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A pipe dream perspective on totally symmetric self-complementary plane partitions

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Abstract. We characterize totally symmetric self-complementary plane partitions (TSSCPP) as bounded compatible sequences satisfying a Yamanouchi-like condition. As such, they are in bijection with certain pipe dreams. Using this characterization and the recent bijection of [Gao-Huang] between reduced pipe dreams and reduced bumpless pipe dreams, we give a bijection between alternating sign matrices and TSSCPP in the reduced, 1432-avoiding case.

Keywords: Alternating sign matrices, plane partitions, pipe dreams

1 Introduction

Plane partitions are three-dimensional analogues of ordinary partitions. Just as partitions in an $a \times b$ are counted by a lovely formula $\binom{a+b}{a}$, plane partitions in an $a \times b \times c$ box are enumerated by MacMahon's product formula $\prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$ [11]. In a 1986 [13], Stanley considered symmetry operations on plane partitions, namely, reflection (transpose), rotation, and complementation. This yielded 10 symmetry classes of plane partitions consisting of plane partitions invariant under combinations of these operations. The plane partitions invariant under all three operations are called *totally symmetric self-complementary* (*TSSCPP*). As in the case of all plane partitions, each symmetry class has a nice enumeration. TSSCPP inside a $2n \times 2n \times 2n$ box was shown in 1994 by Andrews [1] to be counted by $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. This was, at the time, the conjectured [12] number of $n \times n$ *alternating sign matrices* (*ASM*). The 1996 proofs of this conjecture [16, 9] sparked a search for a natural, explicit bijection between TSSCPP and ASM. Partial bijections have been found on small subsets, including the permutation case [14], the case of two monotone triangle diagonals [6, 4], and the 312-avoiding case [2]. This paper interprets TSSCPP

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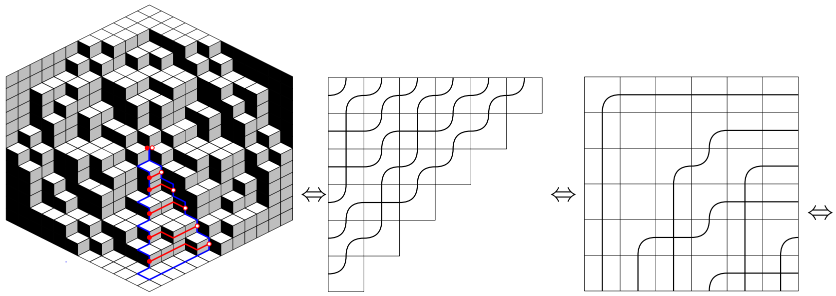


$$\Leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 1: An example of the bijection of this paper. From left to right the objects are: TSSCPP, pipe dream, bumpless pipe dream, ASM. The pipe dream and bumpless pipe dream both have permutation 135264, which avoids 1432.

as *pipe dreams* to extend the bijection of [14] to what appears to be a larger subset than any previous partial bijection. Our main theorem is below; see Figure 1 for an example and Section 2 for definitions. Given $\pi \in S_n$, let $TSSCPP^{red}(\pi)$ denote the set of TSSCPP whose associated pipe dream is reduced and has permutation π, and let $ASM^{red}(\pi)$ denote the set of ASM whose associated *bumpless pipe dream* is reduced and has permutation π. Notice that the reducedness property on TSSCPPs is definitionally inherited from this property on pipe dreams.

Theorem 1.1. *Let $\pi \in S_n$. There is an explicit weight-preserving injection φ from $TSSCPP^{red}(\pi)$ to $ASM^{red}(\pi)$. If π avoids* 1432*, then φ is a bijection.*

The paper is organized as follows. Section 2 contains background on the relevant objects, including the permutation case TSSCPP bijection of [14] and the bijection of [8] between reduced pipe dreams and reduced bumpless pipe dreams, which are important ingredients in our proof. Section 3 proves Theorem 3.3 characterizing TSSCPP as pipe dreams subject to a Yamanouchi-like condition. Section 4 concerns our main result and its proof. This is an extended abstract only; the full version is posted on the arXiv.

2 Background

In this section, we review relevant definitions and bijections from the literature. Subsections 2.1, 2.2, 2.4, and 2.5 review definitions of ASM, TSSCPP, bumpless pipe dreams, and pipe dreams, respectively. Subsections 2.3 and 2.6 contain less-familiar bijections that are important for our main results.

