

An embedding of the skein action on set partitions into the skein action on matchings

Jesse Kim^{*1}

¹Department of Mathematics, University of California San Diego, CA

Abstract. Rhoades defined a skein action of the symmetric group on noncrossing set partitions which generalized an action of the symmetric group on matchings. The \mathfrak{S}_n -action on matchings is made possible via the Ptolemy relation, while the action on set partitions is defined in terms of a set of skein relations that generalize the Ptolemy relation. The skein action on noncrossing set partitions has seen applications to coinvariant theory and coordinate rings of partial flag varieties. We will show how Rhoades' \mathfrak{S}_n -module can be embedded into the \mathfrak{S}_n -module generated by matchings, thereby explaining how Rhoades' generalized skein relations all arise from the Ptolemy relation.

1 Introduction

This extended abstract concerns an action of \mathfrak{S}_n on the vector space spanned by the set of noncrossing set partitions of $[n] := \{1, 2, \dots, n\}$ first defined by Rhoades [6]. This action is defined in terms of three *skein relations*, the simplest of which is the *Ptolemy relation* shown below.

$$\begin{array}{c} \bullet & & \bullet \\ & \diagdown & / \\ & \bullet & \\ & / & \diagdown \\ \bullet & & \bullet \end{array} \mapsto \begin{array}{c} | \\ \bullet \\ | \\ \bullet \end{array} \begin{array}{c} | \\ \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet & \bullet \\ \hline \bullet & \bullet \end{array}$$

The original motivation for defining this action was to give an algebraic proof of certain cyclic sieving results on noncrossing set partitions, first proven by Reiner, Stanton, and White [5] and Pechenik [4] via direct enumeration. Rhoades' action has since been found within coinvariant rings and coordinate rings of certain partial flag varieties [1, 3], strengthening the claim that it is an action worth studying in its own right. Rhoades' action generalizes a similar action on the vector space with basis given by noncrossing matchings. In this extended abstract, we will show how Rhoades' action can be found within the action on noncrossing matchings and thereby explain how the three skein relations all arise from the Ptolemy relation.

*jvkim@ucsd.edu.

More precisely, A *matching* on $[n]$ is a collection of disjoint size-two subsets of $[n]$. Let $M(n)$ denote the set of all matchings on $[n]$. The symmetric group \mathfrak{S}_n acts naturally on $M(n)$ as follows. If $\sigma \in \mathfrak{S}_n$ and $m = \{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$ is a matching, then

$$\sigma \star m = \{\{\sigma(a_1), \sigma(b_1)\}, \dots, \{\sigma(a_k), \sigma(b_k)\}\}. \quad (1.1)$$

We can extend this action to an action on $\mathbb{C}[M(n)]$, the \mathbb{C} -vector space with basis given by matchings of $[n]$. For our purposes, it will be more useful to consider a sign-twisted version of this action, where

$$\sigma \circ m = \text{sign}(\sigma) \{\{\sigma(a_1), \sigma(b_1)\}, \dots, \{\sigma(a_k), \sigma(b_k)\}\}. \quad (1.2)$$

A matching is *noncrossing* if it does not contain two subsets $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$. The action on matchings does not descend to an action on $NCM(n)$, the set of all noncrossing matchings of $[n]$, since permuting elements in a noncrossing matching could introduce crossings. However, we can linearize and define an action on $\mathbb{C}[NCM(n)]$, the \mathbb{C} -vector space with basis given by noncrossing matchings of $[n]$. For any noncrossing matching m and adjacent transposition $s_i = (i, i + 1)$, define

$$s_i \cdot m = \begin{cases} s_i \circ m & s_i \circ m \text{ is noncrossing} \\ m + m' & \text{otherwise} \end{cases}. \quad (1.3)$$

Here \circ denotes the action on all matchings and m' is the matching where the subsets of m containing i and $i + 1$, call them $\{i, a\}$ and $\{i + 1, b\}$ have been replaced with $\{i, i + 1\}$ and $\{a, b\}$ and all other subsets remain the same. In other words, $s_i \circ m$, m , and m' form a trio of matchings that differ only in a Ptolemy relation. There exists an \mathfrak{S}_n -equivariant linear projection $p_M : \mathbb{C}[M(n)] \rightarrow \mathbb{C}[NCM(n)]$ given for any matching m by

$$m \mapsto w^{-1} \cdot (w \circ m), \quad (1.4)$$

where w is any permutation for which $w \circ m$ is noncrossing. The kernel of this projection is spanned by sums of matchings which differ only by a Ptolemy relation, i.e.

