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A Generalized RSK for Enumerating Linear Series on n-pointed Curves

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Abstract. We give a combinatorial proof of a recent geometric result of Farkas and Lian on linear series on curves with prescribed incidence conditions. The result states that the expected number of degree-d morphisms from a general genus-g, n-marked curve C to $\mathbb{P}^r$, sending the marked points on C to specified general points in $\mathbb{P}^r$, is equal to $(r+1)^g$ for sufficiently large d. This computation may be rephrased as an intersection problem on Grassmannians, which has a combinatorial interpretation in terms of Young tableaux by the Littlewood-Richardson rule. We give a bijection, generalizing the RSK correspondence, between the tableaux in question and the $(r+1)$-ary sequences of length g, and we explore our bijection's combinatorial properties.

We also apply similar methods to give a combinatorial interpretation and proof of the fact that, in the modified setting in which $r = 1$ and several marked points map to the same point in $\mathbb{P}^1$, the number of morphisms is still 2^g for sufficiently large d.

Keywords: RSK algorithm, Young tableaux, Schubert calculus, linear series, curves

1 Introduction

In a recent paper [8], Farkas and Lian provide enumerative formulas for the number of maps from a curve C to a complex projective space $\mathbb{P}^r$ with specified incidence conditions. In particular, let C be a general curve of genus g, and let $x_1, \ldots, x_n$ be distinct general points on C. Also choose distinct general points $y_1, \ldots, y_n$ in $\mathbb{P}^r$. Then we write $L_{g,r,d}$ for the number of degree d morphisms $f : C \to \mathbb{P}^r$ for which $f(x_i) = y_i$ for all $i = 1, 2, \ldots, n$, which is finite precisely when $nr = dr + d + r - rg$.

For sufficiently large d and setting $n = (dr + d + r - rg)/r$ (assuming r divides d), it was shown in [8] that

$$L_{g,r,d} = (r+1)^g. \tag{1.1}$$

In addition, $L_{g,r,d}$ is equal to a certain intersection of Schubert cycles in the Grassmannian. The latter formula has the following combinatorial interpretation, as we will show in Section 3.

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Definition 1.1. Define an L**-tableau with parameters** (g, r, d) to be a way of filling the boxes of an $(r+1) \times (d-r)$ grid with rg "red" integers and $(d-r)(r+1) - rg$ "blue" integers such that:

- The red integers are left-and-bottom justified, and weakly increase up columns and strictly increase across rows. They consist of the numbers $1, 2, \ldots, g$ each occurring exactly r times.
- The blue integers are right-and-top justified, and strictly increase up columns and weakly increase across rows. Their values are from $\{0, 1, \ldots, r\}$.

Example 1.2. The following is an L-tableau with parameters $(4, 3, 9)$. We write the "red" numbers as black font with a red shaded background for clarity.

2	4	1	3	3	3
1	3	4	1	2	2
1	2	3	0	1	1
1	2	3	4	0	0

The work in [8] shows that the number of L-tableaux with parameters (g, r, d) is equal to $(r+1)^g$ whenever $d \geq rg + r$ (and $r|d$, so that n is an integer) via geometric methods, and asks for a combinatorial proof. Our first main result resolves this open problem by finding a combinatorial proof, of the following stronger result.

Theorem 1.3. *The number of L-tableaux with parameters (g, r, d) is $(r+1)^g$ when $d \geq g + r$.*

Note that in this purely combinatorial setting we only require $d \geq g + r$ rather than $d \geq rg + r$, and d is not necessarily divisible by r.

Our methods generalize the RSK algorithm (see Section 2.1). In particular, in the case $r = 1$, the pair of red and blue tableaux correspond directly under the RSK bijection to the binary sequences of length g (see [10]). For $r > 1$ we introduce an intermediate bijection (see Definition 3.7) to reduce to the RSK correspondence once again.

Remark 1.4. In the case $r = 1$, there are several known proofs of the fact that $L_{g,1,d} = 2^g$ for sufficiently large d, including via *scattering amplitudes* for $d = g + 1$ [15] and by establishing recursions using the boundary geometry of the moduli space of *Hurwitz covers* [2]. Neither of these proofs were combinatorial in nature, though the recursions arising in the latter paper by Cela-Pandharipande-Schmitt [2] are related to Dyck paths and other Catalan objects for small d.

