Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #70, 12 pp.

Proceedings of the 35^{th} Conference on Formal Power Series and Algebraic Combinatorics (Davis)

The multispecies zero range process and modified Macdonald polynomials

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Abstract. In a previous part of this work (FPSAC 2021), we gave a new tableau formula for the modified Macdonald polynomials $\widetilde{H}_\lambda(X;q,t)$, using a weight on tableaux involving the *queue inversion* (quinv) statistic. In this paper we explicitly describe the connection between these combinatorial objects and a class of multispecies totally asymmetric zero range processes (mTAZRP) on a ring, with site-dependent jump-rates. We construct a Markov chain on the space of tableaux which projects to the mTAZRP, and whose stationary distribution can be expressed in terms of quinv-weighted tableaux. We also obtain interesting symmetry properties of the mTAZRP probabilities under permutation of the jump rates between the sites. Finally, we give explicit formulas for particle densities and correlations of the process purely in terms of modified Macdonald polynomials.

Keywords: zero range process, Markov chain, tableaux, modified Macdonald polynomials

1 Introduction

Over the past several years, there has been a growing body of research devoted to exploring the relationships of particle models and integrable systems with symmetric (and associated nonsymmetric) functions; see [2] for many examples. These connections have led to new combinatorial objects, new positive formulas [6, 7, 10, 1], and deeper understanding of both the symmetric functions and the associated particle processes. One such connection, coming from [5], was the remarkable discovery that the symmetric Macdonald polynomial $P_\lambda(X;q,t)$ specializes to the partition function of the multispecies asymmetric simple exclusion process (ASEP) on a ring.

The *modified Macdonald polynomials* $\widetilde{H}_\lambda(X;q,t)$, introduced by Garsia and Haiman [11], are a transformed version of the P_λ's, obtained via plethysm from their integral

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form. Following the discovery of the link between $P_\lambda(X;q,t)$ and the partition function of the multispecies ASEP, a natural question was whether there exists a related statistical mechanics model for which some specialization of $\widetilde{H}_\lambda(X;q,t)$ is equal to its partition function. In this extended abstract, we describe such a model, which turns out to be a *multispecies totally asymmetric zero range process (mTAZRP)*, in a form which had previously been considered in [14].

The mTAZRP is a multispecies variant of the more general zero range process (ZRP), which is an important example of a non-equilibrium exactly-solvable interacting particle system. It was introduced by Spitzer in an influential paper [13] that initiated the mathematical study of interacting particle systems, along with the related asymmetric simple exclusion process. The ZRP describes particles hopping to adjacent sites on a graph that allows multiple particles per site, such that the rate of hopping from one site to another depends solely on the content at the site of origin. Among the many reasons that the ZRP is of great interest to physicists is that it is an excellent toy model for the physics of phase-separation and condensation; see [8] for a review.

In this work we study the stationary distribution of the mTAZRP by considering fillings of the diagram of the partition λ with entries in $\{1,\ldots,n\}$ – see, for example, Figure 2(b). Let $\mathrm{Tab}(\lambda,n)$ denote the set of such fillings. We define a function $\mathcal{P}$ (see Definition 3.5) from $\mathrm{Tab}(\lambda,n)$ to the configurations of the mTAZRP. In earlier work [4], we introduced the "queue inversion" statistic, $\mathrm{quinv} : \mathrm{Tab}(\lambda,n) \mapsto \mathbb{N}$ (see Definition 3.2) and used it to obtain a new formula for the modified Macdonald polynomial $\widetilde{H}_\lambda(X;q,t)$.

The first main result in this article is that the stationary distribution of the mTAZRP on n sites with particle types given by a partition λ is the projection under the map $\mathcal{P}$ of the distribution of the quinv statistic on $\mathrm{Tab}(\lambda,n)$:

Theorem 1.1. *Consider the mTAZRP on n sites with particle content determined by the partition λ. Then the stationary probability of a configuration w in that mTAZRP equals*

$$\frac{1}{\mathcal{Z}_{(\lambda,n)}} \sum_{\substack{\sigma\in\mathrm{Tab}(\lambda,n)\\ \mathcal{P}(\sigma)=w}} t^{\mathrm{quinv}(\sigma)} x^\sigma,$$

where

$$\mathcal{Z}_{(\lambda,n)} = \sum_{\sigma\in\mathrm{Tab}(n,\lambda)} t^{\mathrm{quinv}(\sigma)} x^\sigma = \widetilde{H}_\lambda(x_1,\ldots,x_n;1,t).$$

In this sense, the modified Macdonald polynomial $\widetilde{H}_\lambda$ specializes at $q=1$ to the *partition function* for the mTAZRP model: all the stationary probabilities may be written as rational functions in $\mathbb{N}[t;x_1,\ldots,x_n]$ with denominator $\widetilde{H}_\lambda(x_1,\ldots,x_n;1,t)$.

