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The Kromatic Symmetric Function: A K-theoretic analogue of X_G

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Abstract. Schur functions are a basis of the symmetric function ring that represent Schubert cohomology classes for Grassmannians. Replacing the cohomology ring with K-theory yields a rich combinatorial theory of inhomogeneous deformations, where Schur functions are replaced their K-analogues, the basis of *symmetric Grothendieck functions*. We introduce and initiate a theory of the *Kromatic symmetric function* $\overline{X}_G$, a K-theoretic analogue of the chromatic symmetric function X_G of a graph G. The Kromatic symmetric function is a generating series for graph colorings in which vertices may receive any nonempty set of distinct colors such that neighboring color sets are disjoint.

Our main result lifts a theorem of Gasharov (1996) to this setting, showing that when G is a claw-free incomparability graph, $\overline{X}_G$ is a positive sum of symmetric Grothendieck functions. This result suggests a topological interpretation of Gasharov's theorem. We then show that the Kromatic symmetric functions of path graphs are not positive in any of several K-analogues of the e-basis of symmetric functions, demonstrating that the Stanley–Stembridge conjecture (1993) does not have such a lift to K-theory and so is unlikely to be amenable to a topological perspective. We also define a vertex-weighted extension of $\overline{X}_G$ and show that it admits an edge deletion-contraction relation. Finally, we give a K-analogue for $\overline{X}_G$ of the classic monomial-basis expansion of X_G.

Keywords: chromatic symmetric function, K-theory, Stanley-Stembridge conjecture, deletion-contraction, symmetric Grothendieck function

1 Introduction

The *chromatic symmetric function* X_G of a graph G was introduced by R. Stanley [28] as a generalization of G.D. Birkhoff's *chromatic polynomial* [4]. While the chromatic polynomial enumerates proper graph colorings by the number of colors used, X_G also records

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how many times each color is used. A recent boom of research regarding X_G has focused on the *Stanley–Stembridge conjecture* [31], which proposes (in a reformulation by M. Guay-Paquet [17]) that unit interval graphs have chromatic symmetric functions that expand positively in the e-basis of the ring Sym of symmetric functions. In the last few years, various special cases of this conjecture have been established through direct combinatorial analysis, including the cases of *lollipop graphs* [12] and many *claw-free* graphs [18]. Another approach has been to consider various generalizations of the chromatic symmetric function and corresponding lifts of the Stanley–Stembridge conjecture. Examples of this latter approach include the *chromatic quasisymmetric function* and *Shareshian–Wachs conjecture* of [27] (further studied in, e.g., [1, 2, 6, 7]), and D. Gebhard–B. Sagan's [16] chromatic symmetric function in noncommuting variables combined with notions of (e)*-positivity* and *appendable* (e)*-positivity* (further studied in, e.g., [3, 10, 13]). Our work provides a novel generalization of X_G in the same vein.

An important appearance of the ring of symmetric functions Sym is as the cohomology of complex *Grassmannians* (parameter spaces for linear subspaces of a vector space) or more precisely for the classifying space BU. Here, the *Schubert classes* derived from a natural cell decomposition of BU are represented by the *Schur function* basis s_λ of Sym. A richer perspective into the topology of BU is obtained by replacing cohomology with a generalized cohomology theory. In particular, there has been much focus on studying the associated combinatorics of the *K-theory* ring (see, e.g., [5, 23, 25, 32]). In this context, many of the classical objects of symmetric function theory are seen to have interesting K-analogues, often resembling "superpositions" of classical objects. For example, classical *semistandard Young tableaux* are replaced by *set-valued tableaux* (allowing multiple labels per cell), while Schur functions are replaced by *Grothendieck polynomials* $\overline{s}_\lambda$ (inhomogeneous deformations of s_λ).

Our work introduces a K-analogue of the chromatic symmetric function X_G, enumerating colorings of the graph G that assign a nonempty set of distinct colors to each vertex such that adjacent vertices receive disjoint sets. While our *Kromatic symmetric function* $\overline{X}_G$ is new, similar functions have been previously considered. The first such function was originally discussed by Stanley [29] in the context of graph analogues of symmetric functions, with connections to the real-rootedness of polynomials. Recently, as part of his effort to refine Schur-positivity results and the Stanley–Stembridge conjecture, B.-H. Hwang [19] studied a similar quasisymmetric function for graphs endowed with a fixed map $\alpha : V(G) \to \mathbb{N}$ that dictates the size of the set of colors each vertex receives. To connect chromatic quasisymmetric functions of vertex-weighted graphs to *horizontal-strip LLT polynomials*, F. Tom [33] has considered a variant for fixed α with repeated colors allowed. Our work appears to be the first to connect these ideas to the combinatorics of K-theoretic Schubert calculus. (However, [24] is similar in spirit to our work, developing a K-theoretic analogue of the *Postnikov–Shapiro algebra* [26], an apparently unrelated invariant of graphs).

