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A unipotent realization of the chromatic quasisymmetric function

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Abstract. The chromatic quasisymmetric function is a t-analogue of Stanley's chromatic symmetric function, and has recently been at the center of a number of exciting developments in algebraic combinatorics. This extended abstract contributes to this trend, describing a novel realization of certain chromatic quasisymmetric functions as characters of the finite general linear group $\mathrm{GL}_n(\mathbb{F}_q)$. Additional results tie these characters to other aspects of the chromatic quasisymmetric function: point counting in Hessenberg varieties over $\mathbb{F}_q$, realizing the plethystic connection with unicellular LLT polynomials, and re-interpreting positivity conjectures.

Keywords: Chromatic quasisymmetric function; unicellular LLT polynomial; unipotent group; combinatorial Hopf algebra

1 Introduction

The chromatic symmetric function sits at a nexus of disparate areas of mathematics. At face value, this symmetric function encodes the coloring problem of a graph as an analogue of the chromatic polynomial [18]. However, through a well-known equivalence between the ring of symmetric functions and the representation theory of the symmetric groups (see e.g. [14]), some chromatic symmetric functions are also complex characters of the symmetric group [7]. Moreover, by way of a t-analogue known as the chromatic quasisymmetric function, Brosnan and Chow [3] and Guay-Paquet [10] independently proved that the characters corresponding to indifference graphs are afforded by symmetric group modules on the cohomology rings of regular semisimple Hessenberg varieties, as predicted by a conjecture of Shareshian and Wachs [17]. Thus, certain questions about graphs, representation theory, and algebraic geometry coincide in the structure of these symmetric functions, and vice versa.

Based on the paper [6], this extended abstract describes yet another connection to the chromatic quasisymmetric function, this time from the general linear group $\mathrm{GL}_n(\mathbb{F}_q)$ over the finite field with q elements. The maximal unipotent subgroup $\mathrm{UT}_n(\mathbb{F}_q)$ of

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$\mathrm{GL}_n(\mathbb{F}_q)$ has a family of characters indexed by indifference graphs, and Theorem 3.1 shows that and up to a factor of $(q-1)^n$ the induction functor gives a map

$$\left\{ \begin{array}{c} \text{indifference graph} \\ \text{indexed characters} \end{array} \right\} \xrightarrow{\mathrm{Ind}_{\mathrm{UT}_n(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)}} \left\{ \begin{array}{c} \text{chromatic quasisymmetric functions for} \\ \text{indifference graphs evaluated at } t = q \end{array} \right\}$$

where unipotently supported class functions of $\mathrm{GL}_n(\mathbb{F}_q)$ are identified with symmetric functions via the Hall algebra. Section 3 describes this result and its proof using a Hopf algebraic interpretation of induction established in a second paper of the author [5].

The remaining sections of this abstract explore implications of Theorem 3.1. While the theorem is reminiscent of the Brosnan–Chow–Guay-Paquet theorem, the underlying association between symmetric functions and $\mathrm{GL}_n(\mathbb{F}_q)$ characters in this paper is quite different and offers a complementary perspective. Moreover, the relevant characters are elementary in nature and may be more accessible from a module theoretic standpoint.

Section 4 gives a geometric interpretation of the characters in Theorem 3.1: their values count the points of Hessenberg varieties associated to an ad-nilpotent ideal over $\mathbb{F}_q$. Even over $\mathbb{C}$, Hessenberg varieties associated to ad-nilpotent ideals are markedly different from the ones in the Brosnan–Chow–Guay-Paquet theorem, and there is no known module structure on their cohomology rings. However, Precup and Sommers [16] have given a geometric connection between the Poincaré polynomials of these Hessenberg varieties and the chromatic quasisymmetric function, and Corollary 4.4 shows how Theorem 3.1 can be seen as a representation theoretic manifestation of this phenomenon.

Section 5 revisits a well-known relationship between chromatic quasisymmetric functions and the family of unicellular LLT polynomials [4] from the perspective of $\mathrm{GL}_n(\mathbb{F}_q)$. The unipotent characters of $\mathrm{GL}_n(\mathbb{F}_q)$ give another realization of symmetric functions, and any character can be projected onto its unipotent summands to produce a symmetric function. For the characters in Theorem 3.1, this association turns out to be a twisted version of the relationship between chromatic quasisymmetric functions and unicellular LLT polynomials, and Theorem 5.3 gives a map

$$\left\{ \begin{array}{c} \text{indifference graph} \\ \text{indexed characters} \end{array} \right\} \xrightarrow{\omega \circ \text{projection} \circ \mathrm{Ind}_{\mathrm{UT}_n(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)}} \left\{ \begin{array}{c} \text{unicellular LLT polyno-} \\ \text{mials evaluated at } t = q \end{array} \right\}.$$

No previous connection between LLT polynomials and $\mathrm{GL}_n(\mathbb{F}_q)$ representation theory was known, though similar associations exist for quantum groups [13], affine Hecke algebras [8], and the symmetric groups [10, 12].