Our construction is very closely related to the approach involving *multiline queues*, which was first introduced for the TASEP [9] and has subsequently been used to describe the stationary distributions of a variety of multispecies interacting particle systems. In particular, a generalized version of multiline queues was used in [7] to interpolate between $P_\lambda(X;q,t)$ and probabilities of the multispecies ASEP.

We prove Theorem 1.1 by constructing a Markov chain on the state space $\mathrm{Tab}(\lambda, n)$, which has two key properties: (i) it projects to the mTAZRP via the function $\mathcal{P}$; (ii) the stationary probability of a filling σ is proportional to the weight $t^{\mathrm{quinv}(\sigma)}x^\sigma$.

A key point of interest is the presence of the parameters $x_1, \ldots, x_n$ representing site-dependent rates in the TAZRP. In contrast, for the ASEP for example, existing results cover only the case where all rates are equal, involving the specialization of the Macdonald polynomials to $x_1 = \cdots = x_n = 1$. By exploiting symmetries of fillings, we obtain interesting new symmetry properties for the mTAZRP under permutation of the state-dependent jump rates.

Theorem 1.2. *Consider the stationary distribution of the mTAZRP on n sites with particle content determined by λ, and parameters $x_1, \ldots, x_n, t$. The distribution of the configuration restricted to sites $1, \ldots, \ell$ is symmetric in the variables $\{x_{\ell+1}, \ldots, x_n\}$.*

The article is organized as follows. The mTAZRP is formally defined in Section 2. Section 3 contains the necessary background on fillings of diagrams and tableaux statistics, the definition of the Markov chain on $\mathrm{Tab}(\lambda, n)$ for strict partitions λ, as well as the sketch of the proof of our main results for that case. (The Markov chain for general partitions is much more technical and is explained in the longer version of this extended abstract [3].) Section 4 is devoted to observables. In Section 4.1, we study the symmetry properties of the mTAZRP and explain the ideas behind Theorem 1.2. Finally, we end with formulas for single-site densities and currents of particles of a given species in Section 4.2.

2 Definition of the mTAZRP

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing sequence of positive integers. When all parts of λ are distinct, λ is called a *strict partition*.

We consider a one-dimensional lattice with n sites (labelled $1, 2, \ldots, n$) with periodic boundary conditions, containing k particles of types (or species) $\{\lambda_1, \ldots, \lambda_k\}$. Our convention is that particles of larger species are stronger. The set of configurations of the model consists of all possible arrangements of the k particles amongst the n sites. Each site may contain an arbitrary number of particles, and particles of the same type are indistinguishable. We may identify configurations as multiset compositions of type λ with n parts; that is, sequences $w = (w_1, \ldots, w_n)$ where for $1 \leq j \leq n$, w_j is a (possibly empty) multiset, such that the union of all the parts $\biguplus_{j=1}^n w_j$ is equal to the multiset λ. We denote the set of such configurations by $\mathcal{T}(\lambda, n)$.

The system evolves as a continuous-time Markov chain with a global parameter t and site-dependent parameters $x_1, \ldots, x_n$. Any transition of the system consists of a

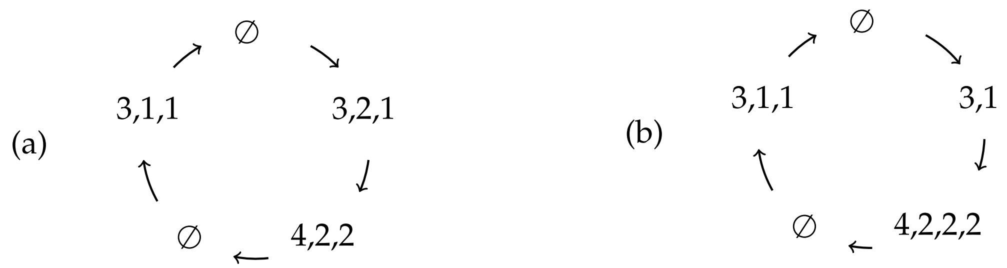

Figure 1: (a) The configuration in this figure is $(\cdot|321|422|\cdot|311)$ (read clockwise starting from the topmost site), and is an element of $\mathcal{T}(\lambda,5)$ for $\lambda=(4,3,3,2,2,2,1,1,1)$. The arrows show the direction of hopping. (b) The configuration is $(\cdot|31|4222|\cdot|311)$ after a particle of type 2 hops from site 2 to site 3. This hop occurs with rate $x_2^{-1}t$.

single particle jumping from site j to site $j+1$, for some $1 \le j \le n$ (sites are considered cyclically mod n). The transition rates are defined as follows. If the number of particles of type a at site j is c_a, and the number of particles of type larger than a at site j is d_a, then a particle of type a from site j jumps with rate $x_j^{-1}t^{d_a}\sum_{i=0}^{c_a-1}t^i$. See Figure 1 for an example.