Our second result enumerates a related set of maps. Set $r = 1$, fix an integer k with $1 \leq k \leq \min(n, d)$, and consider choices of points $y_1, \ldots, y_n \in \mathbb{P}^1$ such that

$$y_1 = y_2 = \cdots = y_k.$$

Write $L'_{g,d,k}$ for the number, assuming that $n = (dr + d + r - rg)/r = 2d + 1 - g$, of maps $f : C \to \mathbb{P}^1$ such that $f(x_i) = y_i$ for all i. We use a similar interpretation in terms of a family of Young tableaux that we call **L'-tableaux**, starting from an intersection theoretic formula in Grassmannians for $L'_{g,d,k}$ given by Farkas and Lian [8], to enumerate these maps for sufficiently high degree curves.

Theorem 1.5. *If $d \geq g + k$, we have $L'_{g,d,k} = 2^g$.*

Notice that the formula 2^g coincides with $(r+1)^g$ at $r = 1$. Indeed, it is remarked in [8] that the simple formula $L'_{g,d,k} = 2^g$ may be explained geometrically in the same way that the formula $L_{g,r,d} = (r+1)^g$ does. Further details and a more general formula for maps with arbitrary ramification profiles were given by Cela and Lian [1]. However, the formula from [8] coming from Grassmannians is stated as a *difference* of two sums of Schubert class intersection products (see Section 4), and there was previously no stated enumerative combinatorial interpretation of the latter formula, or for that matter a direct explanation for why the difference should be positive. We provide such an interpretation in order to prove Theorem 1.5.

In Section 2 below we recall the necessary background results and notation from tableaux theory and intersection theory on Grassmannians. In Section 3 we translate the geometric formulas for $L_{g,r,d}$ into a Young tableaux enumeration problem and give a sketch of the proof of Theorem 1.3. In Section 4 we define L'-tableaux for $L'_{g,d,k}$ and sketch the proof of Theorem 1.5. Full details can be found in our paper [10].

2 Background

We briefly recall several facts about Young tableaux, linear series, and Grassmannians.

2.1 The RSK correspondence on words

The **Young diagram** of a partition λ is the left- and bottom-justified grid of unit squares in the first quadrant (called *boxes*) for which there are λ_i boxes in the i-th row from the bottom for all i. A **semistandard Young tableau**, or **SSYT**, of shape λ is a way of filling the boxes of the Young diagram with nonnegative integers such that the rows are weakly increasing from left to right and the columns are strictly increasing from bottom to top. An SSYT is **standard**, written SYT, if the entries are $1, 2, \ldots, n$ each used exactly once. The **shape** of a Young tableau is the underlying Young diagram, and its **content** is the tuple $(m_1, m_2, \ldots)$ where m_i is the number of times i appears in the tableau.

The RSK correspondence, in its most general form, is a bijection between two-line arrays and pairs of semistandard Young tableaux of the same shape (see [9] for an excellent overview). In the case that the bottom row of the two-line array consists of the numbers

$1, 2, 3, \ldots, n$ in order, the top row may be any sequence, and we obtain the following special case of the RSK correspondence.

Proposition 2.1 (RSK for words)**.** *Let $A(r,n) = \{0,1,2,3,\ldots,r\}^n$ denote the set of all length-n sequences with entries from $\{0,1,2,\ldots,r\}$. Let $B(r,n)$ be the set of all pairs (P,Q) such that P is a semistandard Young tableau with letters in $\{0,1,2,\ldots,r\}$, Q is a standard Young tableau, and P and Q have the same shape of size n. There is an explicit bijection, called the RSK correspondence, from $A(r,n)$ to $B(r,n)$ for all r,n.*

We will refer to the length-n sequences of letters from $\{0,1,\ldots,r\}$ as $(r+1)$**-ary sequences**, generalizing the notion of a **binary sequence** from $\{0,1\}$. We will use the following property that follows from the definition of semistandard Young tableaux.

Remark 2.2. Since P has letters in $\{0,1,2,3,\ldots,r\}$, and columns are strictly increasing, any pair (P,Q) in $B(r,n)$ can have height at most $r+1$.