$$\begin{aligned} & \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}, \dots, \{a_{2k-1}, a_{2k}\}\} \\ & + \{\{a_1, a_3\}, \{a_2, a_4\}, \{a_5, a_6\}, \dots, \{a_{2k-1}, a_{2k}\}\} \\ & + \{\{a_1, a_4\}, \{a_2, a_3\}, \{a_5, a_6\}, \dots, \{a_{2k-1}, a_{2k}\}\} \end{aligned} \quad (1.5)$$

for any distinct $a_1, \dots, a_{2k} \in [n]$. This projection can be thought of as a way to “resolve” crossings in a matching and obtain a sum of noncrossing matchings.

Analogously, let $\Pi(n)$ denote the family of all set partitions of $[n]$, and let $\mathbb{C}[\Pi(n)]$ be the \mathbb{C} -vector space with basis given by $\Pi(n)$. We can define an action of \mathfrak{S}_n on $\mathbb{C}[\Pi(n)]$ analogous to the action on $\mathbb{C}[M(n)]$. A set partition is *noncrossing* if there do not exist

distinct blocks A and B and elements $a, c \in A, b, d \in B$ with $a < b < c < d$. Let $NCP(n)$ denote the set of all noncrossing set partitions, and let $\mathbb{C}[NCP(n)]$ be the corresponding vector space. Rhoades [6] defined an action of \mathfrak{S}_n on $\mathbb{C}[NCP(n)]$ as follows. For any noncrossing set partition π and adjacent transposition s_i ,

$$s_i \cdot \pi = \begin{cases} -\pi & i \text{ and } i+1 \text{ are in the same block of } \pi \\ -\pi' & \text{at least one of } i \text{ and } i+1 \text{ is in a singleton block of } \pi \\ \sigma(\pi') & i \text{ and } i+1 \text{ are in different size 2 or larger blocks of } \pi \end{cases}$$

where π' is the set partition obtained by swapping which blocks i and $i+1$ are in, and σ is defined for any almost-noncrossing (i.e. the crossing can be removed by a single adjacent transposition) partition π by $\sigma(\pi) = \pi + \pi_2 - \pi_3 - \pi_4$ where, if the crossing blocks in σ are $\{i, a_1, \dots, a_k\}$ and $\{i+1, b_1, \dots, b_l\}$, then π_2, π_3 and π_4 are obtained from π by replacing these blocks with

- $\{i, i+1\}$ and $\{a_1, \dots, a_k, b_1, \dots, b_l\}$ for π_2
- $\{i, i+1, a_1, \dots, a_k\}$ and $\{b_1, \dots, b_l\}$ for π_3
- $\{i, i+1, b_1, \dots, b_l\}$ and $\{a_1, \dots, a_k\}$ for π_4

when $k, l \geq 2$. If $k = 1$ then $\pi_4 = 0$ instead and if $l = 1$ then $\pi_3 = 0$ instead. The sum of partitions given by $\sigma(\pi)$ is best described with a picture, see Figure 1. The three possibilities (depending on whether $k, l \geq 2$) are the three skein relations mentioned earlier. A more detailed description of this action can be found in [6]. Rhoades showed that we again have an \mathfrak{S}_n -equivariant linear projection $p_\Pi : \mathbb{C}[\Pi(n)] \rightarrow \mathbb{C}[NCP(n)]$ given for any set partition π by

$$\pi \mapsto w^{-1} \cdot (w \circ \pi), \tag{1.6}$$

where w is any permutation for which $w \circ \pi$ (here \circ denotes the action of \mathfrak{S}_n on all set partitions) is noncrossing. The kernel of this projection is generated by the set of all elements of the form $w \circ (s_i \circ \pi + \sigma(\pi))$ (the skein relations) for any permutation w and almost noncrossing set partition π .