Example 2.1. $\mathcal{T}((3,1,1),3)$ consists of the following 18 states:

$$(311|\cdot|\cdot),(31|1|\cdot),(31|\cdot|1),(3|11|\cdot),(3|1|1),(3|\cdot|11),(11|3|\cdot),(11|\cdot|3),(1|31|\cdot),$$
$$(1|1|3),(1|3|1),(1|\cdot|31),(\cdot|311|\cdot),(\cdot|31|1),(\cdot|3|11),(\cdot|11|3),(\cdot|1|31),(\cdot|\cdot|311).$$

Examples of transitions of the mTAZRP on $\mathcal{T}((3,1,1),3)$ are:

- The jumps from $(\cdot|311|\cdot)$ are to $(\cdot|11|3)$ with rate x_2^{-1}, and to $(\cdot|31|1)$ with rate $x_2^{-1}(t+t^2)$;
- The jumps from $(\cdot|1|31)$ are to $(3|1|1)$ with rate x_3^{-1}, to $(1|1|3)$ with rate $x_3^{-1}t$, and to $(\cdot|\cdot|311)$ with rate x_2^{-1}.

3 Fillings of λ and a Markov chain on these fillings

Let $\lambda=(\lambda_1,\ldots,\lambda_k)$ be a partition. The *diagram of type* λ, which we denote $\mathrm{dg}(\lambda)$, consists of the cells $\{(r,i), 1\le i\le k, 1\le r\le\lambda_i\}$, which we depict using k bottom-justified columns, where the i'th column from left to right has λ_i boxes. See Figure 2(a) for an illustration of $\mathrm{dg}(3,2,1,1)$. The cell (r,i) corresponds to the cell in the i'th column in the r'th row of $\mathrm{dg}(\lambda)$, where rows are labeled from bottom to top.

A *filling* of type (λ, n) is a function $\sigma : \mathrm{dg}(\lambda) \to [n]$ defined on the cells of $\mathrm{dg}(\lambda)$ (where $[n] = \{1, 2, \ldots, n\}$). We also refer to a diagram together with a filling of it as a *tableau*. Let $\mathrm{Tab}(\lambda, n)$ be the set of fillings of type (λ, n), and $\mathrm{Tab}(\lambda)$ the set of fillings $\sigma : \mathrm{dg}(\lambda) \to \mathbb{Z}^+$.

$$\text{(a)}\ \begin{array}{|c|c|c|c|}\cline{1-1} 1 \\ \cline{1-2} 3 & 2 \\ \hline 7 & 6 & 5 & 4 \\ \hline\end{array} \qquad\qquad \text{(b)}\ \begin{array}{|c|c|c|c|c|}\cline{1-1} 3 \\ \cline{1-3} 1 & 1 & 2 \\ \hline 1 & 3 & 2 & 3 & 1 \\ \hline\end{array} \qquad\qquad \text{(c)}\ \begin{array}{|c|c|c|c|c|c|c|}\cline{1-3} u & u & \\ \cline{1-5} \ell & \ell & x & u & u \\ \hline & & & \ell & \ell & \ell & \ell \\ \hline\end{array}$$

Figure 2: (a) The diagram $\mathrm{dg}(3, 2, 1, 1)$ with the cells labeled according to the reading order in Definition 3.1. Our convention is that the cell labelled 5 has coordinates $(1, 3)$. (b) A tableau of type $\lambda = (3, 2, 2, 1, 1)$ and $n = 3$. The weight of this filling is $x_1^4 x_2^2 x_3^3 t^7$. (c) The cells in the lower and upper arm of the cell x are labeled 'ℓ' and 'u', respectively.

For σ in $\mathrm{Tab}(\lambda, n)$ (or in $\mathrm{Tab}(\lambda)$), and a cell $x = (r, i)$ of $\mathrm{dg}(\lambda)$, we write $\sigma(x) = \sigma(r, i)$ and call this the *content* of the cell x in the filling σ. Define $\mathrm{South}(x)$ and $\mathrm{North}(x)$ to be the cells $(r - 1, i)$ and $(r + 1, i)$ (respectively), directly below the cell x in the same column, if those cells exist.

Definition 3.1. Define the *reading order* on the cells of a tableau to be along rows from top to bottom and from right to left within each row, as in Figure 2(a).

Definition 3.2. Given a diagram $\mathrm{dg}(\lambda)$, a *triple* consists of either

- three cells $(r + 1, i)$, (r, i) and (r, j) with $i < j$; or
- two cells (r, i) and (r, j) with $i < j$ and $\lambda_i = r$ (in which case the cell $(r + 1, i)$ doesn't exist). Such a configuration is called a *degenerate triple*.