Finally, Section 6 describes the $\mathrm{GL}_n(\mathbb{F}_q)$ representation theoretic meaning of two "positivity conjectures" about the chromatic quasisymmetric function and unicellular LLT polynomial. Each conjecture postulates that when the symmetric function in question is expressed in a chosen basis, each coefficient will be nonnegative. For the chromatic quasisymmetric functions, the modified Stanley–Stembridge conjecture [17, Conjecture 1.3] (see also [20]) concerns positivity in the elementary symmetric function basis, and is entirely open. For the LLT polynomials, positivity in the Schur basis has been established by Grojnowski and Haiman [8], but no positive combinatorial formula for the coefficients is known [11]. While no immediate progress is made on either conjecture in this abstract, the $\mathrm{GL}_n(\mathbb{F}_q)$ interpretations may be a useful starting point for future work.

The remainder of this abstract is organized as follows. Section 2 gives general background material. Section 3 contains the main result, Theorem 3.1, and Section 4 relates this result to Hessenberg varieties. Section 5 concerns the unicellular LLT polynomial. Finally, Section 6 discusses the aforementioned positivity conjectures.

2 Preliminaries

This section contains preliminary material on Hopf algebras and symmetric functions (Section 2.1), chromatic quasisymmetric functions (Section 2.2), the character theory of $\mathrm{UT}_n(\mathbb{F}_q)$ (Section 2.3), and its relation to $\mathrm{GL}_n(\mathbb{F}_q)$ characters (Section 2.4).

2.1 (Quasi-)Symmetric functions and combinatorial Hopf algebras

This section describes the Hopf algebraic techniques required to prove the main result. Throughout, the term *Hopf algebra* will refer to a graded connected $\mathbb{C}$-vector space

$$H_\bullet = \bigoplus_{n\geq 0} H_n \qquad \text{such that} \qquad H_0 \cong \mathbb{C},$$

equipped with a $\mathbb{C}$-bialgebra structure and a compatible antipode map. This extra structure is not used prominently in this work, which instead focuses on Hopf algebra homomorphisms; these can be thought of as a well-behaved subclass of graded linear maps.

Two particularly important examples now follow. A *composition* of $n \in \mathbb{Z}_{\geq 0}$ is a sequence of positive integers $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $\alpha_1 + \cdots + \alpha_k = n$. Call each α_i a *part* of α. The *monomial quasisymmetric function* associated to the composition α is

$$M_\alpha = \sum_{i_1 < \cdots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_\ell}^{\alpha_{\ell(\alpha)}} \in \mathbb{C}[[\mathbf{x}]].$$

where $\mathbf{x} = \{x_1, x_2, \ldots\}$ is an infinite, totally ordered set of indeterminates. With grading given by degree, the *Hopf algebra of quasisymmetric functions* is

$$\mathcal{Q}\mathrm{Sym} = \mathbb{C}\text{-span}\{M_\alpha \mid \alpha \text{ is a composition}\}.$$

A *partition of n* is a composition of n with non-increasing parts. Let

$$\mathcal{P} = \bigsqcup_{n\geq 0} \mathcal{P}_n \qquad \text{with} \qquad \mathcal{P}_n = \{\text{partitions of } n\}.$$

The *Hopf algebra of symmetric functions* is the subspace

$$\mathrm{Sym} = \mathbb{C}\text{-span}\{m_\lambda \mid \lambda \in \mathcal{P}\} \subseteq \mathcal{Q}\mathrm{Sym} \qquad \text{with} \qquad m_\lambda = \sum_{\mathrm{sort}(\alpha)=\lambda} M_\alpha,$$

where $\mathrm{sort}(\alpha)$ is the partition obtained by listing the parts of α in non-increasing order.

This abstract will use several standard bases of Sym found in [14]: the elementary symmetric functions $\{e_\lambda \mid \lambda \in \mathcal{P}\}$ [14, I.2], the Schur functions $\{s_\lambda \mid \lambda \in \mathcal{P}\}$ [14, I.3], and the degree-shifted Hall-Littlewood symmetric functions $\{\widetilde{P_\lambda}(\mathbf{x};t) \mid \lambda \in \mathcal{P}\}$ [14, IV.4], which depend on an additional parameter t. We will also use the involution $\omega : \mathrm{Sym} \to \mathrm{Sym}$, given by $\omega(s_\lambda) = s_{\lambda'}$, where λ' denotes the transpose of λ.