Example 2.3. The 4-ary sequence $0,2,1,1,0,3,0,0,1$ in $A(3,9)$ corresponds under RSK to the following pair of tableaux in $B(3,9)$:

$$\begin{array}{|c|c|c|c|c|}\cline{1-1} 2 \\ \cline{1-3} 1 & 1 & 3 \\ \hline 0 & 0 & 0 & 0 & 1 \\ \hline \end{array} \;,\quad \begin{array}{|c|c|c|c|c|}\cline{1-1} 5 \\ \cline{1-3} 3 & 7 & 8 \\ \hline 1 & 2 & 4 & 6 & 9 \\ \hline \end{array}$$

2.2 Linear series and calculations in the Grassmannian

The definitions of $L_{g,r,d}$ and $L'_{g,r,d}$ may be made more rigorous via the theory of linear series on curves. (See [7] for an introduction to this topic.) A **linear series** of type $\mathfrak{g}^r_d$ on a smooth curve C can be thought of as a r-dimensional linear family of sets of d points on C. More formally, this data is encoded by a pair $(\mathcal{L}, V)$ where $\mathcal{L}$ is a line bundle of degree d and $V \subseteq H^0(C,\mathcal{L})$ is a dimension $r+1$ space of sections, which in turn corresponds to a map $\phi_{(\mathcal{L},V)}$ from C to $\mathbb{P}^r$.

There has been much study of enumeration of linear series with prescribed *ramification* conditions at specified points, including the work of Eisenbud and Harris [6], Osserman [12], Chan and Pflueger [5], Chan, López Martín, Pflueger, and Teixidor i Bigas [3], Larson, Larson, and Vogt [11], and others. Many of these results relied on the combinatorial tools of Young tableaux, symmetric function theory, and other aspects of algebraic combinatorics, and in [4] led to new results in algebraic combinatorics as well.

Here we consider the related problem of enumerating linear series with prescribed *incidence* conditions at specified marked points. Write $G^r_d(C)$ to denote the moduli space of $\mathfrak{g}^r_d$'s on C, and let $x_1, \ldots, x_n$ be general marked points on C. Then the values $L_{g,r,d}$ may alternatively be defined as the degree of the evaluation map $\mathrm{ev}_{(x_1,\ldots,x_n)}$ from $G^r_d(C)$ to the moduli space P^n_r of n points in $\mathbb{P}^r$, given by evaluating the maps $\phi_{(\mathcal{L},V)}$ at the points

$x_1, \ldots, x_n$. In [8], a degeneration argument, starting with a reduction to genus 0, is used to reduce the problem of computing the degree of this evaluation map to an intersection problem in Schubert calculus. For genus 0 curves C, $H^0(C, \mathcal{L})$ in general has dimension $d+1$ as a complex vector space for sufficiently high d. Therefore, the $r+1$-dimensional subspaces $V \subseteq H^0(C, \mathcal{L})$ sweep out a copy of the Grassmannian $\mathrm{Gr}(r+1, d+1)$.

The Grassmannian $\mathrm{Gr}(r+1, d+1)$, defined as the moduli space of $r+1$-dimensional subspaces of $\mathbb{C}^{d+1}$, has a well-known Schubert decomposition (with respect to a given flag) into Schubert varieties X_λ. (See [9, Ch. 9] for background on Schubert calculus.) Here, λ ranges over all partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ whose Young diagram fits inside an $(r+1) \times (d+1)$ grid.

The Schubert varieties give rise to a basis of *Schubert classes* $\sigma_\lambda := [X_\lambda]$ of its Chow ring $A^\bullet(\mathrm{Gr}(r+1, d+1))$. With respect to this basis, it is shown in [8] that whenever either $d \geq rg + r$, $d = r + \frac{rg}{r+1}$, or $r = 1$, we have

$$L_{g,r,d} = \int_{\mathrm{Gr}(r+1,d+1)} \sigma_{1^r}^g \cdot \left[\sum_{\alpha_0 + \cdots + \alpha_r = (r+1)(d-r) - rg} \left(\prod_{i=0}^{r} \sigma_{\alpha_i} \right) \right]. \tag{2.1}$$

Here the notation σ_{1^r} is shorthand for $\sigma_{(1,1,1,\ldots,1)}$ where the tuple $(1, 1, \ldots, 1)$ has length r, and σ_{α_i} is shorthand for $\sigma_{(\alpha_i)}$. The integral indicates that the sum of products of Schubert cycles in question expands in the Schubert basis as a constant multiple of $\sigma_{(d+1)^r} := \sigma_{(d+1,d+1,\ldots,d+1)}$, and the integral is defined to be this constant coefficient.

The corresponding result from [8] for $L'_{g,d,k}$ is stated in Section 4.

