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Foata-like bijections and science fiction

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Abstract. A central open problem in algebraic combinatorics is to find a combinatorial formula for the Kostka-Macdonald polynomials $\widetilde{K}_{\lambda\mu}(q,t)$, which describe the expansion of the Macdonald polynomial $\widetilde{H}_\mu(Z;q,t)$ in the Schur basis. Haiman proved that $\widetilde{K}_{\lambda\mu}(q,t)$ has nonnegative integer coefficients by proving that the dimension of the Garsia-Haiman module $\mathcal{H}_\mu$ equals $n!$, demonstrating an intricate relationship between the Kostka-Macdonald polynomials and this module. This relationship was further expounded upon by Bergeron and Garsia, whose "science fiction" heuristics conjecture certain intersection properties of Garsia-Haiman modules which mirror observed symmetries in the Kostka-Macdonald coefficients. The most potent of these heuristics is the $\frac{n!}{k}$ conjecture, which asserts that the dimension of the intersection of k Garsia-Haiman modules should have dimension $\frac{n!}{k}$. We solve the special case of the $\frac{n!}{k}$ conjecture where the indexing partitions have hook shape by constructing an explicit basis for the intersection, using two maps in the spirit of Foata's.

Keywords: Macdonald polynomials, Garsia-Haiman modules, bijective combinatorics

1 Introduction

The (transformed) Macdonald polynomials $\{\widetilde{H}_\mu(Z;q,t) : \mu \text{ an integer partition}\}$ form a notable basis for the ring of symmetric functions, as they simultaneously generalize the Schur functions, Hall-Littlewood polynomials, and Jack symmetric functions, among others. As such, they have become a central object of study since their introduction by Macdonald in [11]. Of particular interest is their expansion in the basis of Schur functions $\{s_\lambda(Z)\}$:

$$\widetilde{H}_\mu(Z;q,t) = \sum_\lambda \widetilde{K}_{\lambda\mu}(q,t) s_\lambda(Z).$$

A priori we have $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{Q}(q,t)$, but Macdonald's *positivity theorem* asserts that $\widetilde{K}_{\lambda\mu}(q,t)$ is in fact a polynomial in q and t with nonnegative integer coefficients. It remains an important open problem to provide a combinatorial proof of Macdonald positivity — e.g. writing $\widetilde{K}_{\lambda\mu}(q,t)$ as a (q,t)-weighted sum over some set of combinatorial objects — but Haiman [10] resolved the positivity theorem algebraically, using a representation-theoretic analog $\mathcal{H}_\mu$ of $\widetilde{H}_\mu(Z;q,t)$.

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For μ an integer partition of $n \in \mathbb{N}$, the *Garsia-Haiman module* $\mathcal{H}_\mu$ is a bigraded symmetric group module defined as follows (see [7] for full details). Let $X = x_1, x_2, \ldots, x_n$, $Y = y_1, y_2, \ldots, y_n$, and define

$$\mathfrak{I}_\mu = \{f \in \mathbb{Q}[X,Y] : f(\tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_n}; \tfrac{\partial}{\partial y_1}, \ldots, \tfrac{\partial}{\partial y_n})\Delta_\mu = 0\}$$

to be the ideal of polynomials whose corresponding differential operator annihilates a certain $\mathfrak{S}_n$-alternating, doubly homogeneous polynomial $\Delta_\mu \in \mathbb{Q}[X,Y]$, so that $\mathfrak{I}_\mu$ is $\mathfrak{S}_n$-invariant. The Garsia-Haiman module is defined as $\mathcal{H}_\mu = \mathbb{Q}[X,Y]/\mathfrak{I}_\mu$; it inherits the diagonal action of $\mathfrak{S}_n$ on $\mathbb{Q}[X,Y]$, and is afforded a bigrading since Δ_μ is doubly homogeneous, so we may write

$$\mathcal{H}_\mu = \bigoplus_{i,j} (\mathcal{H}_\mu)_{i,j}.$$

Haiman made the following connection between $\mathcal{H}_\mu$ and the Macdonald polynomials:

Theorem 1 ([9])**.** *If the dimension of $\mathcal{H}_\mu$ is $n!$, then the bigraded Frobenius series of $\mathcal{H}_\mu$ given by*

$$Frob_{\mathcal{H}_\mu}(Z; q, t) = \sum_{i,j} q^i t^j ch((\mathcal{H}_\mu)_{i,j}),$$

where ch is the Frobenius map which sends the irreducible $\mathfrak{S}_n$-module S^λ to the Schur function s_λ, equals the transformed Macdonald polynomial $\widetilde{H}_\mu(Z; q, t)$.