Let us write $a = \sigma(r + 1, i)$, $b = \sigma(r, i)$, and $c = \sigma(r, j)$, for the contents of the cells of the triple, so that we can depict a triple along with its contents as

$$\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array} \cdots \boxed{c}\,, \qquad \text{or} \qquad \begin{array}{c} \emptyset \\ \boxed{b} \end{array} \cdots \boxed{c} \quad \text{(degenerate).}$$

We say that a triple is a *queue inversion triple*, or a *quinv triple* for short if

$$a \leq b < c \quad \text{or} \quad c < a \leq b \quad \text{or} \quad b < c < a,$$

meaning that its entries are oriented counterclockwise when they are read in increasing order, with ties being broken with respect to reading order. If the triple is degenerate with content b, c, it is a quinv triple if and only if $b < c$.

Definition 3.3. The *weight* of a filling σ is $x^\sigma t^{\mathrm{quinv}(\sigma)}$, where:

- $x^\sigma = \prod_{u \in \mathrm{dg}(\lambda)} x_{\sigma(u)}$ is the monomial corresponding to the content of σ,
- $\mathrm{quinv}(\sigma)$ is the total number of quinv triples in σ.

See Figure 2(b) for an example of a filling and its weight. In the prequel to this article [4], we used the quinv statistic to obtain a new formula for the modified Macdonald polynomials.

Theorem 3.4 ([4, Theorem 2.6])**.** *Let λ be a partition. Then*

$$\widetilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \mathrm{Tab}(\lambda)} x^\sigma t^{\mathrm{quinv}(\sigma)} q^{\mathrm{maj}(\sigma)}.$$

where $\mathrm{maj}(\sigma)$ *is the major index statistic, defined in [12].*

The main focus of this extended abstract will be to obtain formulas for the stationary probabilities and correlations of the mTAZRP on $\mathcal{T}(\lambda, n)$ using weights on the fillings in $\mathrm{Tab}(\lambda, n)$.

We associate some set of fillings in $\mathrm{Tab}(\lambda, n)$ to each state of the TAZRP by mapping the bottom row of a filling in $\mathrm{Tab}(\lambda, n)$ to a state of a TAZRP of type λ on n sites. The correspondence is given as a function $\mathcal{P} : \mathrm{Tab}(\lambda, n) \to \mathcal{T}(\lambda, n)$, as follows.

Definition 3.5. Let $\sigma \in \mathrm{Tab}(\lambda, n)$. Then $\mathcal{P}(\sigma) = (w_1, \ldots, w_n)$, where for $1 \leq j \leq n$,

$$w_j = \{\lambda_i \in \lambda \mid \sigma(1, i) = j\}$$

is the multiset of the heights of the columns of T whose bottom-most entry is j.

Example 3.6. We show all six tableaux corresponding to $(\cdot|21|1) \in \mathcal{T}((2,1,1),3)$:

$$\begin{array}{|c|cc}\hline 1 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{2} \\ \hline\end{array} \qquad \begin{array}{|c|cc}\hline 1 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array} \qquad \begin{array}{|c|cc}\hline 2 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array} \qquad \begin{array}{|c|cc}\hline 3 & \multicolumn{2}{c}{} \\ \hline 2 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

For the rest of this section, we shall focus on the case where λ is a strict partition, i.e. has all its parts distinct. This corresponds to the property that all particles of the mTAZRP have distinct types. In particular, fillings of $\mathrm{dg}(\lambda)$ have no degenerate triples when λ is strict. The case of a general partition λ is much more complicated and involves a technical work-around to treat the degenerate triples, which is given in full detail in the full version of this article [3].

We begin with some definitions to prepare for the description of the Markov chain.

Definition 3.7. Let $u = (r, c)$ be a cell of $\mathrm{dg}(\lambda)$. The *lower arm* of u, denoted $\mathrm{Larm}(u)$, consists of all cells to its left in the same row plus all cells to its right in the row below:

$$\mathrm{Larm}(u) := \{(r, j) \in \mathrm{dg}(\lambda) \mid j < c\} \cup \{(r-1, j) \in \mathrm{dg}(\lambda) \mid j > c\}.$$

Similarly, we define the *upper arm* of u, denoted $\mathrm{Uarm}(u)$, to consist of all cells to its left in the row above plus all cells to its right in the same row:

$$\mathrm{Uarm}(u) := \{(r+1,j) \in \mathrm{dg}(\lambda) \mid j < c\} \cup \{(r,j) \in \mathrm{dg}(\lambda) \mid j > c\}.$$

See Figure 2(c). Note that what we call the lower arm is the statistic 'arm' in [12]. Observe that $v \in \mathrm{Larm}(u)$ if and only if $u \in \mathrm{Uarm}(v)$. Now we define

$$\mathrm{down}(\sigma,u) := \#\{y \in \mathrm{Larm}(u) : \sigma(y) = \sigma(u)\}$$

for a cell u such that $\sigma(\mathrm{South}(u)) \neq \sigma(u)$, and its analog

$$\mathrm{up}(\sigma,u) := \#\{y \in \mathrm{Uarm}(u) : \sigma(y) = \sigma(u)\}$$

for a cell u such that $\sigma(\mathrm{North}(u)) \neq \sigma(u)$.