2.3 The iterated Pieri rule

The intersections of Schubert cycles on the Grassmannian may be calculated via symmetric function theory, as products of Schubert classes correspond to products of *Schur functions*. Indeed, let $\{s_\lambda\}$ be the classical Schur function basis of the ring of symmetric functions, where λ ranges over all partitions (see [9] or [14, Ch. 7]). Then the integral in equation (2.1) is equal to the coefficient of $s_{(d+1)^{(r+1)}}$ in the expansion

$$s_{1^r}^g \cdot \left[\sum_{\alpha_0 + \cdots + \alpha_r = (r+1)(d-r) - rg} \left(\prod_{i=0}^{r} s_{\alpha_i} \right) \right] \tag{2.2}$$

The coefficients of products of Schur functions expressed in the Schur basis are called **Littlewood-Richardson coefficients**. In particular we can write

$$s_\lambda \cdot s_\mu = \sum c_{\lambda\mu}^{\nu} s_\nu$$

where the Littlewood-Richardson coefficients $c_{\lambda\mu}^{\nu}$ are all nonnegative integers. In our setting we will only need to focus on the cases when one of λ or μ is either a horizontal row or vertical column, in which case the Littlewood-Richardson coefficients are described by the *Pieri rules* (see, for instance, [9, pp. 24–25]) as follows.

Suppose λ and μ are partitions for which the Young diagram of μ fits inside that of λ. The **skew shape** λ/μ is the set of boxes that are in λ but not in μ. A skew shape is a **horizontal strip** if no two of its boxes are in the same column, and it is a **vertical strip** if no two of its boxes are in the same row.

Proposition 2.4 (Pieri rules)**.** *For $\mu = (\alpha)$ a single-row partition, the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu} = c^{\nu}_{\lambda,(\alpha)}$ is equal to 1 if ν/λ is a horizontal strip, and 0 otherwise.*

For $\mu = (1^r)$ a single-column partition, the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu} = c^{\nu}_{\lambda,(1^r)}$ is equal to 1 if ν/λ is a vertical strip, and 0 otherwise.

Proposition 2.4 gives us a rule for multiplying any Schur function by either $s_{(1^r)}$ or s_α, and expanding the result again as a sum of Schur functions. In particular,

$$s_\lambda \cdot s_{(\alpha)} = \sum_{\nu/\lambda \in \mathrm{Horz}(\alpha)} s_\nu \qquad s_\lambda \cdot s_{(1^r)} = \sum_{\nu/\lambda \in \mathrm{Vert}(\mathrm{r})} s_\nu$$

where $\mathrm{Horz}(\alpha)$ and $\mathrm{Vert}(r)$ are the sets of all horiztonal strips of size α and vertical strips of size r respectively. We can iterate to give a rule for any product of row or column Schur functions as follows.

Corollary 2.5 (Iterated Pieri rule)**.** *Let $\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(k)}$ be partitions, each of which is either a horizontal row or a vertical column. Then*

$$s_{\mu^{(1)}} \cdot s_{\mu^{(2)}} \cdot \cdots \cdot s_{\mu^{(k)}} = \sum c^{\nu}_{\mu^{(1)}\cdots\mu^{(k)}} s_\nu$$

where $c^{\nu}_{\mu^{(1)}\cdots\mu^{(k)}}$ is equal to the number of ways to extend $\mu^{(1)}$ by horizontal or vertical strips (as indicated by each $\mu^{(i)}$) of sizes $|\mu^{(2)}|, \ldots, |\mu^{(k)}|$ such that the total resulting shape is ν.

In order to keep track of the horizontal and vertical strips, we will label the squares of the strip corresponding to $\mu^{(i)}$ by i for each i. This results in a tableau-like object that enumerates the generalized Pieri coefficients $c^{\nu}_{\mu^{(1)}\cdots\mu^{(k)}}$ above.