In this extended abstract, having introduced the *Kromatic symmetric function*, we begin to develop its combinatorial theory. Further details and proofs are given in the full manuscript [8]. We show that the Kromatic symmetric function $\overline{X}_G$ for any graph G expands positively in a K-theoretic analogue (that we also introduce) of the *monomial basis* of Sym. In this expansion, the coefficients enumerate coverings of the graph by (possibly overlapping) *stable sets*. We further extend the definition of $\overline{X}_G$ to a vertex-weighted setting, where we give a deletion-contraction relation analogous to that developed by the first and last authors [9] for the vertex-weighted version of X_G.

Our main result is that the Kromatic symmetric function of a *claw-free incomparability graph* expands positively in the symmetric Grothendieck basis $\overline{s}_\lambda$ of Sym, lifting to K-theory a celebrated result of V. Gasharov [15] that such graphs have Schur-positive chromatic symmetric functions. While all known proofs of Gasharov's theorem are representation-theoretic or purely combinatorial, the existence of our K-theoretic analogue suggests that both results likely also have an interpretation in terms of the topology of Grassmannians. Precisely, for each claw-free incomparability graph G, there should be a subvariety of the Grassmannian whose cohomology class is represented by X_G and whose K-theoretic structure sheaf class is represented by $\overline{X}_G$. It would be very interesting to have an explicit construction of such subvarieties.

On the other hand, we show that the Kromatic symmetric functions $\overline{X}_{P_n}$ of path graphs P_n generally do not expand positively in either of two K-theoretic deformations we propose for the e-basis of Sym. This fact suggests that the Stanley–Stembridge conjecture, if true, is not naturally interpreted in terms of the cohomology of Grassmannians and is unlikely to be amenable to such topological tools from Schubert calculus. We hope these observations can play a similar role to [11] in limiting the range of potential avenues of attack on the Stanley–Stembridge conjecture.

This extended abstract is organized as follows. In Section 2, we provide an overview of the background and notation used from symmetric function theory (Section 2.1), K-theoretic Schubert calculus (Section 2.2), and graph theory (Section 2.3). In Section 3, we formally introduce the Kromatic symmetric function $\overline{X}_G$ and give its basic properties, including a formula for the expansion in a new K-analogue of the monomial-basis of Sym and a deletion-contraction relation for a vertex-weighted generalization. We also give our main theorem that the Kromatic symmetric functions of claw-free incomparability graphs expand positively in symmetric Grothendieck functions, lifting the main result of [15]. In Section 4, we introduce two different K-theoretic analogues of the e-basis of Sym and show that the Kromatic symmetric function $\overline{X}_{P_3}$ of a 3-vertex path graph P_3 is not positive in either analogue, casting doubt on hopes for a Schubert calculus-based approach to the Stanley–Stembridge conjecture.

2 Background

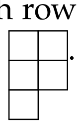
Throughout this work, $\mathbb{N}$ denotes the set of (strictly) positive integers. We write $[n]$ for the set of positive integers $\{1, 2, \ldots, n\}$. If S is any set, 2^S denotes the power set of all subsets of S.

2.1 Partitions and Symmetric Functions

In this section, we give a brief overview of necessary background material necessary. Further details can be found, e.g., in the textbooks of Stanley [30], Manivel [22], and Macdonald [21].

An ***integer partition*** $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a finite nonincreasing sequence of positive integers. We define $\ell(\lambda)$ to be the length of the sequence λ (so above, $\ell(\lambda) = k$). Define $r_i(\lambda)$ to be the number of occurrences of i as a part of λ (so, for example, $r_1(2,1,1) = 2$). If

$$\sum_{i=1}^{\ell(\lambda)} \lambda_i = n,$$

we say that λ is a partition of n, and we write $\lambda \vdash n$. The ***Young diagram of shape*** λ is a set of squares called ***cells***, left- and top-justified (i.e., in "English notation"), so that the ith row from the top contains λ_i cells. For example, the Young diagram of shape $(2,2,1)$ is . Let $C(\lambda)$ denote the set of cells of the Young diagram of shape λ. If $\mathsf{c} \in C(\lambda)$ is a cell of the Young diagram of shape λ, we write $\mathsf{c}^{\uparrow}$ for the cell immediately above c (assuming it exists), $\mathsf{c}^{\rightarrow}$ for the cell immediately right of c, etc. We write λ^{T} for the ***transpose*** of λ, the integer partition whose Young diagram is obtained from that of λ by exchanging rows and columns.

