $\mathrm{KR}([n], d)$ by

$$Q(F)_1 = \left\{ (v,w) \,\middle|\, \epsilon^d_{(v,w)} \in F \right\}.$$

They proved as Theorem 3 that $E$ is a subset of in $Q(F)_1$ for some facet $F$ of $\mathrm{KR}([n], d)$ if and only if there exists a function $\rho \colon E \to \mathbb{R}$ such that $\rho(v) - \rho(w) = d(v,w)$ for $(v,w) \in E$.

When $d = d_G$ for a graph $G$ with the vertex set $[n]$, $\mathrm{KR}([n], d_G) = \mathrm{SE}(G)$, the third condition in Corollary 1.4 is closely related to their condition. In fact, if $F$ is a facet of $\mathrm{SE}(G)$ represented by a luff subquiver $R$ of $D(G)$ by $F = \mathrm{DE}(R)$, then $R_1 \subset Q(F)$ and Theorem 3 of Gordon-Petrov says that there exists a function $\rho \colon [n] \to \mathbb{R}$ such that $\rho(v) - \rho(w) = 1$ for $(v,w) \in R_1$.

On the other hand, Corollary 1.4 gives a complete characterization of lluf subquivers $R$ of $D(G)$ such that $\mathrm{DE}(R)$ is a facet of $\mathrm{KR}([n], d_G)$, while Gordon-Petrov's Theorem 3 does not.

This article is organized as follows: After fixing notation and terminology in Section 2, we see sketches of proofs for Theorem 1.1 in Section 3.1, and for Theorem 1.2 in Section 3.2. We end this paper with sample computations in Section 4. This article is an extended abstract of the preprint [10]. We will omit details. See [10] for the details.

## 2 Notation and terminology

Here we fix notation and terminology for quivers. A quiver $Q$ is said to be

1. *acyclic* if there do not exist $v_0, \ldots, v_l \in Q_0$ such that $l > 1$, $v_l = v_0$, and $(v_t, v_{t+1}) \in Q_1$ for $t = 0, \ldots, l-1$,

2. *asymmetric* if $(w, v) \notin Q_1$ for any $(v, w) \in Q_1$, and

3. *symmetric* if $(v, w) \in Q_1$ for all $(w, v) \in Q_1$.

Note that a quiver may not be symmetric nor asymmetric.

The sequence $(v_0, v_1, \ldots, v_l)$ is called a *directed cycle* if $l > 1$, $v_l = v_0$, $(v_t, v_{t+1}) \in Q_1$ for $t = 0, \ldots, l-1$, and $v_i \neq v_j$ for any pair $(i, j)$ with $0 \leq i < j < l$. By definition, a quiver is acyclic if and only if it does not contain a directed cycle.

We define the *underlying graph* of a quiver $Q$ to be the (undirected) graph obtained from $Q$ by using all vertices of $Q$ and by replacing all directed edges of $Q$ with undirected edges. Underlying graphs may have multiple edges. The underlying graph of $Q$ is simple if and only if $Q$ is asymmetric.

In order to describe faces of directed edge polytopes, we need subquivers. A quiver $R$ is called a *subquiver* of $Q$ if $R_0 \subset Q_0$ and $R_1 \subset Q_1$. We say that a subquiver $R$ of $Q$ is

1. *proper* if $R_1$ is a proper subset of $Q_1$,

2. *full* if $R_1 = \{ (v, w) \in Q_1 \mid v, w \in R_0 \}$, and

3. *lluf* if $R_0 = Q_0$.

We make use of (undirected) walks to define connectivity of quivers. Let $Q$ be a quiver. An *undirected walk* from $v_0$ to $v_l$ in $Q$ is a sequence $(v_0, v_1, \ldots, v_l)$ of vertices in $Q$ such that $(v_t, v_{t+1}) \in Q_1$ or $(v_{t+1}, v_t) \in Q_1$ for all $t$. An undirected walk $(v_0, v_1, \ldots, v_l)$ is called

1. *closed* if $v_0 = v_l$, and

2. an *undirected cycle* if it is closed and $v_i \neq v_j$ for any pair $(i, j)$ with $0 \leq i < j < l$.