Example 2.6. The coefficient $c^{(4,2,1)}_{(2),(1,1),(3)}$ is equal to 2, because there are two ways to fill the boxes of shape $(4,2,1)$ with a horizontal strip of two 1's, a vertical strip of two 2's extending it, and a horizontal strip of three 3's that extends the shape again:

$$\begin{array}{|c|ccc}\cline{1-1} 2 & & & \\ \cline{1-2} 2 & \multicolumn{1}{c|}{3} & & \\ \hline 1 & 1 & 3 & \multicolumn{1}{c|}{3} \\ \hline\end{array} \qquad \begin{array}{|c|ccc}\cline{1-1} 3 & & & \\ \cline{1-2} 2 & \multicolumn{1}{c|}{3} & & \\ \hline 1 & 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}$$

In the special case when we have all horizontal strips, we recover the well-known notion of a **semistandard Young tableau**, or **SSYT**: a filling of the boxes of a (possibly skew) Young diagram with numbers such that the rows are weakly increasing from left to right and the columns are strictly increasing from bottom to top. For the case of all vertical strips, we say a **transposed semistandard Young tableau** is a filling of a Young diagram with strictly increasing *rows* and weakly increasing *columns*. We similarly obtain transposed SSYT's in the case of all vertical strips.

3 L-tableaux and enumeration by $(r+1)^g$

Corollary 2.5, combined with the fact that the integral in equation (2.1) is the coefficient of $s_{(d-r)^{r+1}}$ in the corresponding product (2.2), shows that

$$L_{g,r,d} = \sum_{\alpha_0+\cdots+\alpha_r+rg=(r+1)(d-r)} c^{(d-r)^{r+1}}_{(1^r),(1^r),\ldots,(1^r),(\alpha_0),\ldots,(\alpha_r)}$$

where the subscripts include g copies of (1^r). This summation is therefore the number of ways to form a transposed SSYT using each of the numbers $1, 2, \ldots, g$ exactly r times, and then extend it to fill the rest of the $(r+1) \times (d-r)$ grid with a semistandard Young tableau using the numbers $0, 1, \ldots, r$ in some varying amounts $\alpha_0, \ldots, \alpha_r$ each.

This precisely matches the definition of L-tableaux given in Definition 1.1 in the introduction.

Remark 3.1. The preprint [8] mistakenly uses an ordinary (not transposed) SSYT for the red tableau; a correction will appear in a later version of their work and appears in [10].

See Example 1.2. Our discussion thus far, starting from Equation (2.1), has shown:

Proposition 3.2. *The number of L-tableau with parameters (g, r, d) is equal to $L_{g,r,d}$ whenever either $d \geq rg + r$, $d = r + \frac{rg}{r+1}$, or $r = 1$.*

We now note that we can "truncate" by removing some of the right-hand columns of the grid to reduce to a simpler case.

Lemma 3.3 (Truncation)**.** *For any g, r, d with $d \geq g + r$, the number of L-tableaux with parameters (g, r, d) is equal to the number of L-tableaux with parameters $(g, r, g + r)$.*

We omit the proof and refer to [10] for details. The idea is that any column to the right of the g-th column is filled entirely with the blue numbers $0, 1, 2, \ldots, r$ in order and therefore is already determined.

Lemma 3.3 tells us that in order to understand $L_{g,r,d}$ for $d \geq g + r$, it suffices to study the case $d = g + r$. We will restrict to this case throughout the remainder of this section.

Remark 3.4. When $d = g + r$, the rectangle containing the L-tableaux is size $(r+1)g = rg + g$. The red tableau has size rg and so the blue tableau has size g.

We now prove Theorem 1.3. We first define the following sets of tableaux.

Definition 3.5. Let $\mathrm{TrSSYT}(g,r)$ be the set of all transposed SSYT's of content $(r^g) = (r,r,\ldots,r)$ and height $\leq r+1$.

Note that $\mathrm{TrSSYT}(g,r)$ is the set of all possible 'red' tableaux in Definition 1.1. We will refer to them as **red tableaux** throughout this section.

Definition 3.6. Define a $180°$**-rotated SYT** to be the result of rotating a standard Young tableaux $180°$ in the plane. We write $\mathrm{SYT}^{180°}(g,r)$ for the set of all $180°$-rotated SYT of size g and height $\leq r+1$. We informally call such a tableau a **purple tableau**, as it will be used to relate the red and blue tableaux of Definition 1.1 (see Example 3.8).

Given a red tableau, note that each number $1,\ldots,g$ occurs once in every row except one. Relatedly, a purple tableau in the position of the blue tableau will have each number $1,\ldots,g$ in exactly one row. This leads us to define a bijection between the two as follows.

Definition 3.7 (Red to purple bijection)**.** Let $R \in \mathrm{TrSSYT}(g,r)$ be a red tableau. We define a $180°$-rotated tableau $\varphi(R)$ in the upper right corner of a rectangle by the following iterative process. We add boxes labeled $1,2,\ldots,g$ in order, where on the ith step we place a box labeled i as far to the right as possible in the unique row that does **not** contain an i in R.