Let $S_{\mathbb{N}}$ denote the set of all permutations of the set $\mathbb{N}$ fixing all but finitely-many elements. A ***symmetric function*** $f \in \mathbb{C}[\![x_1, x_2, \ldots,]\!]$ is a power series of bounded degree such that for each permutation $\sigma \in S_{\mathbb{N}}$, we have $f(x_1, x_2, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$. The set $\mathrm{Sym} \subset \mathbb{C}[\![x_1, x_2, \ldots]\!]$ of symmetric functions forms a $\mathbb{C}$-vector space. Furthermore, if Λ^d denotes the set of symmetric functions that are homogeneous of degree d, then each Sym^d is a vector space, and

$$\mathrm{Sym} = \bigoplus_{d=0}^{\infty} \mathrm{Sym}^d$$

as graded vector spaces.

The dimension of Sym^d as a $\mathbb{C}$-vector space is equal to the number of integer partitions of d, and many bases of symmetric functions are conveniently indexed by integer partitions. Below we provide some commonly used bases that will be used in this paper.

Definition 2.1. The following are bases of Sym:

- the ***monomial symmetric functions*** $\{m_\lambda\}$, defined as
$$m_\lambda = \sum x_{i_1}^{\lambda_1} \dots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}},$$
where the sum ranges over all distinct monomials formed by choosing distinct positive integers $i_1, \dots, i_{\ell(\lambda)}$;
- the ***augmented monomial symmetric functions*** $\{\widetilde{m}_\lambda\}$, defined as
$$\widetilde{m}_\lambda = \left(\prod_{i=1}^{\infty} r_i(\lambda)! \right) m_\lambda;$$
- the ***elementary symmetric functions*** $\{e_\lambda\}$, defined by
$$e_n = \prod_{i_1 < \cdots < i_n} x_{i_1} \dots x_{i_n}; \quad e_\lambda = e_{\lambda_1} \dots e_{\lambda_{\ell(\lambda)}};$$
- and the ***complete homogeneous symmetric functions*** $\{h_\lambda\}$, defined by
$$h_n = \prod_{i_1 \leq \cdots \leq i_n} x_{i_1} \dots x_{i_n}; \quad h_\lambda = h_{\lambda_1} \dots h_{\lambda_{\ell(\lambda)}}.$$

The space of symmetric functions is equipped with a natural inner product $\langle \cdot, \cdot \rangle$; it may be defined by $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$, where $\delta_{\bullet,\bullet}$ denotes the *Kronecker delta function*.

We will also need the basis of Schur functions. A ***Young tableau*** of shape λ is a function $T : C(\lambda) \to \mathbb{N}$, typically visualized by writing the value $T(\mathsf{c})$ in the cell c. A Young tableau T of shape λ is ***semistandard*** if for each cell $\mathsf{c} \in C(\lambda)$, we have $T(\mathsf{c}) \leq T(\mathsf{c}^{\rightarrow})$ and $T(\mathsf{c}) < T(\mathsf{c}^{\downarrow})$ whenever the cells in question exist. We write $\mathrm{SSYT}(\lambda)$ for the set of all semistandard Young tableaux of shape λ. The ***Schur function*** s_λ is defined by
$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T, \quad \text{where} \quad x^T = \prod_{\mathsf{c} \in C(\lambda)} x_{T(\mathsf{c})}.$$

As λ ranges over integer partitions, the Schur functions are another basis of Sym. The inner product on Sym also satisfies $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu^\intercal}$.

When $f \in \mathrm{Sym}$ is a symmetric function and $\{b_\lambda\}$ is a basis of symmetric functions indexed by integer partitions λ, the notation $[b_\mu]f$ denotes the coefficient of b_μ when f is expanded in the b-basis. A symmetric function $f \in \mathrm{Sym}$ is said to be ***b-positive*** if $[b_\mu]f$ is nonnegative for every integer partition μ.