The paper [1] gives classification of all homomorphisms from an arbitrary Hopf algebra H to $\mathcal{Q}\mathrm{Sym}$ via certain algebra homomorphisms $\zeta : H \to \mathbb{C}$, called *linear characters* herein in order to avoid confusion with the group characters in this work. The Hopf algebra $\mathcal{Q}\mathrm{Sym}$ has a linear character known as the *first principal specialization*,

$$\begin{aligned} \mathrm{ps}_1 : \quad \mathcal{Q}\mathrm{Sym} &\longrightarrow \mathbb{C} \\ M_\alpha &\longmapsto \begin{cases} 1 & \text{if } \alpha = () \text{ or } (n) \text{ for } n \geq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Theorem 2.1 ([1] Theorem 4.1)**.** *Let H be a Hopf algebra. Then the map*

$$\begin{aligned} \{\textit{homomorphisms } H \to \mathcal{Q}\mathrm{Sym}\} &\longrightarrow \{\textit{linear characters of } H\} \\ \Psi &\longmapsto \mathrm{ps}_1 \circ \Psi \end{aligned}$$

is a bijection. In particular, for each linear character ζ of H, there is a unique homomorphism $\Psi : H \to \mathcal{Q}\mathrm{Sym}$ for which $\zeta = \mathrm{ps}_1 \circ \Psi$.

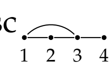
2.2 Indifference graphs and the chromatic quasisymmetric function

This section will describe the chromatic quasisymmetric function of a graph, and go on to define a special class of graphs for which this function is particularly well behaved.

Let γ be a simple, undirected graph with vertex set $[n]$ and edge set $E(\gamma)$. A *coloring* of γ is a function $\kappa : [n] \to \mathbb{Z}_{>0}$. A coloring κ of γ is *proper* if $\kappa(i) \neq \kappa(j)$ for all $\{i,j\} \in E(\gamma)$. The *γ-ascent number* of a coloring κ is

$$\mathrm{asc}_\gamma(\kappa) = \left|\{\{i,j\} \in E(\gamma) \mid i < j \text{ and } \kappa(i) < \kappa(j)\}\right|. \tag{2.2}$$

For example, if $\kappa : [5] \to \mathbb{Z}_{>0}$ is given by $\kappa(1) = 2$, $\kappa(2) = 5$, $\kappa(3) = 1$, and $\kappa(4) = 5$,

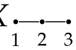
asc  $(\kappa) = \left|\{\{1,2\},\{3,4\}\}\right| = 2.$

In this example, κ is a proper coloring of the given graph.

The *chromatic quasisymmetric function* of γ is

$$X_\gamma(\mathbf{x};t) = \sum_{\substack{\kappa:[n]\to\mathbb{Z}_{>0} \\ \text{proper}}} t^{\mathrm{asc}_\gamma(\kappa)} x_{\kappa(1)}x_{\kappa(2)}\dots x_{\kappa(n)} \in \mathcal{Q}\mathrm{Sym}[t],$$

so that $X_\gamma(\mathbf{x};t)$ is a polynomial in an indeterminate t whose coefficients—by properties of the ascent statistic—are quasisymmetric functions. For example,

X  $(\mathbf{x};t) = t\, M_{(2,1)} + t\, M_{(1,2)} + (t^2+4t+1)\, M_{(1^3)}$. (2.3)

Evaluating the indeterminate t in $X_\gamma(\mathbf{x};t)$ at a complex number gives an actual quasisymmetric function. For example, taking $t = 1$ gives the ordinary chromatic symmetric function of γ as defined by Stanley in [18], and the main result of this paper concerns the evaluation of $X_\gamma(\mathbf{x};t)$ at $t = q$, the order of the finite field $\mathbb{F}_q$.

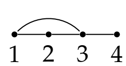
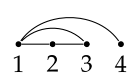
The coefficients of $X_\gamma(\mathbf{x};t)$ are known to be symmetric under certain conditions. An *indifference graph* of size $n \geq 0$ is a simple, undirected graph γ on the vertex set $[n] = \{1,\dots,n\}$ with edge set $E(\gamma)$ satisfying

$$\text{for each } \{i,l\} \in E(\gamma): \qquad \big\{\{j,k\} \mid i \leq j < k \leq l\big\} \subseteq E(\gamma).$$

The empty graph on $\emptyset$ is the unique indifference graph of size zero. Let

$$\mathcal{IG} = \bigsqcup_{n\geq 0} \mathcal{IG}_n \qquad \text{with} \qquad \mathcal{IG}_n = \{\text{indifference graphs on } [n]\}.$$

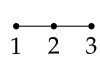
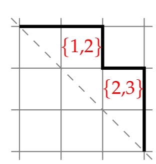
For example,

$\gamma =$  $\in \mathcal{IG}_4$ but $\sigma =$  $\notin \mathcal{IG}_4$,

as $\{1,4\} \in E(\sigma)$ but $\{3,4\} \notin E(\sigma)$.