We say a quiver $Q$ is *connected* if, for any pair $(v, w)$ of vertices of $Q$, there exists an undirected walk from $v$ to $w$ . A connected maximal subquiver of $Q$ is called a *connected component* of $Q$. The set of all connected components of $Q$ is denoted by $\pi_0(Q)$. The number $|Q_0| - |\pi_0(Q)|$ is denoted by $c(Q)$ and is called the *coconnectivity* of $Q$. Note that a quiver is connected if and only if the underlying graph is connected. We say that a luff subquiver $R$ of $Q$ is *component-wise full* if each connected component $C$ of $R$ is a full subquiver of $Q$.

# 3 Sketch of proof

Here we give a sketch of proofs of Theorems 1.1 and 1.2. We omit the details. See [10] for the details.

## 3.1 Dimension

Here we give a sketch of proofs of Theorem 1.1. Note that even if $Q$ is connected, a subquiver $R$ representing a face of $\mathrm{DE}(Q)$ may not be connected. It turns out that the number of connected components is closely related to the dimension of $\mathrm{DE}(R)$. In fact, an upper bound is given by the coconnectivity $c(R)$.

**Lemma 3.1.** *Define a vector subspace $V_Q$ of $\mathbb{R}^{Q_0}$ by*

$$V_Q = \bigcap_{R \in \pi_0(Q)} \kappa_{R_0}^{\perp},$$

*where $\kappa_{R_0}^{\perp}$ is the hyeperplane orthogonal to $\kappa_{R_0} = \sum_{v \in R_0} \boldsymbol{e}_v$ in $\mathbb{R}^n$. Then $\mathrm{DE}(Q) \subset V_Q$ and we have $\dim \mathrm{DE}(Q) \leq |Q_0| - |\pi_0(Q)| = c(Q)$.*

It turns out that $\dim \mathrm{DE}(Q)$ varies depending on the existence of a rank function, since such a function defines another hyperplane that contains $\mathrm{DE}(Q)$.

**Definition 3.2.** For a quiver $Q$, a map $\rho\colon Q_0 \to \mathbb{R}$ is called a *rank function* of $Q$ if it satisfies $\rho(v) + 1 = \rho(w)$ for each edge $(v, w) \in Q_1$.

*Remark* 3.3. The following are equivalent for a quiver $Q$:

1. $Q$ has a rank function $\rho$.
2. $Q$ is asymmetric and satisfies

   $$|\{\, t \mid (v_t, v_{t+1}) \in Q_1 \,\}| = |\{\, t \mid (v_{t+1}, v_t) \in Q_1 \,\}|$$

   for each undirected closed walk $(v_0, v_1, \ldots, v_n)$ in $Q$.
3. $Q$ is asymmetric and satisfies

   $$|\{\, t \mid (v_t, v_{t+1}) \in Q_1 \,\}| = |\{\, t \mid (v_{t+1}, v_t) \in Q_1 \,\}|$$

   for each undirected cycle $(v_0, v_1, \ldots, v_n)$ in $Q$.
4. $Q$ is the Hasse diagram of a graded poset $(Q_0, \leq)$ with the rank function $\rho$.

Let $Q$ have a rank function $\rho$. Since $Q_0 = [n]$, we regard a rank function as a vector in $\mathbb{R}^n$. The affine hyperplane $\rho^\perp + \varepsilon_{(v,w)} = \left\{ \delta + \varepsilon_{(v,w)} \;\middle|\; \delta \in \rho^\perp \right\}$ is independent of choice of an egde $(v, w) \in Q_1$. We define $H_\rho$ to be $\rho^\perp + \varepsilon_{(v,w)}$ for some edge $(v, w) \in Q_1$. The hyperplane $H_\rho$ contains $\mathrm{DE}(Q)$, and is transversal to the hyperplanes defined by connected components of $Q$. We have the following upper bound of $\dim \mathrm{DE}(Q)$.