2.1 Alternating sign matrices

In this subsection, we define alternating sign matrices (see e.g. [12]).

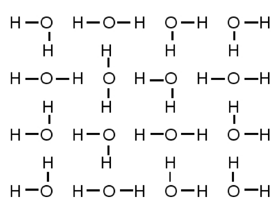
Definition 2.1. An **alternating sign matrix** (**ASM**) is a square matrix with entries in $\{0, 1, -1\}$ such that the rows and columns each sum to 1 and the nonzero entries alternate in sign across each row and across each column.

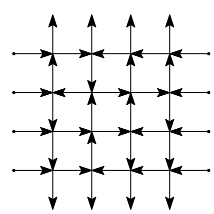
Alternating sign matrices are in bijection with configurations of the *six-vertex / square ice model* of statistical physics with *domain wall boundary conditions*; this was an essential element of the enumeration proof of [9]. The 3×3 alternating sign matrices are below.

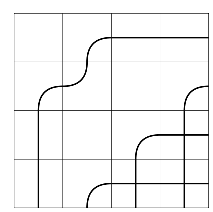
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Below, left and center-left are an alternating sign matrix and its corresponding square ice configuration; the horizontal molecules correspond to $+1$, the vertical molecules correspond to -1, and all other molecules correspond to 0. Center-right is its six-vertex configuration, where the six molecule configurations are replaced by directed edges. Below, right is the corresponding *bumpless pipe dream*, which will be discussed shortly.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$







2.2 Totally symmetric self-complementary plane partitions

In this subsection, we define plane partitions and their symmetry classes (see e.g. [13]).

Definition 2.2. A **plane partition** t is a rectangular array of nonnegative integers $(t_{i,j})_{i,j\geq 1}$ such that $t_{i,j} \geq t_{i',j'}$ if $i \leq i'$, $j \leq j'$. We say t is in an $a \times b \times c$ **bounding box** if $t_{i,j} = 0$ whenever $i > a$ or $j > b$ and $t_{i,j} \leq c$ for all i, j. Let $PP(a \times b \times c)$ denote the set of plane partitions in an $a \times b \times c$ bounding box.

Remark 2.3. We can also view $t \in PP(a \times b \times c)$ as a finite set of positive integer lattice points (i, j, k) with $1 \leq i \leq a$, $1 \leq j \leq b$, and $1 \leq k \leq c$ such that if $(i, j, k) \in t$ and $1 \leq i' \leq i$, $1 \leq j' \leq j$, $1 \leq k' \leq k$ then $(i', j', k') \in t$. This well-known bijection is given as $(i, j, k) \in t$ if and only if $t_{i,j} \geq k$. We will use both of these characterizations in the next definition.

Definition 2.4. A plane partition t is **symmetric** if $t_{i,j} = t_{j,i}$ for all i, j. t is **cyclically symmetric** if whenever $(i, j, k) \in t$ then $(j, k, i) \in t$ as well. t is **totally symmetric** if it is both symmetric and cyclically symmetric, so that whenever $(i, j, k) \in t$ then all six permutations of (i, j, k) are also in t. The **complement** t^C of t inside a given bounding box $a \times b \times c$ is defined as $t^C_{i,j} = c - t_{a-i+1,b-j+1}$ for all $1 \leq i \leq a, 1 \leq j \leq b$. That is, $t^C_{i,j}$ equals the number of empty cubes above $t_{a-i+1,b-j+1}$ in the bounding box. A plane partition t is **self-complementary** inside a given bounding box if $t = t^C$. A **totally symmetric self-complementary plane partition (TSSCPP)** is a plane partition which is both totally symmetric and self-complementary.

Note that for there to exist a self-complementary plane partition in an $a \times b \times c$ bounding box, the volume abc of the box must be an even number. In addition, cyclic symmetry requires $a = b = c$, therefore, we need $a = b = c = 2n$ for there to exist a TSSCPP inside an $a \times b \times c$ bounding box.

Definition 2.5. Let TSSCPP(n) denote the set of TSSCPP inside a $2n \times 2n \times 2n$ box.

2.3 TSSCPP boolean triangles and the permutation case bijection

In this subsection, we review the characterization from [14] of TSSCPP as boolean triangles and the bijection of the same paper between permutation matrices and TSSCPP boolean triangles whose entries weakly decrease along rows.