Rhoades was able to determine the \mathfrak{S}_n -irreducible structure of this module. In particular, the span of all singleton-free noncrossing set partitions with exactly k blocks is an \mathfrak{S}_n -irreducible of shape $(k, k, 1^{n-2k})$, and the span of all noncrossing set partitions with exactly s singletons and exactly k non-singleton blocks is isomorphic to an induction product of $S^{(k, k, 1^{n-2k-s})}$ with the sign representation of \mathfrak{S}_s . Similarly, if we restrict the action on $\mathbb{C}[NCP(n)]$ to the span of noncrossing perfect matchings, i.e. noncrossing matchings of $[2n]$ with exactly n pairs, then this action gives a model for the \mathfrak{S}_n -irreducible of shape (n, n) called the SL_2 -web basis. If we instead restrict to the span of noncrossing matchings with exactly k pairs, we get a submodule isomorphic to the

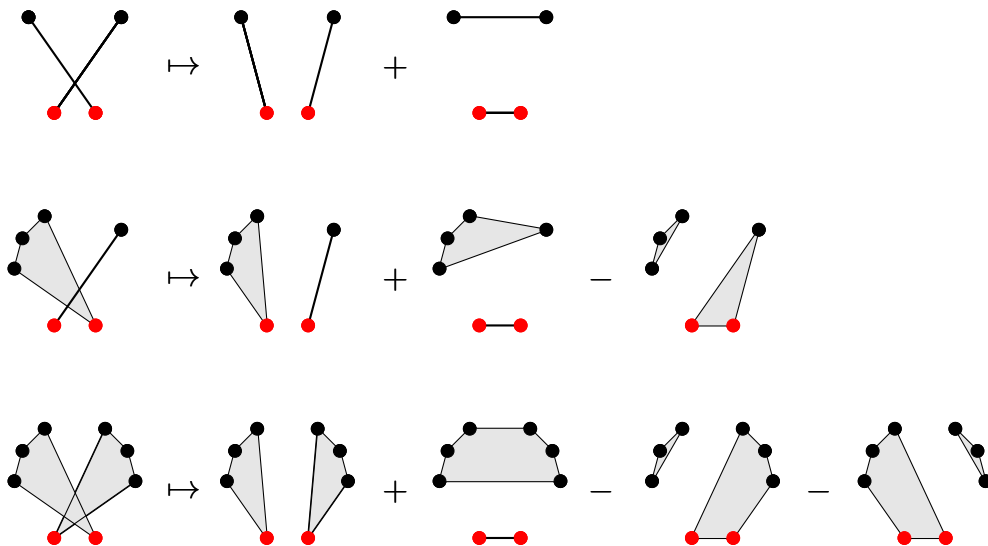


Figure 1: The three skein relations defining the action of \mathfrak{S}_n on $\mathbb{C}[NCP(n)]$. The red vertices are adjacent indices $i, i + 1$ and the shaded blocks have at least three elements. The symmetric 3-term relation obtained by reflecting the middle relation is not shown.

induction product of $S^{(k,k)}$ and a sign representation of \mathfrak{S}_{n-2k} . By the Pieri rule, this induction product is a direct sum of three irreducible submodules, exactly one of which is isomorphic to $S^{(k,k,1^{n-2k})}$. Therefore there exists an embedding of $\mathbb{C}[NCP(n)_0]$, the span of all singleton-free noncrossing set partitions in $\mathbb{C}[NCP(n)]$, into $\mathbb{C}[NCM(n)]$. If we additionally require that our embedding send set partitions with k blocks to matchings with k pairs, then this embedding is unique. Our main result is to explicitly describes the embedding as follows:

Theorem 1. *The linear map $f_n : \mathbb{C}[NCP(n)_0] \rightarrow \mathbb{C}[NCM(n)]$ defined by*

$$f_n(\pi) = \sum_{m \in M_\pi(n)} m$$

is a \mathfrak{S}_n -equivariant embedding of vector spaces. Here $M_\pi(n)$ is defined to be the set of all matchings m in $M(n)$ for which each block of π contains exactly one pair in m .

For an example of this map, if $\pi = \{\{1, 2, 3\}, \{4, 5\}\}$ then

$$f_n(\pi) = \{\{1, 2\}, \{4, 5\}\} + \{\{1, 3\}, \{4, 5\}\} + \{\{2, 3\}, \{4, 5\}\}$$

is a sum of 3 matchings in $\mathbb{C}[NCM(n)]$.