**Lemma 3.4.** *If $Q$ has a rank function, then* $\dim(\mathrm{DE}(Q)) \leq c(Q) - 1$.

In order to obtain lower bounds of $\dim \mathrm{DE}(Q)$, we consider the case of a quiver whose underlying graph is acyclic. Then the dimension of the vector space spanned by $\{ \varepsilon_{(v,w)} \mid (v, w) \in Q_1 \}$ is given by $c(Q)$. Thus we obtain

$$\dim \mathrm{DE}(Q) = \dim \mathrm{aff} \left( \varepsilon_{(v,w)} \;\middle|\; (v, w) \in Q_1 \right) = c(Q) - 1,$$

where aff denotes the affine hull. Hence we have the following.

**Lemma 3.5.** *For any quiver $Q$, we have* $\dim(\mathrm{DE}(Q)) \geq c(Q) - 1$.

For those quivers that do not have rank functions, we have the following lower bound.

**Lemma 3.6.** *If $Q$ does not have a rank function, then* $\dim(\mathrm{DE}(Q)) \geq c(Q)$.

Now Theorem 1.1 follows from Lemmas 3.1 and 3.4 to 3.6.

### 3.2 Facets

Let $R$ be a lluf subquiver of $Q$ so that both $\mathrm{DE}(R)$ and $\mathrm{DE}(Q)$ are contained in $\mathbb{R}^{Q_0}$. In order to prove Theorem 1.2, we would like to know when $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$ and $\dim(\mathrm{DE}(R)) = \dim(\mathrm{DE}(Q)) - 1$.

We first obtain the following relation between the coconnectivities of $Q$ and $R$ by the dimension condition.

By Theorem 1.1, we have the following:

**Lemma 3.7.** *Let $R$ be a lluf subquiver of $Q$. If $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, then $c(R) = c(Q)$ or $c(R) = c(Q) - 1$.*

If a facet $\mathrm{DE}(R)$ of $\mathrm{DE}(Q)$ satisfies $c(R) = c(Q)$, then $Q$ has no rank function but $R$ has a rank function. In this case, we obtain Lemma 3.8, which implies nessesary condition of Theorem 1.2 for the case where $c(R) = c(Q)$:

**Lemma 3.8.** *Let $R$ be a lluf subquiver of $Q$ with $c(R) = c(Q)$. If $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, then the subquiver $R$ has a rank function $\rho \in \mathbb{R}^{Q_0}$ such that*

$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0.$$

*for $(v, w), (v', w') \in Q_1 \setminus R_1$ and $Q$ does not have a rank function.*

If a facet $\mathrm{DE}(R)$ of $\mathrm{DE}(Q)$ satisfies $c(R) > c(Q)$, then both $Q$ and $R$ have a rank function or both has no rank function. In this case, we have the following:

**Lemma 3.9.** *Let $R$ be a subquiver of $Q$ with $c(R) = c(Q) - 1$. If $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, then $R$ is a component-wise full subquiver of $Q$.*

If $Q/R$ has a directed cycle, then we have a pair $(R^1, R^2)$ of connented components of $R$ with an edge of $Q$ connecting $R^1$ to $R^2$ and an edge of $Q$ connecting $R^2$ to $R^1$. The hyperplanes corrsponding to those edges separates the connected components $R^1, R^2$. Hence we have the following:

**Lemma 3.10.** *Let $R$ be a subquiver of $Q$ with $c(R) = c(Q) - 1$. If $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$, then $Q/R$ is acyclic.*

Lemmas 3.9 and 3.10 imply nessesary condition of Theorem 1.2 for the case where $c(R) + 1 = c(Q)$

By the calculation of the supporting hyperplanes, we have Lemmas 3.11 and 3.12.