Definition 2.6 (Def 2.12 of [14]). A **TSSCPP boolean triangle** of order n is a triangular integer array $b = \{b_{i,j}\}$ for $1 \leq i \leq n-1$, $n - i \leq j \leq n-1$ with entries in $\{0, 1\}$ such that the diagonal partial sums satisfy the following inequality for all $1 \leq j < i \leq n-1$:

$$1 + \sum_{k=j+1}^{i} b_{k,n-j-1} \geq \sum_{k=j}^{i} b_{k,n-j}. \tag{2.1}$$

Call this the (i, j)-inequality, in which $n - j$ and $n - j - 1$ are the diagonals being compared and i indicates the row index of where the sums stop.

We give below the indexing of a generic TSSCPP boolean triangle.

$$\begin{array}{ccccccccc}
 & & & & b_{1,n-1} & & & & \\
 & & & b_{2,n-2} & & b_{2,n-1} & & & \\
 & & b_{3,n-3} & & b_{3,n-2} & & b_{3,n-1} & & \\
 & & & & \vdots & & & & \\
b_{n-1,1} & & & b_{n-1,2} & \cdots & b_{n-1,n-2} & & & b_{n-1,n-1}
\end{array}$$

Below are a non-example and an example of a TSSCPP boolean triangle.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 0 & & 0 & & \\
 & 1 & & 0 & & 0 & & 1 & \\
0 & & 0 & & 0 & & 1 & & 0
\end{array}
\qquad\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 0 & & 0 & & \\
 & 1 & & 0 & & 1 & & 1 & \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

In the left triangle, the $(4, 1)$-inequality is not satisfied, since $\sum_{k=1}^{4} b_{k,n-1} = 3$ while $\sum_{k=2}^{4} b_{k,n-2} = 1$. In the triangle on the right, all (i, j)-inequalities are satisfied.

Proposition 2.7 (Prop 2.13 of [14])**.** *TSSCPP boolean triangles of order n are in bijection with* $TSSCPP(n)$*.*

The bijection proceeds by taking the fundamental domain of the TSSCPP, transforming it into a nest of non-intersecting lattice paths, and then recording the two different types of steps in each path as 0 and 1. The diagonal partial sum condition (2.1) is equivalent to the requirement that the paths do not intersect. See [14] for details.

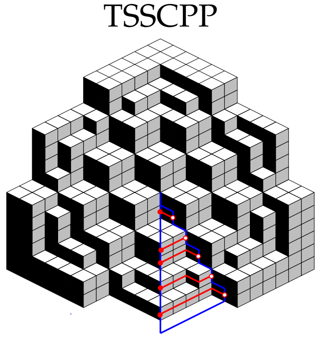
We now review the characterization of a certain subset of TSSCPP boolean triangles.

Definition 2.8 (Def 3.1 of [14])**.** A **permutation TSSCPP boolean triangle** is a TSSCPP boolean triangle with weakly decreasing rows.

That is, the entries equal to one in a permutation TSSCPP boolean triangle are all left-justified. The terminology 'permutation' in the above definition is justified by the statistic-preserving bijection in the theorem below.

Theorem 2.9 (Theorem 3.5 of [14])**.** *There is a natural, statistic-preserving bijection* Φ *between* $n \times n$ *permutation matrices with inversion number p and permutation TSSCPP boolean triangles of order n with p zeros.*

TSSCPP



$\leftrightarrow$

Permutation TSSCPP Boolean triangle

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

$\leftrightarrow$

Permutation Matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The injection φ in Theorem 1.1 extends this bijection under the mild transformation of flipping the resulting matrix vertically (or reversing the one-line notation of the permutation). Thus, in Theorem 1.1 we instead map the TSSCPP above, to the vertical reflection of the matrix shown above.

2.4 Bumpless pipe dreams

In this subsection, we define bumpless pipe dreams and describe the bijection with alternating sign matrices.

Definition 2.10. A **bumpless pipe dream** [10] of size n is a tiling of an $n \times n$ grid of squares by the following six types of tiles: , , , , , , such that n pipes traveling from the south border to the east border are formed. We denote the set of bumpless pipe dreams of size n as $\mathrm{BPD}(n)$. We say a bumpless pipe dream is **reduced** if no two pipes cross twice. We associate a permutation to each reduced bumpless pipe dream by labeling the pipes $1, \ldots, n$ from left to right on the south border and read off the pipe labels from top to bottom on the east border. Let $\mathrm{BPD}^{red}(\pi)$ denote the set of all reduced bumpless pipe dreams with permutation π.