Proposition 2.4 ([17, Theorem 4.5])**.** *For an indifference graph $\gamma \in \mathcal{IG}_n$, the coefficients of each power of t in $X_\gamma(\mathbf{x};t)$ is a symmetric function.*

The indifference graphs of size $n \geq 0$ are enumerated by the nth Catalan number, as seen in the following bijection, found in [19, Sol. 187]. Associate to each $\gamma \in \mathcal{IG}_n$ the southeast lattice path from $(0,0)$ to $(n,-n)$ which lies directly above the diagonal $x = -y$ and the unit squares centered at $(j-\frac{1}{2}, \frac{1}{2}-i)$ for each $\{i,j\} \in E(\gamma)$. For example,

 $\longleftrightarrow$ 

$= (EESESS)$. (2.5)

2.3 Linear algebraic groups and class functions

For a finite group G, the space of complex valued class functions on G is

$$\mathsf{cf}(G) = \{\psi : G \longrightarrow \mathbb{C} \mid \psi(g) = \psi(hgh^{-1}) \text{ for all } g, h \in G\}.$$

Under the usual inner product $\langle \cdot, \cdot \rangle : \mathsf{cf}(G) \otimes \mathsf{cf}(G) \to \mathbb{C}$, there are two orthogonal bases of $\mathsf{cf}(G)$: the irreducible characters of G and the conjugacy class identifier functions,

$$\{\delta_K \mid K \in \mathtt{Cl}\} \qquad \text{with} \qquad \delta_K(g) = \begin{cases} 1 & \text{if } g \in K. \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The remainder of this section will construct a subspace of class functions of a particular family of groups. Fix a prime power q, let $\mathbb{F}_q$ denote the field with q elements, and let $\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{F}_q)$. The *unipotent upper triangular group* is the subgroup

$$\mathrm{UT}_n = \{g \in \mathrm{GL}_n \mid \ (g - 1_n)_{i,j} \neq 0 \text{ only if } i < j \ \}$$

where $1_n \in \mathrm{GL}_n$ is the identity matrix. The set $\mathcal{IG}_n$ indexes a family of normal subgroups in UT_n known as *normal pattern subgroups* [15, Lemma 4.1]: for $\gamma \in \mathcal{IG}_n$, let

$$\mathrm{UT}_\gamma = \{g \in \mathrm{UT}_n \mid \ g_{i,j} = 0 \text{ if } \{i, j\} \in E(\gamma)\ \},$$

where $E(\gamma)$ denotes the edge set of γ. If π is the Dyck path corresponding to γ, then UT_γ is the subset of elements of UT_n with nonzero entries occurring only on the diagonal or above the path π. For example using the graph and Dyck path from Equation (2.5),

$$\mathrm{UT}_{\underset{1\ 2\ 3}{\bullet\!-\!\bullet\!-\!\bullet}} = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The characters appearing in the main results below span a subspace of $\mathsf{cf}(\mathrm{UT}_n)$ which was initially constructed in [2]. For $\gamma \in \mathcal{IG}_n$, let

$$\overline{\chi}^\gamma = \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{UT}_n}(\mathbb{1}),$$

the character of the UT_n-module $\mathbb{C}[\mathrm{UT}_n/\mathrm{UT}_\gamma]$, so that

$$\overline{\chi}^\gamma = q^{|E(\gamma)|}\delta_{\mathrm{UT}_\gamma}. \tag{2.7}$$

The characters $\overline{\chi}^\gamma$ are linearly independent, and span a self-dual subspace

$$\mathsf{scf}(\mathrm{UT}_n) = \mathbb{C}\text{-span}\{\overline{\chi}^\gamma \mid \gamma \in \mathcal{IG}_n\},$$

known as the subspace of *superclass functions*.

Remark 2.8. *The space* $\mathsf{scf}(\mathrm{UT}_n)$ *comes from a* supercharacter theory *of* UT_n *[2], and has a number of interesting bases indexed by* $\mathcal{IG}_n$. *One of these—the "supercharacter basis"—is typically denoted by* χ^γ *and has the property that* $\overline{\chi}^\gamma = \sum_{\sigma \subseteq \gamma} \chi^\sigma$, *from which the notation* $\overline{\chi}^\gamma$ *is derived. More details can be found in the extended version of this abstract [6, Section 2.3].*

2.4 Homomorphisms between Hopf algebras of class functions

In [21, III], Zelevinsky defines a graded connected Hopf algebra on the space

$$\mathsf{cf}(\mathrm{GL}_\bullet) = \bigoplus_{n\geq 0} \mathsf{cf}(\mathrm{GL}_n),$$

with structure maps coming from the parabolic induction and restriction functors. The paper [5] defines a similar Hopf structure on the spaces

$$\mathsf{scf}(\mathrm{UT}_\bullet) = \bigoplus_{n\geq 0} \mathsf{scf}(\mathrm{UT}_n), \qquad \text{and} \qquad \mathsf{cf}(\mathrm{UT}_\bullet) = \bigoplus_{n\geq 0} \mathsf{cf}(\mathrm{UT}_n),$$

in which the former is a sub-Hopf algebra of the latter. This section will describe several homomorphisms involving these Hopf algebras.