**Lemma 3.11.** *Let $R$ be a component-wise full subquiver of $Q$. If $Q/R$ is acyclic, then $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$.*

**Lemma 3.12.** *Let $R$ be a lluf proper subquiver of a quiver $Q$. If $R$ has a rank function $\rho \in \mathbb{R}^{Q_0}$ such that*

$$(\rho(v) - \rho(w) + 1)(\rho(v') - \rho(w') + 1) > 0$$

*for any $(v, w), (v', w') \in Q_1 \setminus R_1$, then $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$.*

# 4 Examples

Here we apply our main theorem to some special quivers, and consider the $f$-vector, i.e., the sequence of the number $f_d$ of the faces of dimension $d$.

## 4.1 Quivers related to forests

First we consider the case of an asymmetric quiver $Q$ with no undirected closed walk, i.e., the case where the underlying graph of $Q$ is a forest. In this case, we have

$$|\mathrm{vert}(\mathrm{DE}(Q))| = |Q_1| = c(Q).$$

On the other hand, $Q$ has a rank function and we have $\dim(\mathrm{DE}(Q)) = c(Q) - 1$ by Theorem 1.1. Hence $\mathrm{DE}(Q)$ is a simplex of dimension $|Q_1| - 1 = c(Q) - 1$. We also have $f_d = \binom{|Q_1|}{d+1}$, and $\mathrm{DE}(R)$ is a face of $\mathrm{DE}(Q)$ for any lluf subquiver $R$ of $Q$.

Next we consider a forest $G = (G_0, G_1)$ and $\mathrm{SE}(G)$. Since a quiver whose underlying graph is $G$ has a rank function, it is a facet of $\mathrm{SE}(G)$. Hence $f_d = \binom{|G_1|}{d+1} 2^{d+1}$.

## 4.2 Quivers related to cycles

Here we consider a quiver related to the cycle with $m$ edges and $m$ vertices. In this case, we regard the vertex set $Q_0$ as $\mathbb{Z}/m\mathbb{Z} = \{ \bar{1}, \dots, \overline{m} = \bar{0} \}$. Moreover, we define $O^+$ to be the set $\{ (\overline{i-1}, \bar{i}) \mid \bar{i} \in Q_0 \}$ of 'clock-wise' edges, and $O^-$ to be the set $\{ (\overline{i+1}, \bar{i}) \mid \bar{i} \in Q_0 \}$ of 'anticlock-wise' edges.

First we consder the quiver whose underlying graph is a cycle.

*Example* 4.1. Let $Q$ be an asymmetric quiver whose underlying graph is the cycle with $2n+1$ edges. In this case, $Q$ has no rank function. Hence $\mathrm{DE}(Q)$ is a $2n$ dimensional polytope with $2n+1$ vertices. Thus $\mathrm{DE}(Q)$ is a simplex of dimension $2n$.

Next consider the case of even cycle.

*Example* 4.2. We consider an asymmetric quiver $Q$ whose underlying graph is an even cycle. Let $m = 2n$. We define $Q_1^+ = Q_1 \cap O^+$, $Q_1^- = Q_1 \cap O^-$. If $|Q_1^+| = |Q_1^-| = n$, then $Q$ has a rank function. It follows from Theorem 1.1 that $\dim(\mathrm{DE}(Q)) = 2n-2$. Since $|\mathrm{vert}(\mathrm{DE}(Q))| = |Q_1| = 2n$, $\mathrm{DE}(Q)$ is not a simplex.

Let $R$ be a lluf subquiver of $Q$ whose directed edge polytope $\mathrm{DE}(R)$ is a facet of $\mathrm{DE}(Q)$. Since $R$ also has a rank function, we have

$$2n-3 = \dim(\mathrm{DE}(R)) = |Q_0| - |\pi_0(R)| - 1 = 2n-1-|\pi_0(R)|,$$

which implies that $|Q_1 \setminus R_1| = 2$. Let $Q_1 \setminus R_1 = \{ e', e'' \}$. The acyclicity of $Q/R$ following from Theorem 1.3 allows us to assume that $e' \in Q_1^+$ and $e'' \in Q_1^-$. This is a characterization of facets of $\mathrm{DE}(Q)$.