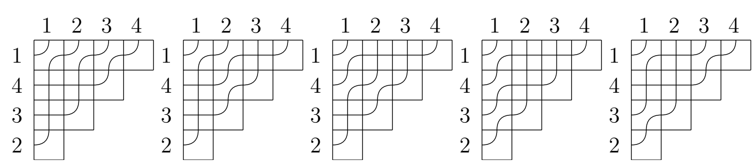
There is a natural bijection between $\mathrm{BPD}(n)$ and $\mathrm{ASM}(n)$, as described in [15]. To obtain an ASM from a BPD we simply replace each with a 1, each with a -1, and the rest with 0s. For the inverse map it is not difficult to see the positions of and uniquely determine a bumpless pipe dream. See p.3 for an example.

2.5 Pipe dreams

In this subsection, we define pipe dreams and bounded compatible sequences and describe the bijection between them.

Definition 2.11. A **pipe dream** [3] of size n is a tiling of a square $n \times n$ grid of squares with two kinds of tiles, the cross-tile and elbow-tile , such that the positions on or below the main (anti)diagonal only consist of elbow-tiles. We think of a pipe dream as n pipes, labelled $1, \ldots, n$ traveling from the north border and exiting from the west border. We denote the set of pipe dreams of size n as $\mathrm{PD}(n)$. We say a pipe dream is **reduced** if no two pipes cross twice. We associate a permutation to each reduced pipe dream by reading from top to bottom the labels of each pipe on the west border of the pipe dream. Let $\mathrm{PD}^{red}(\pi)$ denote the set of reduced pipe dreams with permutation π.

The set of pipe dreams for a fixed permutations are connected by *chute* and *ladder* moves. For precise definitions see [3]. When a ladder (or chute) move is bounded by a 2×2 square, we call this move a **simple slide**, as shown below, left.



Above, right shows the set of pipe dreams $\mathrm{PD}^{red}(1432)$. The first four pipe dreams are connected by simple slides; the fifth is not.

Definition 2.12. A **bounded compatible sequence** [5] is a pair $(\mathbf{a}, \mathbf{r})$ where $\mathbf{a} = (a_1, \ldots, a_\ell)$ and $\mathbf{r} = (r_1, \ldots, r_\ell)$ are words of positive integers, satisfying the following conditions:

(a) $r_1 \geq r_2 \geq \cdots \geq r_\ell$,
(b) $a_i \geq r_i$ for all $1 \leq i \leq \ell$,
(c) $r_i > r_{i+1}$ if $a_i \geq a_{i+1}$.

There is a simple bijection between $\mathrm{PD}(n)$ and the set of all bounded compatible sequences where $a_i < n$ for each i, see [3]. Given a bounded compatible sequence $(\mathbf{a}, \mathbf{r})$, we may construct a pipe dream by putting a cross-tile at position $(r_i, a_i + 1 - r_i)$ for each $1 \leq i \leq \ell$ and fill the remaining positions with elbow-tiles. Conversely, given a pipe dream, we may construct a bounded compatible sequence as follows: scan the pipe dream from bottom to top and within each row left to right, and whenever we encounter a cross-tile at position (r, c) we append $(r + c - 1, r)$ to the compatible sequence. For example, the corresponding bounded compatible sequences for the pipe dreams in $\mathrm{PD}^{red}(1432)$ on p.6 are as follows; the vector $\mathbf{a}$ is recorded in the top row and $\mathbf{r}$ in the bottom row.

$$\begin{pmatrix} 3&2&3 \\ 3&2&2 \end{pmatrix}, \begin{pmatrix} 3&2&3 \\ 3&2&1 \end{pmatrix}, \begin{pmatrix} 3&2&3 \\ 3&1&1 \end{pmatrix}, \begin{pmatrix} 3&2&3 \\ 2&1&1 \end{pmatrix}, \begin{pmatrix} 2&3&2 \\ 2&2&1 \end{pmatrix}.$$