2.2 K-theoretic Schubert calculus

The ***Grassmannian*** $\Gamma_k = \mathrm{Gr}_k(\mathbb{C}^\infty)$ is the parameter space of k-dimensional vector subspaces of the space of all eventually-zero sequences of complex numbers. The space Γ_k can be given the structure of a projective Ind-variety and has a cell decomposition into cells Γ_λ indexed by partitions with at most k parts. Each Γ_λ induces a cohomology class $\sigma_\lambda \in H^\star(\Gamma_k)$ and classically we have $H^\star(\Gamma_k) \cong \mathrm{Sym}_k = \mathrm{Sym} \cap \mathbb{C}[x_1, \ldots, x_k]$ with the isomorphism taking the class of the cell σ_λ to the Schur polynomial $s_\lambda(x_1, \ldots, x_k)$.

Each cell-closure in Γ_k also has a structure sheaf, inducing a class in the representable ***K-theory*** ring $K^0(\Gamma_k)$. These K-theoretic classes are represented by inhomogeneous symmetric polynomials called *Grothendieck polynomials* $\bar{s}_\lambda(x_1, \ldots, x_k)$.

A ***set-valued tableau*** of shape λ is a filling T of each cell of $C(\lambda)$ with a nonempty *set* of positive integers. The set-valued tableau T is ***semistandard*** if for each cell $\mathsf{c} \in C(\lambda)$, we have $\max T(\mathsf{c}) \leq \min T(\mathsf{c}^{\rightarrow})$ and $\max T(\mathsf{c}) < \min T(\mathsf{c}^{\downarrow})$ whenever the cells in question exist. In other words, T is semistandard if every Young tableau formed by choosing one number from the set of each cell is semistandard. Let $\mathrm{SV}(\lambda)$ denote the set of all semistandard set-valued tableaux of shape λ. The ***symmetric Grothendieck function*** $\bar{s}_\lambda$ is

$$\bar{s}_\lambda = \sum_{T \in \mathrm{SV}(\lambda)} (-1)^{|T| - \ell(\lambda)} x^T,$$

where $|T| = \sum_{\mathsf{c} \in C(\lambda)} |T(\mathsf{c})|$ and $x^T = \prod_{\mathsf{c} \in C(\lambda)} \prod_{i \in T(\mathsf{c})} x_i$. Note that $\bar{s}_\lambda$ contains terms of degree greater than or equal to $|\lambda|$, and that the sum of all of its lowest-degree terms is equal to s_λ.

2.3 Graphs and Coloring

Here, we recall basic notions, terminology, and notations from graph theory. For further details, see, e.g., the textbooks [14, 34].

A ***graph*** G consists of a set V of ***vertices***, and a set E of unordered pairs of distinct vertices called ***edges***. All graphs in this paper are *simple*, so there are no loops, and no multi-edges. When $\{v_1, v_2\} \in E(G)$, we will typically denote this edge by v_1v_2 and say v_1 and v_2 are ***adjacent***. Two graphs G, G' are ***isomorphic*** if there is a bijection $\phi : V(G) \to V(G')$ such that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(G')$. In this paper, we consider graphs up to isomorphism.

The ***complete graph*** K_d with d vertices is the graph such that $V(K_d) = [d]$, and

$$E(K_d) = \{vw : v, w \in [d], v \neq w\}.$$

The n-vertex ***path*** P_n has vertex set $V(P_n) = [n]$ and edge set $E(P_n) = \{uv : u, v \in [n], v - u = 1\}$. The ***claw*** $K_{1,3}$ has vertex set $V(K_{1,3}) = [4]$ and edge set $E(K_{1,3}) = \{12, 13, 14\}$.

A ***stable set*** (or ***independent set***) of a graph G is a set $S \subseteq V(G)$ of vertices such that for each $v, w \in S$, $vw \notin E(G)$. A ***clique*** of a graph G is a set $S \subseteq V(G)$ of vertices such that for each $v \neq w \in S$, $vw \in E(G)$.

For $\alpha : V(G) \to \mathbb{N}$ a vertex weighting of the graph G, the ***α-clan graph of*** G is the graph $C_\alpha(G)$ obtained by blowing up each vertex v into a clique of $\alpha(v)$ vertices. More formally, $C_\alpha(G)$ has vertex set $V(C_\alpha(G)) = \{(v,i) : v \in V(G), i \in [\alpha(v)]\}$. In $C_\alpha(G)$, the vertices (v,i) and (w,j) are adjacent either if $vw \in E(G)$ or if both $v = w$ and $i \neq j$.