For such a subquiver $R$, $\mathrm{DE}(R)$ is a simplex of dimension $2n-3$. Since faces of a simplex are in one-to-one correspondence to subsets of the vertex set, for a lluf subquiver $S$ of $Q$, $\mathrm{DE}(S)$ is a proper face of $\mathrm{DE}(Q)$ of dimension $d$ if and only if $|S_1 \cap Q_1^+| < n$, $|S_1 \cap Q_1^-| < n$, and $|S_1| = d+1$. Hence

$$f_d = \binom{2n}{d+1} - 2\binom{n}{d+1-n},$$

where the binomial coefficient $\binom{m}{k}$ equals 0 if $m < k$ or $k < 0$.

Next we consider a cycle $G$ and $\mathrm{SE}(G)$.

*Example* 4.3. Let $C_{2n}$ be the cycle with $2n$ edges and $Q = D(C_{2n})$. In this case, we have $Q_1 = O^+ \cup O^-$. The symmetric edge polytope $\mathrm{SE}(C_{2n}) = \mathrm{DE}(Q)$ is a $(2n-1)$-dimensional polytope by Theorem 1.1. The faces of $\mathrm{SE}(C_{2n})$ can be determined as follows.

By (3) of Corollary 1.4, for a lluf subquiver $R$ of $Q$ with

$$\dim(\mathrm{DE}(R)) = \dim(\mathrm{SE}(C_{2n})) - 1 = 2n-2,$$

$\mathrm{DE}(R)$ is a facet of $\mathrm{SE}(C_{2n})$ if and only if $c(R) = c(Q) = 2n-1$ and there exists a function $\rho : \mathbb{Z}/2n\mathbb{Z} \to \mathbb{R}$ such that

$$\rho(\overline{i-1}) - \rho(\bar{i}) = \begin{cases} 1 & ((\overline{i-1}, \bar{i}) \in R_1) \\ -1 & ((\bar{i}, \overline{i-1}) \in R_1) \\ 0 & (\text{otherwise}), \end{cases}$$

which implies that only one of $(\overline{i-1}, \bar{i})$ or $(\bar{i}, \overline{i-1})$ belongs to $R_1$ for each $i$. Let $R_1^+ = R_1 \cap O^+$ and $R_1^- = R_1 \cap O^-$. Then we have $\left|R_1^+\right| = \left|R_1^-\right|$.

Since $\mathrm{DE}(R)$ is of dimension $2n-2$,

$$|R_1| = |\mathrm{vert}(\mathrm{DE}(R))| \geq 2n-1.$$

By the condition on $\rho$, we see that the underlying graph of $R$ must be the whole $C_{2n}$. Hence facets of $\mathrm{SE}(C_{2n})$ are in bijective correspondence to subsets of cardinality $n$ in $O^+$. Hence we have

$$f_{2n-2} = \binom{2n}{n}.$$

Note that facets of $\mathrm{SE}(C_{2n})$ are polytopes in Example 4.2. In particular, faces of codimension 2 in $\mathrm{SE}(C_{2n})$ are simplices of dimension $(2n-3)$, which means that all faces of $\mathrm{SE}(C_{2n})$ except for facets are simplices. In other words, for $d < 2n-2$ and a lluf subquiver $R$ of $Q$, $\mathrm{DE}(R)$ is a face of dimension $d$ in $\mathrm{SE}(C_{2n})$ if and only if $|R_1| = d+1$, $|R_1 \cap O^+| < n$, $|R_1 \cap O^-| < n$, and $R_1^+ \cap (-R_1^-) = \emptyset$. Hence it follows from direct calculation that

$$f_d = \binom{2n}{d+1} 2^{d+1}.$$

We remark that D'Ali, Delucchi, and Michałek [2] also performed the same computation based on the characterization of facets by Higashitani et al. [6].