Let λ be a partition, $u = (r,c) \in \mathrm{dg}(\lambda)$ be a cell, and $\sigma \in \mathrm{Tab}(\lambda,n)$ be a filling of λ. We want to consider the maximal vertical chain of cells upwards from u along which the contents increase by 1 (cyclically mod n) at each step. Let $h_u(\sigma)$ be the row number of the top of the chain. More precisely, $r \leq h_u(\sigma) \leq \lambda_c$, and:

(i) for all i with $r \leq i < h_u(\sigma)$, we have $\sigma(\mathrm{North}(i,c)) - \sigma(i,c) \equiv 1 \pmod{n}$;

(ii) either $h_u(\sigma) = \lambda_c$, or $\sigma(\mathrm{North}(h_u(\sigma),c)) - \sigma(h_u(\sigma),c) \not\equiv 1 \pmod{n}$.

Definition 3.8. Given λ and a cell $u = (r,c)$, we define a map $R_u : \mathrm{Tab}(\lambda,n) \to \mathrm{Tab}(\lambda,n)$ as follows. If $\sigma \in \mathrm{Tab}(\lambda,n)$ with $\sigma(\mathrm{South}(u)) = \sigma(u)$, then let $R_u(\sigma) = \sigma$. Otherwise, the map R_u will increment by 1 all the contents in the maximal contiguous increasing chain above u. That is, for $v \in \mathrm{dg}(\lambda)$,

$$R_u(\sigma)(v) = \begin{cases} \sigma(v)+1 \pmod{n} & v = (i,c) \text{ for } r \leq i \leq h_u(\sigma), \\ \sigma(v) & \text{otherwise.} \end{cases}$$

One may check from the definition that R_u is invertible. Define $R_y^{-1} : \mathrm{Tab}(\lambda,n) \to \mathrm{Tab}(\lambda,n)$ to be the map such that $R_y^{-1} \circ R_u(\sigma) = \sigma$, where $y \in \mathrm{dg}(\lambda)$ that depends on u and σ. See Example 3.10 for examples of both maps.

The following properties of R_y^{-1} are straightforward to check from the definitions:

Lemma 3.9.

(i) Take $\tau \in \mathrm{Tab}(\lambda,n)$, and $u = (r,j) \in \mathrm{dg}(\lambda)$ such that $\tau(u) \neq \tau(\mathrm{South}(u))$; define $\sigma := R_u(\tau)$. Then $y = (h_u(\tau),j)$ is the unique cell such that $\tau = R_y^{-1}(\sigma)$ and this y has the property that $\sigma(y) \neq \sigma(\mathrm{North}(y))$. Moreover,

$$x^{\tau} = x^{\sigma} x_{\tau(u)} x_{\sigma(y)}^{-1}. \tag{3.1}$$

(ii) Conversely, take $\sigma \in \mathrm{Tab}(\lambda, n)$, and $y = (r', j) \in \mathrm{dg}(\lambda)$ such that $\sigma(y) \neq \sigma(\mathrm{North}(y))$; define $\tau := R_y^{-1}(\sigma)$. Then there is a unique u such that $\sigma = R_u(\tau)$; moreover, $\tau(u) \neq \tau(\mathrm{South}(u))$ and $h_u(\tau) = r'$.

Example 3.10. Let $\sigma \in \mathrm{Tab}((3,2,1),3)$ be the tableau

$$\begin{array}{|c|c|c|} \hline 1 \\ \cline{1-2} 1 & 3 \\ \hline 3 & 2 & 3 \\ \hline \end{array}$$

corresponding to the mTAZRP state $\mathcal{P}(\sigma) = (\cdot|2|31)$. Note that $R_{(3,1)}(\sigma) = R^{-1}_{(2,1)}(\sigma) = \sigma$. Below, we show the map R for the cells $(1,1)$ and $(2,2)$, as well as the inverse map R^{-1} for the cells $(3,1)$ and $(2,2)$, with the corresponding maximal chain highlighted.