2.6 Reduced BPD-PD bijection

Both reduced pipe dreams and reduced bumpless pipe dreams give combinatorial formulas for Schubert polynomials $\mathfrak{S}_\pi$, $\pi \in S_\infty$ which are important polynomials in the study of Schubert calculus [3, 10]. Explicitly,

$$\mathfrak{S}_\pi = \sum_{D \in \mathrm{PD}^{red}(\pi)} \prod_{(r,c) \in \mathrm{cross}(D)} x_r = \sum_{D \in \mathrm{BPD}^{red}(\pi)} \prod_{(r,c) \in \mathrm{blank}(D)} x_r.$$

For this reason, there exists a weight-preserving bijection between $\mathrm{PD}^{red}(\pi)$ and $\mathrm{BPD}^{red}(\pi)$, where the weight of a PD or BPD is its monomial contribution to the Schubert polynomial indexed by its permutation.

In [8], such an explicit direct bijection $\varphi : \mathrm{BPD}^{red}(\pi) \to \mathrm{PD}^{red}(\pi)$ is given using an iterative algorithm. To find the image of a BPD under φ, the algorithm computes in each iteration the position of one crossing in the corresponding PD. For a detailed description of this process, see [8, Definition 3.1]. For explicit examples, see [8, Example 3.4]. This bijection is weight-preserving; in particular, for $D \in \mathrm{BPD}^{red}(\pi)$, the number of blank tiles in row k equals the number of cross-tiles in row k of $\varphi(D)$.

Because the bijection is weight-preserving and there is a unique lowest weight monomial that corresponds to the Lehmer code of the permutation in each Schubert polynomial, the permutation BPD is mapped to the **bottom pipe dream**, the unique pipe dream with all crosses left-justified.

3 Characterizing TSSCPP as pseudo-Yamanouchi pipe dreams

This section focuses on our first theorem: a characterization of TSSCPP as a subset of all (reduced and non-reduced) pipe dreams.

3.1 Mapping TSSCPP into pipe dreams

Recall the bijection of Proposition 2.7 from TSSCPP to the TSSCPP boolean triangles of Definition 2.6. As TSSCPP boolean triangles are triangular arrays with entries in $\{0, 1\}$, we can transform them to pipe dreams (reduced and non-reduced), since these are triangular arrays of tiles with two choices for each spot. There are several possibilities for how to do this; we choose to correspond each 1 to a cross-tile ⊞ and each 0 to an elbow-tile ⧄. There are also several choices for orientation of the triangle. We set the following convention.

Given a TSSCPP boolean triangle b of order n, create a triangular array $y_{i,j}$, $1 \leq i \leq n-1$, $1 \leq j \leq n-i$ of zeros and ones where $y_{i,j} = b_{n-i,i+j-1}$. That is, rotate b counterclockwise and justify to the corner.

$$
\begin{array}{lllll}
b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \cdots & b_{n-1,n-1} \\
b_{n-2,2} & b_{n-2,3} & \cdots & b_{n-2,n-1} & \\
b_{n-3,3} & \cdots & b_{n-3,n-1} & & \\
 & \iddots & & & \\
b_{1,n-1} & & & &
\end{array}
\qquad
\begin{array}{lllll}
y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,n-1} \\
y_{2,1} & y_{2,2} & \cdots & y_{2,n-2} & \\
y_{3,1} & \cdots & y_{3,n-3} & & \\
 & \iddots & & & \\
y_{n-1,1} & & & &
\end{array}
$$

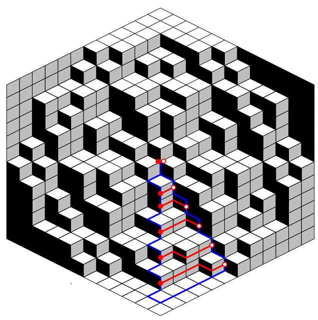
The inequality of Definition 2.6 translates to the following:

$$1 + \sum_{k=1}^{i} y_{j-k,k} \geq \sum_{k=1}^{i+1} y_{j-k+1,k} \quad \text{for all } 1 \leq i < j \leq n-1.$$

Now we turn each 1 into a cross-tile ⊞ and each 0 into an elbow-tile ⧄. We call the pipe dreams that lie in this image the TSSCPP pipe dreams. See Figure 2 for an example. Note that permutation TSSCPP boolean triangles have weakly decreasing rows; this corresponds to left-justified crosses in the associated pipe dream.