Given a vertex $v \in V(G)$, its ***open neighborhood*** $N(v)$ is defined by $N(v) = \{w : vw \in E(G)\}$. The ***contraction*** of a graph G by an edge $vw \in E(G)$, notated G/e, is the graph with vertex set $V(G/e) = V(G) \backslash \{v,w\} \cup \{v^*\}$, and edge set $E(G/e) = E(G) \backslash (vN(v) \cup wN(w)) \cup (v^*N(v) \cup v^*N(w))$.

A ***coloring*** of a graph G is a function $\kappa : V(G) \to \mathbb{N}$. A coloring κ of G is ***proper*** if $\kappa(a) \neq \kappa(b)$ whenever $ab \in E(G)$.

The ***chromatic symmetric function***[28] of a graph G is the power series

$$X_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

where the first sum ranges over all proper colorings κ of G. Note that, for every graph G, $X_G \in \mathrm{Sym}$.

3 The Kromatic Symmetric Function

A ***vertex-weighted graph*** (G, α) consists of a graph G together with a function $\alpha \colon V(G) \to \mathbb{N}$; we call α the ***weight function*** on the vertices of G. A ***proper α-coloring*** of G is a function $\kappa : V(G) \to 2^{\mathbb{N}} \backslash \{\emptyset\}$ assigning to each $v \in V(G)$ a set of $\alpha(v)$ distinct colors in $\mathbb{N}$, subject to the constraint that when $uv \in E(G)$, we have $\kappa(u) \cap \kappa(v) = \emptyset$. Note that these conditions are equivalent to saying that any choice of a single element from each $\kappa(v)$ yields a proper coloring of G. A ***proper set-coloring*** of G is a proper α-coloring for some weight function on the vertices of G.

The ***chromatic symmetric function*** of the vertex-weighted graph (G, α) is

$$X_G^\alpha = \sum_{\kappa} \prod_{v \in V(G)} \prod_{i \in \kappa(v)} x_i,$$

where the first sum runs over all proper α-colorings of G. Note that up to a scalar factor depending only on α, the chromatic symmetric function X_G^α equals the chromatic symmetric function $X_{C_\alpha(G)}$ of the α-clan graph of G.

Definition 3.1. The ***Kromatic symmetric function*** of a graph G is the symmetric power series

$$\overline{X}_G = \sum_{\alpha} X_G^\alpha,$$

where α ranges over all weightings of the vertex set $V(G)$.

In other words, $\overline{X}_G$ enumerates all colorings of G by nonempty sets of colors, such that adjacent vertices receive disjoint sets of colors. Note that $\overline{X}_G$ is not a homogeneous symmetric function, but rather consists of X_G plus terms of degree higher than $|V(G)|$.

Remark 3.2. Stanley [29] considered a function Y_G related to $\overline{X}_G$, although with two differences. Firstly, Y_G uses the rescaled power series $X_{C_\alpha(G)}$ in place of X_G^α. Secondly, Y_G allows $\alpha(v) = 0$, whereas the Kromatic symmetric function $\overline{X}_G$ only considers strictly positive vertex weightings. We are unaware of any further study of the functions Y_G since their introduction in [29].

Although vertex weightings are used in the definition of $\overline{X}_G$, the function $\overline{X}_G$ is independent of any particular vertex weighting. We will find it useful to also consider a vertex-weighted analogue of $\overline{X}_G$. Let α and w be independent vertex weightings on G. Define

$$X^\alpha_{(G,w)} = \sum_\kappa \prod_{v \in V(G)} \left(\prod_{i \in \kappa(v)} x_i \right)^{w(v)},$$

where again the first sum runs over all proper α-colorings of G. Finally, we define the ***vertex-weighted Kromatic symmetric function*** of the vertex-weighted graph (G, w) to be

$$\overline{X}_{(G,w)} = \sum_\alpha X^\alpha_{(G,w)},$$

where the sum is over all weight functions α.

For λ an integer partition, let K_λ denote the vertex-weighted complete graph $(K_{\ell(\lambda)}, w)$, where $w(i) = \lambda_i$ for each i. It is straightforward to see that $X_{K_\lambda} = \widetilde{m}_\lambda$, the augmented monomial symmetric function. Thus, by analogy, we define

$$\overline{\widetilde{m}}_\lambda := \overline{X}_{K_\lambda} = \sum_\alpha \widetilde{m}_{\lambda_1^{\alpha_1}, \ldots, \lambda_{\ell(\lambda)}^{\alpha(\ell(\lambda))}},$$

where the sum is over all vertex weightings α of $K_{\ell(\lambda)}$. We call $\overline{\widetilde{m}}_\lambda$ the ***K-theoretic augmented monomial symmetric function***. To justify this definition, we show that the Kromatic symmetric function of every graph (even every vertex-weighted graph) is a positive sum of K-theoretic augmented monomial symmetric functions.