*Example* 4.4. Consider the case of an odd cycle $C_{2n+1}$. Let $Q = D(C_{2n+1})$. The symmetric edge polytope $\mathrm{SE}(C_{2n+1}) = \mathrm{DE}(Q)$ is a $2n$-dimensional polytope by Theorem 1.1.

For a lluf subquiver $R$ of $Q$, suppose that $\dim(\mathrm{DE}(R)) = 2n-1$. By the same argument as in Example 4.3, $\mathrm{DE}(R)$ is a facet of $\mathrm{SE}(C_{2n+1})$ if and only if $c(R) = c(Q)$, $\left|R_1^+\right| = \left|R_1^-\right| = n$, and $R_1^+ \cap (-R_1^-) = \emptyset$. Hence we have

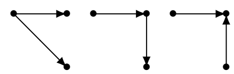
$$f_{2n-1} = (n+1)\binom{2n+1}{n} = \frac{(2n+1)!}{n!n!} = (2n+1)\binom{2n}{n}.$$

Thus, for $d < 2n-2$ and a lluf subquiver $R$ of $Q$, $\mathrm{DE}(R)$ is a face of $\mathrm{SE}(C_{2n+1})$ of dimension $d$ if and only if $|R_1| = d+1$, $\left|R_1^+\right| < n$, $\left|R_1^-\right| < n$, and $R_1^+ \cap (-R_1^-) = \emptyset$. Hence it follows from direct calculation that

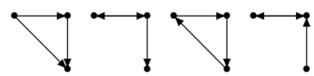
$$f_d = \binom{2n+1}{d+1} 2^{d+1}.$$

## 4.3 Quivers of small size

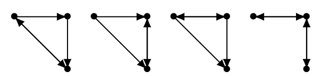
As a final remark, we give examples of $f$-vectors of directed edge polytopes for connected quivers with three vertices. The directed edge polytopes for the following quivers are 1-dimensional simplices:



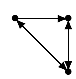
The directed edge polytopes for the following quivers are 2-dimensional simplices:



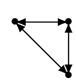
Next we give examples of the case where directed edge polytopes are not simplices. The $f$-vectors for the directed edge polytopes for the following quivers are $(1, 4, 4, 1)$:



The $f$-vector for the directed edge polytope for the following quiver is $(1, 5, 5, 1)$:



The $f$-vector for the directed edge polytope for the following quiver is $(1, 6, 6, 1)$:



In the case of the quivers with three vertices, we have only three kinds of $f$-vectors for derected edge polytopes which are not simplices. In the case of the quivers with four vertices, we, however, have more kinds of $f$-vectors as follows:

$(1, 5, 8, 5, 1)$, $(1, 6, 10, 6, 1)$, $(1, 7, 13, 8, 1)$, $(1, 8, 16, 10, 1)$, $(1, 9, 19, 12, 1)$,
$(1, 9, 17, 10, 1)$, $(1, 10, 20, 12, 1)$, $(1, 6, 11, 7, 1)$, $(1, 7, 14, 9, 1)$, $(1, 8, 17, 11, 1)$,
$(1, 8, 14, 8, 1)$, $(1, 9, 15, 8, 1)$, $(1, 7, 12, 7, 1)$, $(1, 5, 9, 6, 1)$, $(1, 7, 11, 6, 1)$,
$(1, 9, 18, 11, 1)$, $(1, 10, 21, 13, 1)$, $(1, 6, 12, 8, 1)$, $(1, 7, 15, 10, 1)$, $(1, 6, 9, 5, 1)$,
$(1, 10, 19, 11, 1)$, $(1, 11, 22, 13, 1)$, $(1, 12, 24, 14, 1)$, $(1, 8, 13, 7, 1)$, $(1, 4, 4, 1)$,
$(1, 8, 18, 12, 1)$, $(1, 8, 15, 9, 1)$, $(1, 8, 12, 6, 1)$, $(1, 9, 16, 9, 1)$.

## Acknowledgements

The authors thank anonymous referees for helpful suggestions.

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