3.2 A Yamanouchi-like condition on bounded compatible sequences

In this subsection, we prove Theorem 3.3 characterizing TSSCPP pipe dreams. We also prove Lemma 3.2, which will be used in Section 4.



$\leftrightarrow$

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 1 & & & \\ & & 1 & & 0 & & 0 & & \\ & 1 & & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{array} \quad \leftrightarrow \quad \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & & \\ 0 & 1 & & & \\ 0 & & & & \end{array}$$

$\leftrightarrow$

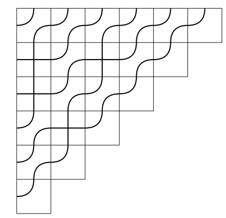


Figure 2: An example of transforming a TSSCPP to a pipe dream.

Definition 3.1. Given a bounded compatible sequence $(\mathbf{a}, \mathbf{r}) = ((a_1, \dots, a_\ell), (r_1, \dots, r_\ell))$, define $\text{count}(k, j)(\mathbf{a})$ (or $\text{count}(k, j)$ when $\mathbf{a}$ is understood) to be the number of j that appear in $a_1, \dots, a_k$. We say that $(\mathbf{a}, \mathbf{r})$ is **pseudo-Yamanouchi** if for all $1 \leq k \leq \ell$, $1 \leq j \leq n-2$, $1 + \text{count}(k, j) \geq \text{count}(k, j+1)$. We also say that a pipe dream is pseudo-Yamanouchi if its corresponding bounded compatible sequence is so.

Lemma 3.2. *For any π, the bottom pipe dream is pseudo-Yamanouchi.*

Proof. The bottom pipe dream is the unique pipe dream of π with all left-justified cross-tiles. Thus the bounded compatible sequence $(\mathbf{a}, \mathbf{r})$ is either empty or $\mathbf{a}$ is made up of increasing runs such that it can be written for some $m \geq 1$ as

$$\mathbf{a} = (j_1, j_1 + 1, \dots, j_1^* - 1, j_1^*, j_2, j_2 + 1, \dots, j_2^* - 1, j_2^*, \dots, j_m, j_m + 1, \dots, j_m^* - 1, j_m^*)$$

where $j_1 > j_2 > \cdots > j_m$ and $j_i^* \geq j_i$ for all $1 \leq i \leq m$. Because $j_1 > j_2 > \cdots > j_m$, each increasing run needs to start with a smaller number than the previous.

Suppose $(\mathbf{a}, \mathbf{r})$ is not pseudo-Yamanouchi. Choose the smallest k such that there exists a j for which $1 + \text{count}(k, j) < \text{count}(k, j+1)$. Since $\text{count}(k, j)$ is a non-negative increasing function of k, it must be that $1 + \text{count}(k-1, j) = \text{count}(k-1, j+1)$ and $a_k = j+1$, since we chose k to be the smallest value with the property. If we are in row $j+1$ ($r_k = j+1$), then this is the first time that $j+1$ has appeared in $\mathbf{a}$, so $\text{count}(k, j+1) = 1$ and thus cannot be greater than $1 + \text{count}(k, j)$. If $r_k > j+1$, then $a_{k-1} = j$, since the cross-tiles are left-justified, and thus $\text{count}(k-1, j) + 1 = \text{count}(k, j)$. But $1 + \text{count}(k-1, j) \geq \text{count}(k-1, j+1) = \text{count}(k, j+1) - 1$. So finally, $\text{count}(k, j) \geq \text{count}(k, j+1) - 1$, which is a contradiction. $\square$

Theorem 3.3. $\text{TSSCPP}(n)$ *is in bijection with the set of pseudo-Yamanouchi pipe dreams in* $\text{PD}(n)$.

Proof. We identify a TSSCPP with the 0-1 triangular array $(y_{i,j})_{1 \leq i \leq n-1, 1 \leq j \leq n-i}$ satisfying the inequalities $1 + \sum_{k=1}^{i} y_{j-k,k} \geq \sum_{k=1}^{i+1} y_{j-k+1,k}$ for all $1 \leq i < j \leq n-1$, as described in

Section 3.1. These inequalities mean the following: for any position (i,j) in the corresponding pipe dream, the number of crosses in the same diagonal as (i,j) at or below row i can be at most one more than the number of crosses in the previous diagonal at or below row i. Therefore it suffices to check this property when (i,j) is a cross to decide whether the pipe dream is TSSCPP.