Example 3.8. If R is the tableau at left below, its corresponding purple tableau $\varphi(R)$ is shown at right below.

2	4	5	6		
1	3	4	5	7	
1	2	3	5	6	7
1	2	3	4	6	7

7	3	1
	6	2
		4
		5

A generalized version of this map was shown to be a bijection in [13]. We also provide a more direct proof in this specific case in [10].

Lemma 3.9. *The map φ is a bijection from $\mathrm{TrSSYT}(g,r)$ to $\mathrm{SYT}^{180°}(g,r)$ for all g,r. Moreover, for any $R \in \mathrm{TrSSYT}(g,r)$, the shapes of R and $\varphi(R)$ are complementary in an $(r+1)\times g$ rectangle.*

We now make precise the notion of a "blue tableau" (see Definition 1.1).

Definition 3.10. Define a $180°$**-rotated semistandard tableau**, or **blue tableau** (with parameters r, g), to be a filling of a $180°$-rotated Young diagram of size g with numbers $0, 1, 2, \ldots, r$ such that the rows are weakly increasing from left to right and columns are strictly increasing from bottom to top.

Lemma 3.11. *The pairs of blue and purple tableaux of the same shape correspond to $(r+1)$-ary sequences of length g bijectively, via inverting the entries of the blue tableau (that is, replacing each entry i by $r - i$), rotating both $180°$, and applying RSK.*

The proof is a straightforward application of RSK and we omit it.

Example 3.12. Consider the pair of blue and purple tableaux below. The corresponding pair (P, Q) is shown at right.

$$\begin{array}{|c|c|c|} \hline 0 & 2 & 3 \\ \hline & 1 & 2 \\ \cline{2-3} & & 1 \\ \cline{3-3} & & 0 \\ \cline{3-3} \end{array} \qquad \begin{array}{|c|c|c|} \hline 7 & 3 & 1 \\ \hline & 6 & 2 \\ \cline{2-3} & & 4 \\ \cline{3-3} & & 5 \\ \cline{3-3} \end{array} \qquad \longrightarrow \qquad \begin{array}{|c|c|c|} \cline{1-1} 3 \\ \cline{1-1} 2 \\ \cline{1-2} 1 & 2 \\ \hline 0 & 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|} \cline{1-1} 5 \\ \cline{1-1} 4 \\ \cline{1-2} 2 & 6 \\ \hline 1 & 3 & 7 \\ \hline \end{array}$$

The corresponding pair (P, Q) is as follows: This pair corresponds via RSK to the $(r+1)$-ary sequence $3, 2, 2, 1, 0, 1, 3$ where $r = 3$.

We finally can produce a bijection between L-tableaux and $(r+1)$-ary sequences.

Proposition 3.13. *The L-tableaux with parameters $(g, r, g+r)$ are in bijection with the $(r+1)$-ary sequences of length g (with letters from the alphabet $\{0, 1, 2, \ldots, r\}$).*

Proof. Each such L-tableaux consists of a red tableau and a blue tableau. The bijection follows from combining Lemma 3.9 with Lemma 3.11, which provide bijections between red tableaux with purple tableaux, and between pairs of blue and purple tableaux with $(r+1)$-ary sequences of length g, respectively. □

Example 3.14. Below is an L-tableau with parameters $(7, 3, 10)$. From our previous examples, we see that it corresponds to the $(r+1)$-ary sequence $3, 2, 2, 1, 0, 1, 3$.

2	4	5	6	0	2	3
1	3	4	5	7	1	2
1	2	3	5	6	7	1
1	2	3	5	6	7	0

There are $(r+1)^g$ sequences of length g in the alphabet $0, 1, 2, \ldots, r$. Combining Proposition 3.13 with truncation (Lemma 3.3), we get as a corollary Theorem 1.3.

Theorem 1.3. *The number of L-tableaux with parameters (g, r, d) is $(r+1)^g$ for all $d \geq r + g$.*

In our full paper [10], we also analyze the case where $r = 1$, which reduces to ordinary RSK, and the case $d = r + \frac{rg}{r+1}$, which recovers a classical theorem of Castelnuovo.