2 The embedding

In order to prove that our map is an embedding, it will be helpful to introduce a multiplicative structure to work with. To do so we will introduce three graded commutative \mathbf{C} -algebras R_n , A_n , and M_n , all with \mathfrak{S}_n -actions. If we forget the multiplicative structure, the underlying \mathfrak{S}_n -modules of R_n , A_n , and M_n will contain a copy of $\mathbf{C}[\Pi(n)]$, $\mathbf{C}[M(n)]$, and $\mathbf{C}[NCM(n)]$ respectively. In the case of M_n , this copy will be all of M_n . The structure of this proof is best explained via a commutative diagram, see Figure 2. We will define a map $h_n \circ \iota_\Pi : \mathbf{C}[\Pi(n)_0] \rightarrow M_n$, and show that its kernel is equal to the kernel of p_Π . The desired embedding f_n then follows from the first isomorphism theorem.

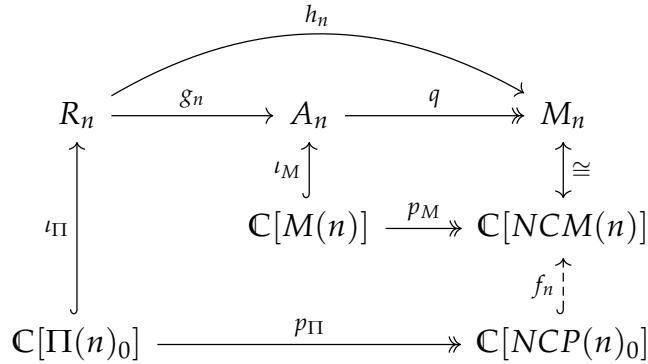


Figure 2: A commutative diagram of the maps used in the following proofs. All maps shown are \mathfrak{S}_n -equivariant linear maps. Maps between R_n , A_n , and M_n are also morphisms of \mathbf{C} -algebras. The desired embedding is shown as a dashed arrow.

We begin with the definition of R_n .

Definition 1. Let $n \in \mathbb{N}$. Define R_n to be the unital graded commutative \mathbf{C} -algebra freely generated by nonempty subsets of $[n]$ where each generator has degree 1. Define a degree-preserving action of \mathfrak{S}_n on R_n by

$$\pi \cdot \{a_1, \dots, a_k\} = \text{sign}(\pi) \{ \pi(a_1), \dots, \pi(a_k) \}$$

for any permutation $\pi \in \mathfrak{S}_n$ and generator $\{a_1, \dots, a_k\} \in R_n$.

The vector space structure of R_n can be thought of as formal sums of multiset collections of subsets of $[n]$ by interpreting each product of generators as a multiset collection of the corresponding subsets. It is in this sense that it contains a copy of $\mathbf{C}[\Pi(n)]$, as set partitions of n are particular collections of subsets of $[n]$. To be precise, there exists an \mathfrak{S}_n -module embedding $\iota_\Pi : \mathbf{C}[\Pi(n)_0] \hookrightarrow R_n$, given by sending any singleton-free set partition π to the product of its blocks.

The ring A_n is a subring of R_n designed to model matchings in much the same way which R_n models set partitions. It is defined as follows.

Definition 2. Let $n \in \mathbb{N}$ and define A_n to be the \mathfrak{S}_n -invariant subalgebra of R_n freely generated by the size two subsets of $[n]$. The subring A_n is invariant under the \mathfrak{S}_n -action of R_n , and thus inherits a graded \mathfrak{S}_n -action from R_n .

Like R_n , the ring A_n can be thought of as the ring of multiset collections of size-two subsets of $[n]$. As matchings are particular collections of size-two subsets of $[n]$, we again have an \mathfrak{S}_n -module embedding $\iota_M : \mathbb{C}[M(n)] \hookrightarrow A_n$, given by

$$\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\} \longmapsto \{a_1, b_1\} \cdots \{a_k, b_k\}$$

for any matching $\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$.

Our final ring, M_n , is defined as a quotient of A_n in the following way.