Due to known identities involving Macdonald polynomials it is necessary that $\mathcal{H}_\mu$ afford a bigraded version of the regular representation, hence the dimension condition, but in fact this is also sufficient.

Haiman proved that $\dim(\mathcal{H}_\mu) = n!$ in [10], thus resolving the Macdonald positivity theorem by realizing $\widetilde{K}_{\lambda\mu}(q,t)$ as encoding the doubly graded multiplicity of the irreducible $\mathfrak{S}_n$-module S^λ in $\mathcal{H}_\mu$. Haiman's proof, however, is algebro-geometric in nature, and does not afford an explicit basis for $\mathcal{H}_\mu$. The construction of such a basis in general remains an open problem, although some special cases have been solved, in particular when μ has hook shape (see [1],[2],[3],[4],[8],[12]).

With an eye towards a combinatorial proof of Macdonald positivity, it is desirable to study $\mathcal{H}_\mu$ more combinatorially. Along these lines, Bergeron and Garsia undertook a speculative study of some of the remarkable intersection properties possessed by the modules $\mathcal{H}_\mu$ in [5]. Their most potent assertion is the following $\frac{n!}{k}$ *conjecture*:

Conjecture 1 ([5])**.** *Let λ be an integer partition of $n+1$, and let $\mu^{(1)}, \ldots, \mu^{(k)}$ be partitions of n each obtained from λ by removing a removable cell from the Young diagram of λ. Then,*

$$\dim\left(\bigcap_{i=1}^{k} \mathcal{H}_{\mu^{(i)}}\right) = \frac{n!}{k}.$$

In the case that λ has hook shape — i.e. $\lambda = (a, 1, 1, \dots 1) = (a, 1^\ell)$ for some $a \geq 2, \ell \geq 1$ — we resolve the $\frac{n!}{k}$ conjecture by constructing an explicit basis for $\mathcal{H}_{(a,1^{\ell-1})} \cap \mathcal{H}_{(a-1,1^\ell)}$.

2 The $\frac{n!}{k}$ conjecture for hook shapes

Adin, Remmel and Roichman define a plethora of different bases for $\mathcal{H}_\mu$ when μ has hook shape in [1]. To tackle the $\frac{n!}{k}$ conjecture in this special case, we adopt a different viewpoint on their *Artin basis*. This basis is indexed by permutations $w \in \mathfrak{S}_n$ and each basis element encodes information about certain inversions in w which depend on the partition μ. We restate this basis in terms of *standard fillings* of μ.

Throughout, we identify a partition $\mu = (\mu_1 \geq \mu_2 \geq \cdots)$ with its Young diagram, opting for French (coordinate) notation, so that μ is depicted diagramatically as an array of bottom- and left-justified cells with μ_i cells in row i. We call a cell $c \in \mu$ *removable* if the diagram $\mu - \{c\}$ still has partition shape.

Definition 1. *For any partition μ of $n \in \mathbb{N}$, a* standard filling *of μ is a bijective map $S : \mu \to \{1, 2, \dots, n\}$, i.e. an assignment of the numbers $1, 2, \dots, n$ to the cells of (the Young diagram of) μ. The set of all standard fillings of μ is denoted $SF(\mu)$.*

By assigning any arbitrary order to the cells of μ, we obtain a bijection between $SF(\mu)$ and $\mathfrak{S}_n$, i.e. $|SF(\mu)| = n!$. For instance, the $3! = 6$ standard fillings of $\mu = (2, 1)$ are:

$$\begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 1 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\cline{1-1} 3 & \\ \hline 2 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\cline{1-1} 1 & \\ \hline 2 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\cline{1-1} 2 & \\ \hline 3 & \multicolumn{1}{c|}{1} \\ \hline\end{array}\,, \quad \begin{array}{|c|c}\cline{1-1} 1 & \\ \hline 3 & \multicolumn{1}{c|}{2} \\ \hline\end{array}\,.$$

For any standard filling S, we let $S_{i,j}$ denote the entry in row i, column j of S. We now hone in on the case where $\mu = (a, 1^\ell)$ has hook shape.