First, we need some additional definitions. We define a ***stable set cover*** C of a graph G to be a collection of stable sets of G such that every vertex of $V(G)$ is in at least one element of C. In symbols, this means that

$$\bigcup_{S \in C} S = V(G);$$

note that this union is not required to be disjoint. We write $\mathsf{SSC}(G)$ for the family of all stable set covers of G. For $C \in \mathsf{SSC}(G)$, if G is endowed with a vertex-weighting w, let

$\lambda(C)$ be the partition of length $|C|$ whose parts are $\sum_{v \in S} w(v)$ for $S \in C$. Finally, let the ***color class*** of the color i in a proper set-coloring κ be $\{v \in V(G) : i \in \kappa(v)\}$, the set of vertices of G that receive color i under κ.

Proposition 3.3. *For any vertex-weighted graph (G, w), we have*

$$\overline{X}_{(G,w)} = \sum_{C \in \mathsf{SSC}(G)} \overline{\widetilde{m}}_{\lambda(C)}.$$

The Kromatic symmetric function for vertex-weighted graphs also admits a deletion-contraction relation, analogous to that of [9] for the chromatic symmetric function.

Given $S \subseteq V(G)$ and $v \in V(G)$ with $v \notin S$, we let $vS \subseteq E(G)$ denote $\{vs : s \in S\}$.

Proposition 3.4. *Let (G, w) be a vertex-weighed graph, and let v_1 and v_2 be vertices such that $e = v_1v_2 \notin E(G)$. Then*

$$\overline{X}_{(G,w)} = \overline{X}_{(G/e,w/e)} + \overline{X}_{(G\cup e,w)} + \overline{X}_{(G\cup e\cup v_1N(v_2),w^1)} + \overline{X}_{(G\cup e\cup v_2N(v_1),w^2)} + \overline{X}_{(G^*,w^*)} \tag{3.1}$$

where

- $w^1(v_1) = w(v_1) + w(v_2)$, *and for all other* $v \in V(G)$, $w^1(v) = w(v)$.
- $w^2(v_2) = w(v_1) + w(v_2)$, *and for all other* $v \in V(G)$, $w^2(v) = w(v)$.
- $G^* = (V(G) \cup v_3, E(G) \cup \{v_1v_2, v_1v_3, v_2v_3\} \cup v_3N(v_1) \cup v_3N(v_2))$.
- $w^*(v_3) = w(v_1) + w(v_2)$, *and for all other* $v \in V(G)$, $w^*(v) = w(v)$.

The deletion-contraction relation of Proposition 3.4 can be used to yield algorithmically the $\overline{\widetilde{m}}_\lambda$-expansion of a Kromatic symmetric function $\overline{X}_{(G,w)}$.

Corollary 3.5. *Arbitrarily applying Proposition 3.4 recursively to a graph (G, w) (repeatedly applying it to any nonedge of each graph formed if possible) terminates in a sum of Kromatic symmetric functions of vertex-weighted complete graphs, yielding the $\overline{\widetilde{m}}_\lambda$ expansion of $\overline{X}_{(G,w)}$.*

Our main result is the following theorem.

Theorem 3.6. *If G is a claw-free incomparability graph, then $\overline{X}_G$ is Grothendieck-positive.*

Proof. (Sketch.) We use the Jacobi-Trudi formula of Iwao [20, Proposition 4.4] to write a *dual Grothendieck function* as a sum of products of complete homogeneous symmetric functions. Then, for any graph G, the inner product of this expression with $\overline{X}_G$ yields a formula for the coefficient of $\overline{s}_\lambda$ in the Grothendieck expansion of $\overline{X}_G$ in terms of its monomial expansion. In the case that G is a claw-free incomparability graph, we then extend Gasharov's [15] theory of *P-arrays* to collect terms in this expansion and show that the coefficient of any $\overline{s}_\lambda$ is nonnegative. $\square$

We would be very interested in a solution to the following.

Problem 3.7. *For each claw-free incomparability graph G, find a corresponding subvariety V_G of the Grassmannian such that the cohomology class of V_G is represented in* Sym *by X_G and the structure sheaf class of V_G is represented by $\overline{X}_G$.*

4 Analogues of the Stanley–Stembridge conjecture

The previous section shows that Schur-positivity of X_G when G is a claw-free incomparability graph lifts to an analogue for $\overline{X}_G$. It is natural to ask if it is similarly possible to lift the ***Stanley-Stembridge conjecture*** — claiming that such X_G are e-positive — to the Kromatic setting. However, it appears that the answer is "no."