In [10, Section 6], Guay-Paquet defines a $\mathbb{C}[t]$-Hopf algebra on the free $\mathbb{C}[t]$-module $\mathbb{C}[t][\mathcal{IG}]$, and specializing $t \mapsto q^{-1}$ gives a Hopf algebra over $\mathbb{C}$; see [5, Section 7].

Theorem 2.9 ([5, Corollary 7.6])**.** *The map $\gamma \mapsto q^{-|E(\gamma)|}\overline{\chi}^{\gamma}$ is an isomorphism from Guay-Paquet's specialized Hopf algebra to $\mathsf{scf}(\mathrm{UT}_\bullet)$.*

A second map comes from the induction functors $\mathrm{Ind}_{\mathrm{UT}_n}^{\mathrm{GL}_n} : \mathsf{cf}(\mathrm{UT}_n) \to \mathsf{cf}(\mathrm{GL}_n)$; let

$$\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}} = \bigoplus_{n\geq 0} \mathrm{Ind}_{\mathrm{UT}_n}^{\mathrm{GL}_n} : \mathsf{cf}(\mathrm{UT}_\bullet) \longrightarrow \mathsf{cf}(\mathrm{GL}_\bullet).$$

Theorem 2.10 ([5, Theorem 6.1])**.** *The map $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ is a Hopf algebra homomorphism.*

The image of $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ is a sub-Hopf algebra, the *class functions with unipotent support*:

$$\mathsf{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) = \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\mathsf{cf}(\mathrm{UT}_\bullet)) \subseteq \mathsf{cf}(\mathrm{GL}_\bullet).$$

The space $\mathsf{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet)$ can also be constructed more explicitly, as follows. An element $g \in \mathrm{GL}_n$ is *unipotent* if g is conjugate to an element of UT_n. The conjugacy classes of unipotent elements in GL_n are indexed by $\mathcal{P}_n$: the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ corresponds to the conjugacy class O_λ of the Jordan matrix

$$J_\lambda = J_{\lambda_1} \oplus J_{\lambda_2} \oplus \cdots \oplus J_{\lambda_\ell} \qquad \text{with} \qquad (J_k)_{i,j} = \begin{cases} 1 & \text{if } j \in \{i, i+1\} \\ 0 & \text{otherwise} \end{cases}$$

Thus, writing $\delta_\lambda = \delta_{O_\lambda}$, standard properties of induction imply that

$$\mathsf{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) = \mathbb{C}\text{-span}\{\delta_\lambda \mid \lambda \in \mathcal{P}\}.$$

Zelevinsky [21, 10.13] (see also [14, IV.4.1]) constructs a Hopf algebra isomorphism

$$\begin{aligned} \mathbf{p}_{\{1\}} : \mathsf{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) &\longrightarrow \mathrm{Sym} \\ \delta_\lambda &\longmapsto \widetilde{P_\lambda}(\mathbf{x}; q) \end{aligned} \tag{2.11}$$

where $\widetilde{P_\lambda}$ is the degree shifted Hall–Littlewood polynomial mentioned in Section 2.1.

3 Realizing $X_\gamma(\mathbf{x};t)$ as a GL_n character

This section gives the main result. Recall the definitions of $\mathcal{IG}_n$ and $X_\gamma(\mathbf{x};t)$ from Section 2.2, the groups UT_γ from Section 2.3, and the maps $\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ and $\mathbf{p}_{\{1\}}$ from Section 2.4.

Theorem 3.1. *For $n \geq 0$ and $\gamma \in \mathcal{IG}_n$,*

$$\mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1}) = (q-1)^n\, \mathbf{p}_{\{1\}}^{-1}\big(X_\gamma(\mathbf{x};q)\big).$$

A full proof of Theorem 3.1 is given in [6, Section 3], but we will give a short sketch below, after a few preliminaries. Theorem 2.1 states that the homomorphism

$$\mathrm{cano}_{\mathrm{CQS}} : \mathsf{scf}(\mathrm{UT}_\bullet) \xrightarrow{\mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}} \mathsf{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet) \xhookrightarrow{\mathbf{p}_{\{1\}}} \mathrm{Sym} \xrightarrow{\text{inclusion}} \mathcal{Q}\mathrm{Sym}$$

is uniquely determined by the linear character $\mathrm{ps}_1 \circ \mathbf{p}_{\{1\}} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}$ of $\mathsf{scf}(\mathrm{UT}_\bullet)$. The proof of Theorem 3.1 amounts to showing that this linear character also corresponds to the homomorphism

$$\begin{array}{rcl} \mathsf{scf}(\mathrm{UT}_\bullet) & \longrightarrow & \mathcal{Q}\mathrm{Sym} \\ \overline{\chi}^\gamma & \longmapsto & (q-1)^n X_\gamma(\mathbf{x};q), \end{array} \tag{3.2}$$

where for each $\gamma \in \mathcal{IG}_n$, $\overline{\chi}^\gamma = \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{UT}_n}(\mathbb{1})$ as in Section 2.3.