Now suppose (a,r) is an entry of a pseudo-Yamanouchi compatible sequence. Then by the definition of the reading order, all crosses that appear at or below row r in the $(a-1)$st diagonal of the corresponding pipe dream appear before (a,r) in the compatible sequence. Therefore the inequality for the $(r, a-r+1)$ position is implied by the pseudo-Yamanouchi property. The converse is true by a similar argument. □

4 Main result

In this section, we prove our main result, Theorem 1.1. The proof uses the following theorem and lemmas, the first of which is due to Yibo Gao. We will need the following terminology. A permutation π **avoids** a permutation π' if there is no subsequence of π having the same relative order as π'.

Theorem 4.1 ([7, Theorem 4.1]). *If $\pi \in S_n$ avoids 1432, then any two reduced pipe dreams of π are connected by simple slides.*

Lemma 4.2. *Suppose $D \in \mathrm{PD}(n)$ is pseudo-Yamanouchi and $D' \in \mathrm{PD}(n)$ is related to D by a simple slide. Then D' is pseudo-Yamanouchi.*

Proof. Suppose $D \in \mathrm{PD}(n)$ is pseudo-Yamanouchi. Let $(\mathbf{a}, \mathbf{r}) = ((a_1, \ldots, a_\ell), (r_1, \ldots, r_\ell))$ be its associated bounded compatible sequence. Suppose for some $1 < i < n$, D has a $\boxplus$ tile at position $(r_i, a_i + 1 - r_i)$ and no $\boxplus$ tiles at positions $(r_i, a_i + 2 - r_i)$, $(r_{i-1}, a_i - r_i)$, or $(r_{i-1}, a_i + 1 - r_i)$. Then a simple slide may be applied to D, resulting in another pipe dream D' with $\boxplus$ tile at position $(r_{i-1}, a_i + 1 - r_i)$ and no $\boxplus$ tiles at positions $(r_i, a_i + 1 - r_i)$, $(r_i, a_i + 2 - r_i)$, or $(r_{i-1}, a_i - r_i)$. That is, the simple slide moves the $\boxplus$ tile up one unit and to the right one unit and there were no other $\boxplus$ tiles in these intermediate squares. This preserves the diagonal but decrements the row index, creating a new bounded compatible sequence $(\mathbf{a}', \mathbf{r}') = ((a'_1, \ldots, a'_\ell), (r'_1, \ldots, r'_\ell))$ such that

$$a'_k = \begin{cases} a_k & k < i \\ a_{k+1} & i \le k < m \\ a_i & k = m \\ a_k & m < k \le \ell \end{cases} \tag{4.1}$$

where $m > i$ is uniquely chosen so that $(\mathbf{a}', \mathbf{r}')$ satisfies the conditions of a bounded compatible sequence. Let $\tilde{j} := a_i$. So $\mathbf{a}$ and $\mathbf{a}'$ differ only in that $\tilde{j}$ has slid to the right from index i to m.

Recall $\mathrm{count}(k,j)(\mathbf{a})$ denotes the number of j that appear in $a_1,\ldots,a_k$. By Theorem 3.3, $(\mathbf{a},\mathbf{r})$ is pseudo-Yamanouchi, so for all $1 \le k \le \ell$, $1 \le j \le n-2$, $1 + \mathrm{count}(k,j)(\mathbf{a}) \ge \mathrm{count}(k,j+1)(\mathbf{a})$. We need only check that $(\mathbf{a}',\mathbf{r}')$ is also pseudo-Yamanouchi. The only values of j we need to consider are $\tilde{j}-1$, $\tilde{j}$, and $\tilde{j}+1$.

By (4.1), $\mathrm{count}(k,j)(\mathbf{a}') = \mathrm{count}(k,j)(\mathbf{a})$ for all values of j when $k < i$ or $k \ge m$. Thus we need only check the pseudo-Yamanouchi inequality for k in the range $i \le k < m$.