We propose two definitions for a lift of the e-basis to the K-theoretic setting. On one hand, e-basis elements in usual symmetric function theory may be defined in terms of fillings of single-column Young diagrams, so we may lift this formula.

Definition 4.1. The ***tableau K-elementary symmetric function*** $\overline{e}_\lambda$ is given by

$$\overline{e}_n = \overline{s}_{1^n} \quad \text{and} \quad \overline{e}_\lambda = \overline{e}_{\lambda_1} \dots \overline{e}_{\lambda_{\ell(\lambda)}}.$$

On the other hand, we may also define $e_n = \frac{1}{n!} X_{K_n}$, and lift this characterization.

Definition 4.2. The ***graph K-elementary symmetric function*** is given by

$$\overline{e}'_n = \frac{1}{n!} \overline{X}_{K_n} \quad \text{and} \quad \overline{e}'_\lambda = \overline{e}'_{\lambda_1} \dots \overline{e}'_{\lambda_{\ell(\lambda)}}.$$

It is reasonable to hope (for extending the Stanley–Stembridge conjecture) that $\overline{X}_G$ is positive in one of these K-theoretic e-bases, whenever G is a claw-free incomparability graph, or even just when G is a unit interval graph. However, one can compute that $\overline{X}_{P_3}$ is not positive in either K-theoretic e-basis, dashing such hopes.

The terms of $\overline{X}_{P_3}$ that are homogeneous of degree 3 must come from tableau or graph K-elementary symmetric functions of degree 3, and have coefficients corresponding to e-expansion of X_{P_3}. Since $X_{P_3} = 3e_3 + e_{21}$, one sees that the terms of $\overline{X}_{P_3}$ for $|\lambda| = 3$ in the $\overline{e}$-basis are $3\overline{e}_3 + \overline{e}_{21}$, and in the $\overline{e}'$-basis are $3\overline{e}'_3 + \overline{e}'_{21}$. However, we now run into problems with the $|\lambda| = 4$ terms. In particular, both $\overline{e}_{21}$ and $\overline{e}'_{21}$ are supported on the monomial $x_1^2 x_2^2$, with two distinct variables each of degree 2. However, it is easy to check that there is no proper set-coloring of P_3 using exactly 1 twice and 2 twice; thus, these monomials must be cancelled by $\overline{e}$ or $\overline{e}'$ terms with strictly negative coefficients.

That this breakdown is so fundamental suggests that it may not be possible to reasonably generalize e-positivity to the Kromatic symmetric function, in stark contrast with the generalization of Schur-positivity given in Theorem 3.6. This suggests that the Stanley–Stembridge is not amenable to a topological interpretation along the lines of Problem 3.7.

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References

[1] A. Abreu and A. Nigro. "Chromatic symmetric functions from the modular law". *Journal of Combinatorial Theory, Series A* **180** (2021), p. 105407.

[2] P. Alexandersson and R. Sulzgruber. "A combinatorial expansion of vertical-strip LLT polynomials in the basis of elementary symmetric functions". *Advances in Mathematics* **400** (2022), p. 108256.

[3] F. Aliniaeifard, V. Wang, and S. van Willigenburg. "The chromatic symmetric function of a graph centred at a vertex". 2021. arXiv:2108.04850.

[4] G. D. Birkhoff. "A determinant formula for the number of ways of coloring a map". *The Annals of Mathematics* **14**.1/4 (1912), pp. 42–46.

[5] A. S. Buch. "A Littlewood-Richardson rule for the K-theory of Grassmannians". *Acta Math.* **189**.1 (2002), pp. 37–78.

[6] S. Cho and J. Hong. "Positivity of Chromatic Symmetric Functions Associated with Hessenberg Functions of Bounce Number 3". *The Electronic Journal of Combinatorics* (2022), P2–19.

[7] L. Colmenarejo, A. H. Morales, and G. Panova. "Chromatic symmetric functions of Dyck paths and q-rook theory". *European Journal of Combinatorics* **107** (2023), p. 103595.

[8] L. Crew, O. Pechenik, and S. Spirkl. "The Kromatic Symmetric Function: A K-theoretic Analogue of X_G". 2023.

[9] L. Crew and S. Spirkl. "A deletion–contraction relation for the chromatic symmetric function". *European Journal of Combinatorics* **89** (2020), p. 103143.