$$R_{(1,1)}(\sigma) = \begin{array}{|c|c|c|} \hline 1 \\ \cline{1-2} 2 & 3 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad R_{(2,2)}(\sigma) = \begin{array}{|c|c|c|} \hline 1 \\ \cline{1-2} 1 & 1 \\ \hline 3 & 2 & 3 \\ \hline \end{array} \qquad R^{-1}_{(3,1)}(\sigma) = \begin{array}{|c|c|c|} \hline 3 \\ \cline{1-2} 1 & 3 \\ \hline 3 & 2 & 3 \\ \hline \end{array} \qquad R^{-1}_{(2,2)}(\sigma) = \begin{array}{|c|c|c|} \hline 1 \\ \cline{1-2} 1 & 2 \\ \hline 3 & 1 & 3 \\ \hline \end{array}$$

rate x_3^{-1}, state $(3|2|1)$; rate x_2^{-1}, state $(\cdot|2|31)$; $u = (3,1)$, state $(\cdot|2|31)$; $u = (1,2)$, state $(2|\cdot|31)$

The inverses of the above maps are given by:

$$R^{-1}_{(2,1)} \circ R_{(1,1)}(\sigma) = R^{-1}_{(2,2)} \circ R_{(2,2)}(\sigma) = R_{(3,1)} \circ R^{-1}_{(3,1)}(\sigma) = R_{(2,2)} \circ R^{-1}_{(2,2)}(\sigma) = \sigma.$$

The next couple of lemmas, whose proofs are combinatorial in nature, will be key ingredients in proofs of our theorems. See [3] for details of the proofs.

Lemma 3.11. *For all $\sigma \in \mathrm{Tab}(\lambda)$ and all k, we have*

$$\sum_{\substack{u \in \mathrm{dg}(\lambda) \\ \sigma(u) = k \neq \sigma(\mathrm{North}(u))}} t^{\mathrm{up}(\sigma,u)} = \sum_{\substack{u \in \mathrm{dg}(\lambda) \\ \sigma(u) = k \neq \sigma(\mathrm{South}(u))}} t^{\mathrm{down}(\sigma,u)}.$$

Lemma 3.12. *Fix a strict partition λ and $n \in \mathbb{N}$. For $\tau \in \mathrm{Tab}(\lambda, n)$, let $u = (r, j) \in \mathrm{dg}(\lambda)$ with $\tau(\mathrm{South}(u)) \neq \tau(u)$, let $\sigma = R_u(\tau)$, and let $y = (h_u(\tau), j)$ be the unique cell such that $\tau = R_y^{-1}(\sigma)$. Then*

$$\mathrm{quinv}(\tau) - \mathrm{quinv}(\sigma) = \mathrm{up}(\sigma, y) - \mathrm{down}(\tau, u). \tag{3.2}$$

We now define the tableau Markov chain with state space $\mathrm{Tab}(\lambda, n)$ for the case where λ is a strict partition. Let $\sigma \in \mathrm{Tab}(\lambda, n)$. We attach an exponential clock with rate $t^{\mathrm{down}(\sigma,u)} x^{-1}_{\sigma(u)}$ to every cell $u \in \mathrm{dg}(\lambda)$ such that $\sigma(\mathrm{South}(u)) \neq \sigma(u)$. When this clock rings, we make a transition to $R_u(\sigma)$. We say that such a transition is *triggered by the cell u*. For $\sigma, \tau \in \mathrm{Tab}(\lambda, n)$, denote by $\mathrm{rate}(\sigma, \tau)$ the rate of the transition from σ to τ.

Theorem 3.13. *The stationary distribution of the tableau Markov chain defined above is proportional to*

$$\mathrm{wt}(\sigma) = t^{\mathrm{quinv}(\sigma)} x^{\sigma}.$$

Proof. Denote all states reached from σ in one step by $\mathrm{Out}(\sigma)$, and all states that reach σ in one step by $\mathrm{In}(\sigma)$:

$$\begin{aligned} \mathrm{Out}(\sigma) &= \{R_u(\sigma) \mid u \in \mathrm{dg}(\lambda),\ \sigma(\mathrm{South}(u)) \neq \sigma(u)\}, \\ \mathrm{In}(\sigma) &= \{\tau \in \mathrm{Tab}(\lambda, n) \mid \sigma \in \mathrm{Out}(\tau)\}. \end{aligned}$$