Definition 3. Define I_n to be the ideal of A_n generated by elements of the following forms

- $\{a, b\} \cdot \{a, b\}$
- $\{a, b\} \cdot \{a, c\}$
- $\{a, b\} \cdot \{c, d\} + \{a, c\} \cdot \{b, d\} + \{a, d\} \cdot \{b, c\}$

for any distinct $a, b, c, d \in [n]$. Then I_n is a homogeneous \mathfrak{S}_n -invariant ideal of A_n , so define M_n to be the graded \mathfrak{S}_n -module $M_n := A_n / I_n$. Let $q : A_n \rightarrow M_n$ be the quotient map.

The first two types of elements listed in the definition of I_n serve the purpose of removing collections of size-two subsets which are not actually matchings. The third is the Ptolemy relation used to define the action of \mathfrak{S}_n on $\mathbb{C}[NCM(n)]$, so quotienting by this ideal gives an \mathfrak{S}_n -module isomorphic to $\mathbb{C}[NCM(n)]$, as per the following argument.

Proposition 1. There is an \mathfrak{S}_n -module isomorphism from $\mathbb{C}[NCM(n)]$ to M_n , given by

$$\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\} \longmapsto \{a_1, b_1\} \cdots \{a_k, b_k\}$$

for any noncrossing matching $\{\{a_1, b_1\}, \dots, \{a_k, b_k\}\}$.

Proof. Let $q : A_n \rightarrow M_n$ be the quotient map. Consider the map $q \circ \iota_M : \mathbb{C}[M(n)] \rightarrow M_n$. The kernel of $q \circ \iota_M$ is spanned by the elements of $\mathbb{C}[M(n)]$ which are sent under ι_M to a multiple of $\{a, b\} \cdot \{c, d\} + \{a, c\} \cdot \{b, d\} + \{a, d\} \cdot \{b, c\}$ for some distinct $a, b, c, d \in [n]$. Therefore the kernel of $q \circ \iota_M$ is equal to the kernel of p_M by 1.5 and the surrounding discussion. The image of $q \circ \iota_M$ is all of M_n . To see this, note that products of generators of A_n form a vector space basis for A_n , and every such basis element is either in the image of ι_M or in I_n . We therefore have

$$\mathbb{C}[NCM(n)] \cong \mathbb{C}[M(n)] / \ker(p_M) = \mathbb{C}[M(n)] / \ker(q \circ \iota_M) \cong \text{im}(q \circ \iota_M) = M_n \quad (2.1)$$

where the isomorphism on the left is induced by the map p_M and the isomorphism on the right is induced by the map $q \circ \iota_M$. Composing these isomorphisms gives the stated map. \square

The following definition is the key idea behind our main theorem.

Definition 4. Let $n \in \mathbb{N}$. Define the \mathbb{C} -algebra map $g_n : R_n \rightarrow A_n$ by

$$g_n(A) = \sum_{\{a,b\} \subseteq A} \{a,b\}$$

for generators $A \in R_n$. Singleton sets are sent to 0 by g_n . Define $h_n := q \circ g_n$ where q is the quotient map $A_n \rightarrow M_n$.

We give the definition in terms of R_n , A_n , and M_n for simplicity and ease of proofs later, but the map we really care about is $h_n \circ \iota_\Pi : \mathbb{C}[\Pi(n)] \rightarrow M_n$. Under this map, a set partition π is sent to the product of its blocks, then each block is sent to the sum of all size-two subsets it contains. After distributing, we get a sum of all ways to pick a size two subset from each block. Composing with the isomorphism between M_n and $\mathbb{C}[NCM(n)]$ we get the sum of all matchings such that each block of π contains exactly one pair of the matching, as in [Theorem 2](#).

The \mathfrak{S}_n -equivariance of h_n is simple to check. We have the following calculation.

Lemma 1. Let $i, j \geq 2$ and let p_1, p_2, \dots, p_i and q_1, q_2, \dots, q_j be in $[n]$. Then the element of R_n

$$\begin{aligned} \kappa_n := & \{p_1, \dots, p_i\} \cdot \{q_1, \dots, q_j\} - \{p_1, \dots, p_{i-1}\} \cdot \{q_1, \dots, q_j, p_i\} \\ & - \{p_1, \dots, p_i, q_j\} \cdot \{q_1, \dots, q_{j-1}\} + \{p_1, \dots, p_{i-1}, q_j\} \cdot \{q_1, \dots, q_{j-1}, p_i\} \\ & + \{p_1, \dots, p_{i-1}, q_1, \dots, q_{j-1}\} \cdot \{p_i, q_j\} \end{aligned} \quad (2.2)$$

lies in the kernel of h_n .