[10] S. Dahlberg. "A new formula for Stanley's chromatic symmetric function for unit interval graphs and e-positivity for triangular ladder graphs". *Sém. Lothar. Combin* **82** (2019).

[11] S. Dahlberg, A. Foley, and S. van Willigenburg. "Resolving Stanley's e-positivity of claw-contractible-free graphs". *J. Eur. Math. Soc. (JEMS)* **22**.8 (2020), pp. 2673–2696.

[12] S. Dahlberg and S. van Willigenburg. "Lollipop and lariat symmetric functions". *SIAM Journal on Discrete Mathematics* **32**.2 (2018), pp. 1029–1039.

[13] S. Dahlberg and S. van Willigenburg. "Chromatic symmetric functions in noncommuting variables revisited". *Advances in Applied Mathematics* **112** (2020), p. 101942.

[14] R. Diestel. *Graph theory*. Fifth ed. Vol. 173. Graduate Texts in Mathematics. Springer, Berlin, 2017, pp. xviii+428.

[15] V. Gasharov. "Incomparability graphs of (3+ 1)-free posets are s-positive". *Discrete Mathematics* **157**.1-3 (1996), pp. 193–197.

[16] D. D. Gebhard and B. E. Sagan. "A chromatic symmetric function in noncommuting variables". *Journal of Algebraic Combinatorics* **13**.3 (2001), pp. 227–255.

[17] M. Guay-Paquet. "A modular relation for the chromatic symmetric functions of $(3+1)$-free posets". 2013. arXiv:1306.2400.

[18] A. M. Hamel, C. T. Hoàng, and J. E. Tuero. “Chromatic symmetric functions and H-free graphs”. *Graphs and Combinatorics* **35**.4 (2019), pp. 815–825.

[19] B.-H. Hwang. “Chromatic quasisymmetric functions and noncommutative P-symmetric functions”. 2022. arXiv:2208.09857.

[20] S. Iwao. “Grothendieck polynomials and the boson-fermion correspondence”. *Algebraic Combinatorics* **3**.5 (2020), pp. 1023–1040.

[21] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, 1998.

[22] L. Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci*. Vol. 6. SMF/AMS Texts and Monographs. Translated from the 1998 French original by John R. Swallow, Cours Spécialisés, 3. American Mathematical Society, Providence, RI and Société Mathématique de France, Paris, 2001, pp. viii+167.

[23] C. Monical, O. Pechenik, and D. Searles. “Polynomials from combinatorial K-theory”. *Canad. J. Math.* **73**.1 (2021), pp. 29–62.

[24] G. Nenashev and B. Shapiro. ““K-theoretic” analog of Postnikov-Shapiro algebra distinguishes graphs”. *J. Combin. Theory Ser. A* **148** (2017), pp. 316–332.

[25] O. Pechenik and A. Yong. “Genomic tableaux”. *J. Algebraic Combin.* **45**.3 (2017), pp. 649–685.

[26] A. Postnikov and B. Shapiro. “Trees, parking functions, syzygies, and deformations of monomial ideals”. *Trans. Amer. Math. Soc.* **356**.8 (2004), pp. 3109–3142.

[27] J. Shareshian and M. L. Wachs. “Chromatic quasisymmetric functions”. *Advances in Mathematics* **295** (2016), pp. 497–551.

[28] R. P. Stanley. “A symmetric function generalization of the chromatic polynomial of a graph”. *Advances in Mathematics* **111**.1 (1995), pp. 166–194.

[29] R. P. Stanley. “Graph colorings and related symmetric functions: ideas and applications A description of results, interesting applications, & notable open problems”. *Discrete Mathematics* **193**.1-3 (1998), pp. 267–286.

[30] R. P. Stanley and S Fomin. “Enumerative combinatorics. Vol. 2, volume 62 of”. *Cambridge Studies in Advanced Mathematics* (1999).

[31] R. P. Stanley and J. R. Stembridge. “On immanants of Jacobi-Trudi matrices and permutations with restricted position”. *J. Combin. Theory Ser. A* **62**.2 (1993), pp. 261–279. DOI.

[32] H. Thomas and A. Yong. “A jeu de taquin theory for increasing tableaux, with applications to K-theoretic Schubert calculus”. *Algebra Number Theory* **3**.2 (2009), pp. 121–148.

[33] F. Tom. Private communication to L. Crew and S. Spirkl. Zoom Meeting, 2021.

[34] D. B. West. *Combinatorial mathematics*. Cambridge University Press, Cambridge, 2021, pp. xx+969.