From Lemma 3.9(ii) we have that $\mathrm{In}(\sigma) = \{R_y^{-1}(\sigma) \mid y \in \mathrm{dg}(\lambda),\ \sigma(\mathrm{North}(y)) \neq \sigma(y)\}$. Then it suffices to show

$$\mathrm{wt}(\sigma) \sum_{\nu \in \mathrm{Out}(\sigma)} \mathrm{rate}(\sigma, \nu) = \sum_{\tau \in \mathrm{In}(\sigma)} \mathrm{wt}(\tau)\, \mathrm{rate}(\tau, \sigma), \tag{3.3}$$

which we can expand as

$$x^{\sigma} t^{\mathrm{quinv}(\sigma)} \sum_{\substack{u \in \mathrm{dg}(\lambda) \\ \sigma(\mathrm{South}(u)) \neq \sigma(u)}} x_{\sigma(u)}^{-1} t^{\mathrm{down}(\sigma,u)} = \sum_{\substack{y \in \mathrm{dg}(\lambda) \\ \sigma(\mathrm{North}(y)) \neq \sigma(y)}} \mathrm{wt}(R_y^{-1}(\sigma))\, \mathrm{rate}(R_y^{-1}(\sigma), \sigma). \tag{3.4}$$

To treat the right-hand side of (3.4), let $\tau = R_y^{-1}(\sigma)$ for some $y \in \mathrm{dg}(\lambda)$ such that $\sigma(\mathrm{North}(y)) \neq \sigma(y)$, and let $u \in \mathrm{dg}(\lambda)$ be such that $R_u(\tau) = \sigma$. From (3.1), we have $x^{\tau} = x^{\sigma} x_{\sigma(y)}^{-1} x_{\tau(u)}$. By Lemma 3.12, we also have

$$\mathrm{wt}(\tau)\, \mathrm{rate}(\tau, \sigma) = x^{\tau} t^{\mathrm{quinv}(\tau)} \cdot x_{\tau(u)}^{-1} t^{\mathrm{down}(\tau,u)} = x^{\sigma} x_{\sigma(y)}^{-1} t^{\mathrm{quinv}(\sigma) + \mathrm{up}(\sigma,y)}. \tag{3.5}$$

Refining over the coefficients of x-monomials and plugging in (3.5), we get that (3.4) is equivalent to

$$x^{\sigma} t^{\mathrm{quinv}(\sigma)} \sum_{k=1}^{n} x_k^{-1} \sum_{\substack{u \in \mathrm{dg}(\lambda) \\ \sigma(\mathrm{South}(u)) \neq \sigma(u) = k}} t^{\mathrm{down}(\sigma,u)} = x^{\sigma} t^{\mathrm{quinv}(\sigma)} \sum_{k=1}^{n} x_k^{-1} \sum_{\substack{y \in \mathrm{dg}(\lambda) \\ \sigma(\mathrm{North}(y)) \neq \sigma(y) = k}} t^{\mathrm{up}(\sigma,y)}. \tag{3.6}$$

Finally, we obtain (3.6) by invoking Lemma 3.11 for each k. □

Recall the projection map $\mathcal{P}$ from $\mathrm{Tab}(\lambda, n)$ to $\mathcal{T}(\lambda, n)$ given in Definition 3.5. From the following lemma, we obtain the first main result of this article, Theorem 1.1.

Lemma 3.14. *Let* $\sigma \in \mathrm{Tab}(\lambda, n)$.

(i) For any cell u in the first row of σ, there exists a transition from the TAZRP state $\mathcal{P}(\sigma) \in \mathcal{T}(\lambda, n)$ to the TAZRP state $\mathcal{P}(R'_u(\sigma))$.

(ii) Let $w = \mathcal{P}(\sigma) \in \mathcal{T}(\lambda, n)$ be the corresponding TAZRP state of σ. Let $1 \leq j \leq n$ be a site, $1 \leq r \leq \lambda_1$ be a particle type such that $w_j = r$. Let w' be the state in $\mathcal{T}(\lambda, n)$ obtained by moving particle r one site to the right. Then

$$\sum_{\substack{u \in \mathrm{dg}(\lambda) \\ \mathcal{P}(R'_u(\sigma)) = w'}} \mathrm{rate}(\sigma, R'_u(\sigma)) = \mathrm{rate}(w, w').$$

4 Observables

4.1 Symmetries in probabilities of configurations in the first ℓ sites

Fix a partition λ and a positive integer n, and consider the TAZRP of type λ on n sites. Fix ℓ with $1 \leq \ell \leq n-1$. We will consider the probability of observing some specific configuration w on the sites $1, \ldots, \ell$. Formally, w can be seen as a configuration of TAZRP(μ, ℓ) for some partition μ consisting of a subset of the parts of λ. We write $\overline{w}$ for the set of configurations of TAZRP(λ, n) whose restriction to $1, \ldots, \ell$ is given by w.