When $i, j > 2$, the element κ_n corresponds to the five-term skein relation depicted in [Figure 1](#). If either i equals 2, then $\{p_1, \dots, p_{i-1}\} = \{p_1\}$ is a one element set and therefore sent to 0 by h_n , removing the term containing $\{p_1\}$ corresponds to the four-term skein relation depicted in [Figure 1](#). Similarly, if j equals 2 or i and j both equal two, removing the terms in κ_n which are individually sent to 0 corresponds to the four or three-term skein relation depicted in [Figure 1](#).

Proof. The proof of this lemma is a mostly straightforward calculation from the definition of h_n . $h_n(\kappa_n)$ can be shown to be a sum of relations in I_n . \square

The preceding lemma allows for the identification of the kernel of $h \circ \iota_\Pi$ and p_Π mentioned in the beginning of this section.

Proposition 2. The kernel of $h \circ \iota_\Pi$ is generated by the set of all elements of the form

$$w \circ (s_i \circ \pi + \sigma(\pi))$$

i.e., the skein relations, for any permutation w and singleton-free almost noncrossing set partition π .

Proof. By Lemma 1, all such elements lie in the kernel. Since restricting $\mathbb{C}[\Pi(n)_0]$ to noncrossing set partitions with exactly k blocks gives an irreducible submodule [6, Proposition 5.2], and each such irreducible is sent to degree k in M_n , it suffices to demonstrate that each irreducible is not sent identically to 0. But for each noncrossing set partition $\pi \in \Pi(n)_0$, since π does not have singleton blocks, $h_n \circ \iota_\Pi(\pi)$ is a nonvanishing sum of noncrossing matchings of $[n]$, which form a basis for M_n by Proposition 1, and is therefore nonzero. \square

We can now prove our main result.

Theorem 2. *The linear map $f_n : \mathbb{C}[NCP(n)_0] \rightarrow \mathbb{C}[NCM(n)]$ defined by*

$$f_n(\pi) = \sum_{m \in M_\pi(n)} m$$

is a \mathfrak{S}_n -equivariant embedding of vector spaces. Here $M_\pi(n)$ is defined to be the set of all matchings m in $M(n)$ for which each block of π contains exactly one pair in m .

Proof. By Proposition 2, the kernel of $h \circ \iota_\Pi$ is equal to the kernel of p_Π . So we have

$$\begin{aligned} \mathbb{C}[NCP(n)_0] &\cong \mathbb{C}[\Pi_0(n)] / \ker(p_\Pi) \\ &= \mathbb{C}[\Pi(n)_0] / \ker(h \circ \iota_\Pi) \cong \text{im}(h \circ \iota_\Pi) \subset M_n \cong \mathbb{C}[NCM(n)] \end{aligned} \quad (2.3)$$

where the isomorphism on the left is induced by p_Π and the middle isomorphism is induced by $h \circ \iota_\Pi$, and the isomorphism on the right is as defined in Proposition 1. Chasing these isomorphisms and inclusions results in the map f_n . \square

3 The image

We have an embedding $f_n : \mathbb{C}[NCP(n)_0] \hookrightarrow \mathbb{C}[NCM(n)]$, so it is a natural question to ask for a description of the image of f_n within $\mathbb{C}[NCM(n)]$. Via the commutative diagram in Figure 2, we have an isomorphism of images

$$\text{im}(h_n) \cong \text{im}(f_n). \quad (3.1)$$

So it is equivalent to describe the image of h_n , and the multiplicative structure of M_n will make describing the image of h_n easier. It turns out to have a simple description, the proof of which requires a few lemmas.