Suppose $i \le k < m$. Since $(\mathbf{a},\mathbf{r})$ and $(\mathbf{a}',\mathbf{r}')$ are related by a simple slide, we know there is no cross in D at position $(r_{i-1}, a_i - r_i)$. That is, $a_{m-1} \ne \tilde{j}-1$. Furthermore, $a_k \ne \tilde{j}-1$ for all $i \le k < m$. Thus $\mathrm{count}(k,\tilde{j}-1)(\mathbf{a}') = \mathrm{count}(k,\tilde{j}-1)(\mathbf{a})$ for all $i \le k < m$ while $\mathrm{count}(k,\tilde{j})(\mathbf{a}') = \mathrm{count}(k,\tilde{j})(\mathbf{a}) - 1$ in this same range. So

$$\mathrm{count}(k,\tilde{j}-1)(\mathbf{a}') = \mathrm{count}(k,\tilde{j}-1)(\mathbf{a}) \ge \mathrm{count}(k,\tilde{j})(\mathbf{a}) - 1 = \mathrm{count}(k,\tilde{j})(\mathbf{a}').$$

So the pseudo-Yamanouchi condition is more than satisfied when comparing diagonals $\tilde{j}-1$ and $\tilde{j}$.

Since $(\mathbf{a},\mathbf{r})$ and $(\mathbf{a}',\mathbf{r}')$ are related by a simple slide, we also know there is no cross in D at position $(r_i, a_i + 2 - r_i)$. That is, $a_{i+1} \ne \tilde{j}+1$. Furthermore, $a_k \ne \tilde{j}+1$ for all $i \le k < m$. Thus $\mathrm{count}(k,\tilde{j}+1)(\mathbf{a}') = \mathrm{count}(k,\tilde{j}+1)(\mathbf{a})$ for all $i \le k < m$ while $\mathrm{count}(k,\tilde{j})(\mathbf{a}') = \mathrm{count}(k,\tilde{j})(\mathbf{a}) - 1$ in this same range. So

$$1 + \mathrm{count}(k,\tilde{j})(\mathbf{a}') = 1 + \mathrm{count}(k,\tilde{j})(\mathbf{a}) \ge \mathrm{count}(k,\tilde{j}+1)(\mathbf{a}) = \mathrm{count}(k,\tilde{j}+1)(\mathbf{a}').$$

Thus, the pseudo-Yamanouchi condition is satisfied on diagonals $\tilde{j}$ and $\tilde{j}+1$.

Therefore, $(\mathbf{a}',\mathbf{r}')$ is pseudo-Yamanouchi, implying D' is pseudo-Yamanouchi. □

Lemma 4.3. *If $\pi \in S_n$ avoids* 1432*, then all reduced pipe dreams of π are pseudo-Yamanouchi.*

Proof. Choose $\pi \in S_n$ that avoids 1432. Using the previous two lemmas, we know that simple slides preserve the pseudo-Yamanouchi property and that all reduced pipe dreams in $\mathrm{PD}^{red}(\pi)$ are connected by simple slides. So we need only show one reduced pipe dream is pseudo-Yamanouchi, and then all of them are. By Lemma 3.2, the bottom (permutation) pipe dream is pseudo-Yamanouchi. Thus the lemma is proved. □

Proof of Theorem 1.1. Let $\pi \in S_n$. The explicit bijection $\varphi : \mathrm{PD}^{red}(\pi) \to \mathrm{BPD}^{red}(\pi)$ of [8] discussed in Section 2.6 is weight-preserving; in particular, for $D \in \mathrm{PD}^{red}(\pi)$, the number of cross-tiles in row k equals the number of blank tiles in row k of $\varphi(D)$. By Theorem 3.3, TSSCPP are characterized as the set of pseudo-Yamanouchi pipe dreams in $\mathrm{PD}(n)$. Thus whenever such pipe dreams are reduced, φ produces a BPD with the same weight. Thus we have an injection $\varphi : TSSCPP^{red}(\pi) \hookrightarrow ASM^{red}(\pi)$ given by transforming the TSSCPP to its corresponding reduced pipe dream, mapping it to a reduced BPD using φ, and then transforming to an ASM using the bijection described in Section 2.4.

Suppose π avoids 1432. Then by Lemma 4.3, all pipe dreams in $\mathrm{PD}^{red}(\pi)$ are pseudo-Yamanouchi, so $TSSCPP^{red}(\pi)$ is in bijection with $\mathrm{PD}^{red}(\pi)$. So the above injection is a bijection between $TSSCPP^{red}(\pi)$ and $ASM^{red}(\pi)$. □

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