Example 4.1. Let $\lambda = (2,2,1,1)$, $n = 4$, $\ell = 2$, and $w = (1|2)$. (Then $\mu = (2,1)$). The configurations of $\mathcal{T}(\lambda, n)$ contributing to $\mathbb{P}_{\lambda,n}(\overline{w})$ are

$$\{(1|2|21|\cdot), (1|2|2|1), (1|2|1|2), (1|2|\cdot|21)\}.$$

In this case, the coefficient of t^3 in $P_{\lambda,n}(\overline{w})$ is $x_1x_2(x_2^2 + x_1x_2)m_2 + (x_1 + 2x_2)m_3 + (4x_1x_2 + 3x_2^2)m_{11} + (3x_1 + 4x_2)m_{21} + m_{31} + 2m_{22})/\widetilde{H}_\lambda(x_1, x_2, x_3, x_4; 1, t)$, where $m_\mu = m_\mu(x_3, x_4)$ is the monomial basis evaluated at the variables x_3, x_4.

The main result we prove in this section is the following.

Proposition 4.2. *Fix a partition λ and positive integer n. For any $0 \leq \ell \leq n$ and configuration w on the first ℓ sites of the TAZRP of type λ on n sites, the probability $\mathbb{P}_{\lambda,n}(\overline{w})$ is symmetric in the variables $\{x_{\ell+1}, \ldots, x_n\}$.*

Proposition 4.2 proves Theorem 1.2. The proof of this result uses LLT polynomials: fixing sites $1, \ldots, \ell$ in the TAZRP is equivalent to fixing a certain set of boxes in the filling to have entries in $\{1, \ldots, \ell\}$. The sum over configurations with those sites fixed can be refined as a sum over fillings of subsets of $\mathrm{dg}(\lambda)$ filled with entries in $\{\ell+1, \ldots, n\}$. Such fillings can be described as certain LLT polynomials in the variables $\{\ell+1, \ldots, n\}$, which are known to be symmetric in those variables. See [3] for more details.

4.2 Densities and currents

Consider the mTAZRP with particle content given by λ and with n sites. We now assume without loss of generality that λ is *packed*, i.e. $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k = 1$ such that $\lambda_i - \lambda_{i-1} \leq 1$.

Let $\tau_i^{(j)}$ be the random variable counting the number of particles of type j at site i in a configuration of the mTAZRP. By convention, we will denote the expectation in the stationary distribution by $\langle \cdot \rangle$. The density of a given species (say j) at a given site (say i) is defined to be the expected number of particles of type j at site i in the stationary distribution and is thus given by $\langle \tau_i^{(j)} \rangle$.

Note the following cyclic symmetry property, which is immediate from the definition of the process.

Proposition 4.3. *The mTAZRP with partition λ on n sites is invariant under simultaneous translation of the sites $i \to i+1$ for $1 \leq i \leq n-1$ and $n \to 1$, and relabelling of the site-dependent parameters $x_i \to x_{i+1}$ for $1 \leq i \leq n-1$ and $x_n \to x_1$.*

As an immediate consequence, we can deduce the following result for the densities.

Proposition 4.4. *Suppose $\langle \tau_1^{(j)} \rangle = r(x_1, \dots, x_n)$. Then for any i, $\langle \tau_i^{(j)} \rangle = r(x_i, \dots, x_n, x_1, \dots, x_{i-1})$.*

By Proposition 4.4, it suffices to compute the densities of all species of particles at the first site. The following theorem is proved using a "coloring" argument based on the fact that to a particle of type j, all particles of lower types are invisible.

Theorem 4.5. *For $1 \leq j \leq k$, the density of the j'th species at the first site is given by*

$$\langle \tau_1^{(j)} \rangle = x_1 \partial_{x_1} \log \left(\frac{\widetilde{H}_{(1^{m_j+\cdots+m_k})}(x_1, \dots, x_n; 1, t)}{\widetilde{H}_{(1^{m_{j+1}+\cdots+m_k})}(x_1, \dots, x_n; 1, t)} \right),$$

where the denominator inside the logarithm is 1 if $j = k$.

Currents are important observables in statistical physics because they are a prime indicator of the lack of reversibility in the system. The *current* of a certain species of particle from site i to site j is defined as the number of particles of that species jumping from i to j per unit of time in the large-time limit, or in the stationary state. The current from i to $i+1$ is the same for all sites i because of particle conservation. Using Proposition 4.3, we obtain:

Theorem 4.6. *Consider the multispecies TAZRP with content λ on n sites. Then, for $1 \leq j \leq k$, the current of particle of species j is given by*

$$[m_j + \cdots + m_k]_t \frac{\widetilde{H}_{(1^{m_j+\cdots+m_k-1})}}{\widetilde{H}_{(1^{m_j+\cdots+m_k})}} - [m_{j+1} + \cdots + m_k]_t \frac{\widetilde{H}_{(1^{m_{j+1}+\cdots+m_k-1})}}{\widetilde{H}_{(1^{m_{j+1}+\cdots+m_k})}},$$

where the modified Macdonald polynomials are evaluated at $q = 1$.

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