Lemma 2. *Let $A \subseteq [n]$. Then $h_n(A)^2 = 0$.*

Proof. Applying the definition of h_n gives

$$h_n(A)^2 = \sum_{\substack{a,b \in [n] \\ a \neq b}} \sum_{\substack{c,d \in [n] \\ c \neq d}} \{a,b\} \cdot \{c,d\} \quad (3.2)$$

Using the defining relation of M_n that

$$\{a,b\} \cdot \{a,c\} = 0$$

we have

$$\sum_{\substack{a,b \in [n] \\ a \neq b}} \sum_{\substack{c,d \in [n] \\ c \neq d}} \{a,b\} \cdot \{c,d\} = \frac{1}{3} \sum_{\substack{a,b,c,d \in [n] \\ a,b,c,d \text{ distinct}}} \{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\}. \quad (3.3)$$

The right hand side of the above equation equals 0 because

$$\{a,b\} \cdot \{c,d\} + \{a,c\} \cdot \{b,d\} + \{a,d\} \cdot \{b,c\} = 0$$

for any distinct $a,b,c,d \in [n]$. □

The proof of the following two lemmas are similar.

Lemma 3. *Let A, B be disjoint subsets of $[n]$. Then*

$$h_n(A) \cdot \left(\sum_{\substack{a \in A \\ b \in B}} \{a,b\} \right) = 0.$$

Lemma 4. *Let B_1, \dots, B_k be the blocks of a singleton free set partition of $[n]$. Then*

$$h_n \left(\prod_{i=1}^k B_i \right) = h_n \left([n] \cdot \prod_{i=1}^{k-1} B_i \right)$$

We can now describe the image of h_n .

Theorem 3. *Let H_n be the ideal of M_n generated by $h_n([n])$. Then*

$$\text{im}(h_n) = H_n.$$

Proof. It is immediate from [Lemma 4](#) that the image of h_n is contained in H_n , so it suffices to show that the vector space dimension

$$\dim(H_n) \leq \dim(\text{im}(h_n)) = \dim(\text{im}(f_n)) = \#NCP(n)_0 \quad (3.4)$$

To bound the dimension of H_n , computation shows that for any $a \in [n]$,

$$h_n([n]) \cdot \{1, a\} = -h_n([n]) \cdot \left(\sum_{\substack{b \in [n] \\ b \neq a, 1}} \{a, b\} \right).$$

Therefore H_n is spanned by elements of the form

$$h_n([n]) \cdot \{a_1, a_2\} \cdot \{a_3, a_4\} \cdots \{a_{2k-1}, a_{2k}\}$$

where the sets $\{a_1, a_2\}, \dots, \{a_{2k-1}, a_{2k}\}$ form a noncrossing matching of $\{2, \dots, n\}$. These elements are not linearly independent, however. Consider the element $\tilde{f}_n(\tilde{\pi})$ of M_n given by

$$\tilde{f}_n(\tilde{\pi}) := \prod_{B \in \tilde{\pi}} h_n(B)$$

for any singleton free noncrossing set partition $\tilde{\pi}$ of $\{2, \dots, n\}$ (i.e. the “image of f_n ” if such a set partition were in the domain of f_n). Computation shows that

$$h_n([n]) \cdot \tilde{f}_n(\tilde{\pi}) = 0$$

The collection of $\tilde{f}_n(\tilde{\pi})$ for singleton-free noncrossing set partitions π of $\{2, \dots, n\}$ is linearly independent. To see this, note that any linear relation among the $\tilde{f}_n(\tilde{\pi})$ would also be a linear relation among $f_{n-1}(\pi)$ where π is the set partition of $[n-1]$ obtained by decrementing the indices in $\tilde{\pi}$. But f_{n-1} is an embedding and singleton-free noncrossing set partitions are linearly independent in $\mathbb{C}[NCP(n-1)_0]$. The dimension of H_n is therefore bounded by

$$\begin{aligned} \dim(H_n) &\leq \#\{\text{noncrossing matchings of } \{2, \dots, n\}\} \\ &\quad - \#\{\text{singleton-free noncrossing set partitions of } \{2, \dots, n\}\}. \end{aligned} \quad (3.5)$$