A relative of the map in Equation (3.2) has been studied by Guay-Paquet in [10]. Translating through the isomorphism of Theorem 2.9 and a $q \leftrightarrow q^{-1}$ interchange property established in [17, Proposition 2.6], we have the following.

Theorem 3.3 ([10, Theorem 57]). *The map $\overline{\chi}^\gamma \mapsto X_\gamma(\mathbf{x};q)$ is a Hopf algebra homomorphism from $\mathsf{scf}(\mathrm{UT}_\bullet)$ to $\mathcal{Q}\mathrm{Sym}$. Writing $\zeta_0 : \mathsf{scf}(\mathrm{UT}_\bullet) \to \mathbb{C}$ for the corresponding linear character,*

$$\zeta_0(\overline{\chi}^\gamma) = \begin{cases} q^{|E(\gamma)|} & \textit{if } \gamma = ([n], \varnothing), \\ 0 & \textit{otherwise.} \end{cases}$$

We now relate ζ_0 to the linear character of $\mathrm{cano}_{\mathrm{CQS}}$. Zelevinsky [21, 10.8] has shown that for a unipotently supported class function $\psi \in \mathsf{cf}_{\mathrm{supp}}^{\mathrm{uni}}(\mathrm{GL}_\bullet)$,

$$\mathrm{ps}_1 \circ \mathbf{p}_{\{1\}}(\psi) = \psi(J_{(n)}),$$

where $J_{(n)}$ is the unipotent Jordan matrix corresponding to the partition (n), as in Section 2.4. Taking $\psi = \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\overline{\chi}^\gamma)$, standard facts about induction and the conjugacy class of $J_{(n)}$ give the following result, proved in [6, Proposition 3.12].

Proposition 3.4. *Let γ be an indifference graph of size $n \geq 0$. Then*

$$\mathrm{ps}_1 \circ \mathbf{p}_{\{1\}} \circ \mathrm{Ind}_{\mathrm{UT}}^{\mathrm{GL}}(\overline{\chi}^\gamma) = \begin{cases} (q-1)^n q^{|E(\gamma)|} & \textit{if } \gamma = ([n], \varnothing), \\ 0 & \textit{otherwise.} \end{cases}$$

Proof sketch of Theorem 3.1. Starting from Theorem 3.3, it can be verified directly that the map in Equation (3.2) is a Hopf algebra homomorphism. By Proposition 3.4, this map has the same linear character as $\mathrm{cano}_{\mathrm{CQS}}$, so by Theorem 2.1, they are the same. □

4 Connections to Hessenberg varieties

This section will describe a relationship between the characters in Theorem 3.1 and Hessenberg varieties. Take $n \geq 0$ and let $\mathbb{K}$ be a field. For each element $A \in \mathrm{Mat}_n(\mathbb{K})$ and subspace $M \subseteq \mathrm{Mat}_n(\mathbb{K})$ which is stable under conjugation by the subgroup of upper triangular matrices $B_n(\mathbb{K}) \subseteq \mathrm{GL}_n(\mathbb{K})$, the *Hessenberg variety* associated to A and M is

$$\mathcal{B}_A^M = \{gB_n(\mathbb{K}) \in \mathrm{GL}_n(\mathbb{K})/B_n(\mathbb{K}) \mid g^{-1}Ag \in M\}.$$

The following results exclusively concern Hessenberg varieties associated to strictly upper triangular subspaces known as *ad-nilpotent ideals*. For $\gamma \in \mathcal{IG}_n$, let

$$\mathfrak{ut}_\gamma(\mathbb{K}) = \{A \in \mathrm{Mat}_n(\mathbb{K}) \mid A_{i,j} \neq 0 \text{ only if } i < j \text{ and } \{i,j\} \notin E(\gamma)\} = \mathrm{UT}_\gamma(\mathbb{K}) - 1_n.$$

Some key examples of Hessenberg varieties of the form $\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{K})}$ are well-known, but a specific study of these varieties is quite recent; see [16] and the references therein.