Definition 2. *Given $S \in SF(a, 1^\ell)$, define a* row inversion *to be a pair $S_{1,i} > S_{1,j}$ where $i < j$. Similarly, define a* column inversion *to be a pair $S_{j,1} > S_{i,1}$ for $i < j$ (note the swapped indices). Denote the collection of row (resp. column) inversions in S by*

$$\begin{aligned} \mathsf{rowInv}(S) &= \{(t, r) : t > r, t \text{ left of } r \text{ in row } 1 \text{ of } S\}, \\ \mathsf{colInv}(S) &= \{(d, c) : d > c, d \text{ above } c \text{ in column } 1 \text{ of } S\}. \end{aligned}$$

Definition 3. *Given $S \in SF(a, 1^\ell)$, define the polynomial $\varphi_S(X, Y) \in \mathbb{Q}[X, Y]$ by*

$$\varphi_S(X, Y) = \prod_{(d,c) \in \mathsf{colInv}(S)} x_d \prod_{(t,r) \in \mathsf{rowInv}(S)} y_r \tag{2.1}$$

(cf. [1], Definition 1.6).

This definition is seen to be equivalent to [1] by associating a standard filling S of μ with its reading word $w_S \in \mathfrak{S}_n$. Note also that $\varphi_S(X,Y)$ encodes only the information of the *larger* elements in column inversions, and of the *smaller* elements in row inversions:

Example 1. *The following standard filling of $\mu = (5,1^4)$:*

$$S = \begin{array}{|c|cccc}\hline 4 & \multicolumn{4}{c}{} \\ \cline{1-1} 1 & \multicolumn{4}{c}{} \\ \cline{1-1} 9 & \multicolumn{4}{c}{} \\ \cline{1-1} 7 & \multicolumn{4}{c}{} \\ \hline 5 & \multicolumn{1}{|c|}{6} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{8} \\ \hline \end{array}$$

has

$$\mathsf{colInv}(S) = \{(\mathbf{4},1),(\mathbf{9},7),(\mathbf{9},5),(\mathbf{7},5)\},$$
$$\mathsf{rowInv}(S) = \{(5,\mathbf{3}),(5,\mathbf{2}),(6,\mathbf{3}),(6,\mathbf{2}),(3,\mathbf{2})\},$$

so $\varphi_S(X,Y) = x_4x_7x_9^2y_2^3y_3^2$.

Theorem 2 ([1], Corollary 1.7)**.** *The set*

$$\{\varphi_S(X,Y) : S \in SF(a,1^\ell)\}$$

forms a basis for the Garsia-Haiman module $\mathcal{H}_{(a,1^\ell)}$.

Equipped with this basis, we prove the following special case of the $\frac{n!}{k}$ conjecture:

Theorem 3 ($\frac{n!}{k}$ for hook shapes)**.** *The $\frac{n!}{k}$ conjecture holds for the hook shape $\lambda = (a,1^\ell)$, i.e.*

$$\dim(\mathcal{H}_{(a,1^{\ell-1})} \cap \mathcal{H}_{(a-1,1^\ell)}) = \frac{n!}{2}.$$

As a hook shape has at most two removable boxes (one at the end of the first row, and one at the end of the first column), this result is as strong as possible for hook shapes since the case $k \geq 3$ is trivial.

The Artin basis of [1] is as "nice" as can be in the sense that, not only do we prove that $\mathcal{H}_{(a,1^{\ell-1})} \cap \mathcal{H}_{(a-1,1^\ell)}$ has the correct dimension, we actually obtain an explicit basis for the intersection by finding exactly $\frac{n!}{2}$ standard fillings of $(a,1^{\ell-1})$ and of $(a-1,1^\ell)$ which yield the same basis element.

To ease the notational burden, we fix once and for all the partitions $\mu = (a,1^{\ell-1})$ and $\rho = (a-1,1^\ell)$, for some $a \geq 2$ and $\ell \geq 1$. Define the following subsets of $SF(\mu)$ and $SF(\rho)$:

$$SF_<(\mu) = \{S \in SF(\mu) : S_{1,1} < S_{1,a}\}, \quad SF_<(\rho) = \{T \in SF(\rho) : T_{\ell+1,1} < T_{1,1}\}. \tag{2.2}$$

Then clearly $|SF_<(\mu)| = |SF_<(\rho)| = \frac{n!}{2}$. We show in turn that

$$\{\varphi_S(X,Y) : S \in SF_<(\mu)\} = \{\varphi_T(X,Y) : T \in SF_<(\rho)\} \tag{2.3}$$

by defining a bijective map $\theta : SF_<(\mu) \to SF_<(\rho)$ which satisfies $\varphi_{\theta(S)} = \varphi_S$.