Noncrossing matchings of $\{2, \dots, n\}$ are in bijection with noncrossing set partitions of $[n]$ in which only the block containing 1 may be a singleton (though it may be larger). Given a noncrossing set partition, simply take the matching that matches the largest and smallest element of each block not containing 1. Singleton-free noncrossing set partitions of $\{2, \dots, n\}$ are in bijection with set partitions of $[n]$ in which $\{1\}$ is the unique singleton block. We therefore have

$$\dim(H_n) \leq \#NCP(n)_0$$

as desired. □

4 Future directions

One of the goals motivating this paper is to find new combinatorially nice bases for \mathfrak{S}_n -irreducibles which arise from existing bases in an analogous way to the skein action. More specifically, suppose we have a basis for S^λ which is indexed by certain structures on the set $[k]$, where $k = |\lambda|$ (e.g. noncrossing perfect matchings, in the case of this paper). We can create a basis for the induction product of S^λ with a sign representation of \mathfrak{S}_{n-k} indexed by all ways to put a certain structure on a k -element subset of $[n]$. The Pieri rule tells us which \mathfrak{S}_n irreducibles this decomposes into. In particular, there will be one copy of $(\lambda, 1^{n-k})$. How do we isolate that irreducible?

It is optimistic to think that there will be a method that works in any sort of generality, but perhaps analogs could be found in certain specific cases. For example, an analog might exist for the web basis for $S^{(k,k,k)}$ introduced by Kuperberg [2]. The web basis consists of planar bipartite graphs embedded in a disk with n boundary vertices all of degree 1, interior vertices are degree 3, all boundary vertices are part of the same bipartition, and no cycles of length less than 6 exist. One potential candidate for a basis for $S^{(k,k,k,1^{n-3k})}$ is as follows.

Conjecture 1. *Let A be the set of all planar bipartite graphs embedded in a disk for which the following conditions hold*

- *There are n vertices on the boundary of the disk, and there exists a bipartition in which all of these vertices are in the same part.*
- *Every interior vertex is connected to a boundary vertex.*
- *Every interior vertex in the same bipartition as the boundary vertices is degree 3. These are called negative interior vertices.*
- *Every interior vertex not in the same bipartition as the boundary vertices is degree at least 3. These are called positive interior vertices.*
- *The number of positive interior vertices minus the number of negative interior vertices is exactly k .*
- *No cycles of length less than 6 exist.*

Then $|A|$ is equal to the dimension of $S^{(k,k,k,1^{n-3k})}$.

The set A can be thought of as consisting of webs for which the condition of interior vertices being degree 3 has been partially relaxed. The conjecture can be shown to hold for $k = 2$ and any n , as well as $n = 10, k = 3$ via direct enumeration. If the above conjecture is true, it suggests the following question.

Question 1. *Does there exist a combinatorially nice action of \mathfrak{S}_n on $\mathbb{C}[A]$ which creates a \mathfrak{S}_n module isomorphic to $S^{(k,k,k,1^{n-3k})}$? If so, what does the unique embedding into $S^{(k,k,k)}$ induced with a sign representation of \mathfrak{S}_{n-3k} look like?*

A positive answer to this question might help elucidate how to apply similar methods more generally.

Acknowledgements

We are very grateful to Brendon Rhoades for many helpful discussions and comments on this project.

References

- [1] J. Kim and B. Rhoades. “Set Partitions, Fermions, and Skein Relations”. *International Mathematics Research Notices* (May 2022). rna110. [DOI](#).
- [2] G. Kuperberg. “Spiders for rank 2 Lie algebras”. *Communications in Mathematical Physics* **180** (1996), pp. 109–151. [DOI](#).
- [3] R. Patrias, O. Pechenik, and J. Striker. “A web basis of invariant polynomials from non-crossing partitions”. *Advances in Mathematics* **408** (2022), p. 108603. [DOI](#).
- [4] O. Pechenik. “Cyclic Sieving of Increasing Tableaux and small Schröder Paths”. *Journal of Combinatorial Theory. Series A* **125** (Sept. 2012). [DOI](#).
- [5] V. Reiner, D. Stanton, and D. White. “The cyclic sieving phenomenon”. *Journal of Combinatorial Theory, Series A* **108.1** (2004), pp. 17–50. [DOI](#).
- [6] B. Rhoades. “A skein action of the symmetric group on noncrossing partitions”. *Journal of Algebraic Combinatorics* **45** (Feb. 2017). [DOI](#).