Proposition 4.1. *Let $n \geq 0$ and $\gamma \in \mathcal{IG}_n$. For $A \in \mathrm{Mat}_n(\mathbb{F}_q)$ with $1_n + A \in \mathrm{GL}_n(\mathbb{F}_q)$,*

$$\mathrm{Ind}_{\mathrm{UT}_\gamma(\mathbb{F}_q)}^{\mathrm{GL}_n(\mathbb{F}_q)}(\mathbb{1})(1_n + A) = (q-1)^n q^{|E(\gamma)|} |\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}|.$$

Proof sketch. The left side is equal to $q^{|E(\gamma)|}$ times the number of cosets $h\mathrm{UT}_n(\mathbb{F}_q) \in \mathrm{GL}_n(\mathbb{F}_q)/\mathrm{UT}_n(\mathbb{F}_q)$ with $h^{-1}(1_n + A)h \in \mathrm{UT}_\gamma(\mathbb{F}_q)$. Each such coset satisfies $hB_n(\mathbb{F}_q) \in \mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}$, with $(q-1)^n$ cosets $h\mathrm{UT}_n(\mathbb{F}_q)$ for each each element of $\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}$. □

The paper [16] gives a similar result for the analogous Hessenberg varieties over $\mathbb{C}$, which involves the modified Poincaré polynomial

$$\mathsf{Poin}_A^{\mathfrak{ut}_\gamma(\mathbb{C})}(t) = \sum_{k \geq 0} \beta_{2k} t^k,$$

where β_i is the ith Betti number of $\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{C})}$. Recall the Hall–Littlewood symmetric function $\widetilde{P_\lambda}(\mathbf{x};t)$ from Section 2.1, and define expressions $d_\lambda^\gamma(t)$ for each partition λ of n by

$$X_\gamma(\mathbf{x};t) = \sum_{\lambda \in \mathcal{P}_n} d_\lambda^\gamma(t) \widetilde{P_\lambda}(\mathbf{x};t). \tag{4.2}$$

Also note that partitions of n index the similarity classes of nilpotent matrices over any field: $\lambda \in \mathcal{P}_n$ corresponds to the class of $J_\lambda - 1_n$, where J_λ is as defined in Section 2.4.

Theorem 4.3 ([16, Equation (4.7)]). *For $n \geq 0$, take $\gamma \in \mathcal{IG}_n$ and $\lambda \in \mathcal{P}_n$. For a nilpotent matrix $A \in \mathrm{Mat}_n(\mathbb{C})$ in the similarity class indexed by λ, $\mathsf{Poin}_A^{\mathfrak{ut}_\gamma(\mathbb{C})}(t) = t^{-|E(\gamma)|} d_\lambda^\gamma(t)$.*

Applying the inverse of the map $\mathbf{p}_{\{1\}}$ defined in Equation (2.11) to Equation (4.2), Theorem 3.1 and Proposition 4.1 give the following.

Corollary 4.4. *For $n \geq 0$, take $\gamma \in \mathcal{IG}_n$ and $\lambda \in \mathcal{P}_n$. Let $A \in \mathrm{Mat}_n(\mathbb{F}_q)$ and $A' \in \mathrm{Mat}_n(\mathbb{C})$ be nilpotent elements belonging to similarity classes indexed by λ. Then $\mathsf{Poin}_{A'}^{\mathfrak{ut}_\gamma(\mathbb{C})}(q) = |\mathcal{B}_A^{\mathfrak{ut}_\gamma(\mathbb{F}_q)}|$.*

5 Realizing $G_\gamma(\mathbf{x};t)$ as a GL_n character

For an indifference graph $\gamma \in \mathcal{IG}$, the *unicellular LLT polynomial* associated to γ is

$$G_\gamma(\mathbf{x};t) = \sum_{\kappa:[n]\to\mathbb{Z}_{>0}} t^{\mathrm{asc}_\gamma(\kappa)} x_{\kappa(1)} x_{\kappa(2)} \dots x_{\kappa(n)} \in \mathbb{C}[[\mathbf{x}]][t],$$

where the sum is over *all* colorings. This is a variation of the polynomials defined by Lascoux, Leclerc, and Thibon in [13], and is due to Carlsson and Mellit [4] (see also [12]). Like $X_\gamma(\mathbf{x};t)$, each $G_\gamma(\mathbf{x};t)$ is a polynomial in t with symmetric function coefficients.

Theorem 5.1 ([4, Proposition 3.5]). *For $n \geq 0$, let $\gamma \in \mathcal{IG}_n$. Using plethystic notation,*

$$(t-1)^n X_\gamma(\mathbf{x};t)\left[\tfrac{\mathbf{x}}{t-1}\right] = G_\gamma(\mathbf{x};t).$$