4 L'-tableaux and enumeration by 2^g

The result from [8] corresponding to $L'_{g,d,k}$ states that if $k+g \le 2d+1$ and $2 \le k \le d$:

$$L'_{g,d,k} = \int_{\mathrm{Gr}(2,d+1)} \sigma_1^g \sigma_{k-1} \left(\sum_{i+j=2d-g-k-1} \sigma_i \sigma_j \right) - \int_{\mathrm{Gr}(2,d)} \sigma_1^g \sigma_{k-2} \left(\sum_{i+j=2d-g-k-2} \sigma_i \sigma_j \right).$$

We first give an interpretation of the left hand integral in the equation above.

Definition 4.1. A **positive L'-tableau with parameters** (g,d,k) is a way of filling a $2 \times (d-1)$ grid with:

- A standard Young tableau of size g in the lower left corner (shaded red),
- A shading of the $k-1$ rightmost boxes in the top row (gray),
- A skew SSYT in two letters $0,1$ on the remaining squares (blue).

By rearranging so that we think of the σ_{k-1} as last in each product, and applying Corollary 2.5, we see that the positive term in equation (4.1) is equal to the number of positive L'-tableaux. The second term in (4.1), which we are subtracting, is similarly given by a set of smaller tableaux that we call *negative* tableaux.

Definition 4.2. A **negative L'-tableau with parameters** (g,d,k) is a filling of a $2 \times (d-2)$ grid with:

- A standard Young tableau of size g in the lower left corner (shaded red),
- A shading of the $k-2$ rightmost boxes in the top row (gray),
- A skew SSYT in two letters $0,1$ on the remaining squares (blue).

Example 4.3. Below are a positive and negative L'-tableau with parameters $(3,7,4)$.

3	0	1			
1	2	0	0	1	1

3	0	1		
1	2	0	0	1

Notice that there exist positive L'-tableaux if and only if $(k-1)+g \le 2(d-1)$, which is slightly stronger than the given condition $k+g \le 2d+1$. In particular if $k+g = 2d$ or $k+g = 2d+1$ we have $L'_{g,d,k} = 0$, so we restrict our attention to the case that $k+g \le 2d-1$. We now prove Theorem 1.5.

Definition 4.4. For fixed g,d,k, write L'_+ and L'_- for the set of positive and negative L' tableaux respectively of type (g,d,k). Also write $\psi : L'_- \to L'_+$ for the map that takes a negative tableau T and adds a blue 1 to the end of the bottom row and a gray box to the end of the top row.

Our above analysis shows that $L'_{g,d,k} = |L'_+| - |L'_-|$, and we analyze this difference combinatorially. We note that ψ is a well-defined injective map, and so

$$L'_{g,d,k} = |L'_+ \setminus \psi(L'_-)|. \tag{4.1}$$

The next proposition characterizes the image $\psi(L'_-)$, and we refer to [10] for our proof.

Proposition 4.5. *A positive tableau T is equal to $\psi(S)$ for some negative tableaux S if and only if the bottom row of T contains a blue* 1.

Applying this proposition and equation (4.1), we obtain the following.

Corollary 4.6. *The quantity $L'_{g,d,k}$ is equal to the number of positive L'-tableaux with parameters (g, d, k) for which the bottom row contains no blue* 1.

For sufficiently large d, we can simplify this characterization even further.

Theorem 1.5. *If $d \geq g + k$, we have $L'_{g,d,k} = 2^g$.*

We refer to [10] for the full proof. As a sketch, we may restrict our attention to the columns before any gray squares appear, and then use the RSK bijection on the red and blue tableaux in those columns to obtain a binary sequence.

Example 4.7. The tableau below at left is a positive L'-tableau with parameters $(3, 7, 4)$ that is not the image of a negative one. Since $g = 3$, we restrict our attention to the first three columns (second image below), then consider the associated pair of tableaux of the same shape by rotating the blue tableaux and inverting the labels. Finally, this pair corresponds under RSK to a unique length 3 binary sequence.

$$\begin{array}{|c|c|c|c|c|c|}\hline 3 & 0 & 1 & & & \\ \hline 1 & 2 & 0 & 0 & 0 & 0 \\ \hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 3 & 0 & 1 \\ \hline 1 & 2 & 0 \\ \hline\end{array} \longrightarrow \left(\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\ ,\ \begin{array}{|c|c}\cline{1-1} 1 & \\ \hline 0 & \multicolumn{1}{c|}{1} \\ \hline\end{array} \right) \longrightarrow 110$$

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