Somewhat surprisingly, the plethystic substitution above has a natural meaning for $\mathrm{GL}_n(\mathbb{F}_q)$. An irreducible character of GL_n is *unipotent* if it is a summand of $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(\mathbb{1})$, where $B_n = B_n(\mathbb{F}_q)$ is as defined in Section 4. These characters are indexed by the partitions of n, and χ^λ will denote the irreducible unipotent character corresponding to $\lambda \in \mathcal{P}(n)$, with $\chi^{(1^n)} = \mathbb{1}$ as in [14]. Zelevinksy [21, 9.4] defines a homomorphism

$$\begin{array}{rccc} \mathbf{p}_{\mathbb{1}}: & \mathrm{cf}(\mathrm{GL}_\bullet) & \longrightarrow & \mathrm{Sym} \\ & \psi & \longmapsto & \sum_\lambda \langle \psi, \chi^\lambda \rangle s_\lambda, \end{array} \tag{5.2}$$

so that for any character ψ of GL_n, the coefficient of s_λ in $\mathbf{p}_{\mathbb{1}}(\psi)$ is the multiplicity of χ^λ in ψ. Recalling the map $\mathbf{p}_{\{1\}}$ from Section 2.4, it is known [14, IV.4] that the composite

$$\mathrm{Sym} \xrightarrow{\mathbf{p}_{\{1\}}^{-1}} \mathrm{cf}^{\mathrm{uni}}_{\mathrm{supp}}(\mathrm{GL}_\bullet) \hookrightarrow \mathrm{cf}(\mathrm{GL}_\bullet) \xrightarrow{\mathbf{p}_{\mathbb{1}}} \mathrm{Sym}$$

can be expressed in plethystic notation as $f \mapsto \omega f[\frac{\mathbf{x}}{t-1}]|_{t=q}$. With Theorem 5.1, this implies the following result; see [6, Theorem 5.1] for a complete proof.

Theorem 5.3. *Let γ be an indifference graph. Then $\mathbf{p}_{\mathbb{1}} \circ \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1}) = \omega G_\gamma(\mathbf{x};q)$.*

6 Positivity conjectures

Recall the bases of Sym given in Section 2.1. An element $f \in \mathrm{Sym}[t]$ is respectively *e-positive* or *s-positive* if there are polynomials $a_\lambda(t) \in \mathbb{Z}_{\geq 0}[t]$ or $b_\lambda(t) \in \mathbb{Z}_{\geq 0}[t]$ for which

$$f = \sum_{\lambda \in \mathcal{P}} a_\lambda(t) e_\lambda \qquad \text{or} \qquad f = \sum_{\lambda \in \mathcal{P}} b_\lambda(t) s_\lambda.$$

For the chromatic quasisymmetric functions in Section 2.2, e-positivity generalizes the Stanley–Stembridge conjecture [20, Conjecture 5.5], which by [9] is the $t = 1$ case of the statement below.

Conjecture 6.1 ([17, Conjecture 1.3])**.** *For each $\gamma \in \mathcal{IG}$, $X_\gamma(\mathbf{x}; t)$ is e-positive.*

In light of Theorem 3.1, there should be a restatement of Conjecture 6.1 in terms of GL_n. However, the map $\mathbf{p}_{\{1\}}$ defined in Section 2.4 does not associate e_λ to a character of GL_n, so some interpretation is required. One possible restatement uses the Steinberg character $\chi^{(n)}$ of GL_n defined in Section 5, which agrees with $\mathbf{p}_{\{1\}}^{-1}(e_n)$ on all unipotent elements. For $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{P}_n$, let St_λ denote the product $\chi^{\lambda_1} \cdots \chi^{\lambda_\ell}$ in $\mathrm{cf}(\mathrm{GL}_\bullet)$.

Conjecture 6.2. *Let $n \geq 0$ and $\gamma \in \mathcal{IG}_n$. There are polynomials $a_\lambda^\gamma(t) \in \mathbb{Z}_{\geq 0}[t]$ such that for each prime power q the character $\eta_\gamma = \sum_{\lambda \in \mathcal{P}_n} a_\lambda^\gamma(q)\,\mathrm{St}_\lambda$ satisfies $(q-1)^n \eta_\gamma(u) = \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})(u)$ for every unipotent element $u \in \mathrm{GL}_n(\mathbb{F}_q)$.*

By [6, Proposition 6.6], Conjectures 6.1 and 6.2 are equivalent. Ideally, a proof of Conjecture 6.2 would construct a module affording the character η_γ.

For the unicellular LLT polynomials from Section 5, Schur positivity has implications in the study of Macdonald polynomials [12] and is known to hold by [8, Corollary 6.9]. Unfortunately, the proof in [8] does not explicitly construct the Schur coefficients.

Open Problem 6.3 ([11, Open Problem 6.6])**.** *Find a (manifestly positive) combinatorial formula for the Schur coefficients $b_\lambda^\gamma(t)$ of $G_\gamma(\mathbf{x}; t)$.*

Theorem 5.3 implies that $b_\lambda^\gamma(q) = \langle \chi^{\lambda'}, \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})\rangle$ for each prime power q, which is a positive integer, but this not yet imply Schur positivity as defined above.

Open Problem 6.4. *Find a combinatorial formula for $\langle \chi^{\lambda'}, \mathrm{Ind}_{\mathrm{UT}_\gamma}^{\mathrm{GL}_n}(\mathbb{1})\rangle$ as a function of q.*

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