Heuristically, the standard fillings in (2.2) are the "right" candidates for a basis of $\mathcal{H}_\mu \cap \mathcal{H}_\rho$ because no standard filling of ρ has a row inversion whose elements are a units apart, and conversely no standard filling of μ has a column inversion whose elements are $\ell + 1$ units apart.

As a first naïve step, we may define a simple auxiliary map which translates between standard fillings of μ and of ρ:

Definition 4. *For any $S \in SF(\mu)$, define a standard filling $\mathsf{bump}(S) \in SF(\rho)$ by moving each entry in the first column of S up one row, and then pushing each remaining entry in the first row to the left one column.*

Then $\mathsf{bump} : SF(\mu) \to SF(\rho)$ is a bijection whose inverse is given by first pushing the first row to the right, and then pushing the first column down. The problem, however, is that bump does not restrict to a bijection $SF_<(\mu) \to SF_<(\rho)$:

Example 2. *For $\mu = (4, 1^3)$ and $\rho = (3, 1^4)$, we have*

$$\mathsf{bump}\left(\begin{array}{|c|c|c|c|} \cline{1-1} 5 \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \hline 6 & 4 & 3 & 7 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|} \cline{1-1} 5 \\ \cline{1-1} 1 \\ \cline{1-1} 2 \\ \cline{1-1} 6 \\ \hline 4 & 3 & 7 \\ \hline \end{array} .$$

Then $S \in SF_<(\mu)$ since $6 < 7$, yet $\mathsf{bump}(S) \notin SF_<(\rho)$ since $5 > 4$.

In addition to correcting this deficiency, we must also ensure that the row and column inversions in the resulting standard filling align with the original. Thus after performing bump, we must take the following into consideration:

1. There may be row inversions in S, created by $u = S_{1,1}$, which are no longer present in $\mathsf{bump}(S)$. We thus must rearrange the first row so as to reintroduce these lost row inversions.

2. After rearranging the first row of $\mathsf{bump}(S)$, there is a new entry v in row 1, column 1, so we may have introduced some column inversions which were not present in S. Thus we must rearrange the first column both in order to negate these newly introduced column inversions, and in order to ensure that the $(\ell + 1, 1)$-entry is less than the $(1, 1)$-entry, so that the resulting filling lies in $SF_<(\rho)$.

To this end, we define a map arm_u which reintroduces the row inversions removed in 1., and a map leg_v which negates the column inversions introduced in 2.; by appropriately composing these maps with bump, we obtain a map which proves (2.3).

Definition 5. *For a fixed $u \in \mathbb{N}$, consider the set of words*

$$\mathcal{A}_u = \{w_1 w_2 \cdots w_m : m \in \mathbb{N}, w_m > u\}.$$

For any $w = w_1 w_2 \cdots w_m \in \mathcal{A}_u$, define a new word $arm_u(w)$ as follows: let $b_1, b_2, \ldots, b_j$ denote the indices for which $w_{b_i} < u$. If there are no such indices, then define $\mathsf{arm}_u(w) = w$. Otherwise,

1. *Draw a vertical bar immediately to the left of w_{b_i} if either $w_{b_i - 1} > u$, or if $i = 1$ and $b_1 = 1$.*
2. *Within each newly created block which contains at least one of the w_{b_i}'s, move the leftmost entry which is $> u$ to the front of the block, immediately right of the (leftmost) vertical bar.*

Define the resulting word to be $\mathsf{arm}_u(w)$.

In context, for $S \in SF_<(\mu)$ and $u = S_{1,1}$, we wish to perform arm_u on the word $S_{1,2} S_{1,3} \cdots S_{1,a}$ consisting of the entries in the first row of S which lie strictly right of the first column (note that by construction this word lies in $\mathcal{A}_u$). We then determine which entries in row 1 of S are $< u$, as these correspond precisely to the row inversions in S that are lost in $\mathsf{bump}(S)$ when u gets bumped into the second row. A larger number is then shuffled to the left of each of these entries in order to reintroduce the lost row inversions.

Example 3. *Let $w = 49263187 \in \mathcal{A}_5$. To compute $\mathsf{arm}_5(w)$ we draw vertical bars to the left of 4, 2, and 3, and within each block shuffle the leftmost number > 5 to the front:*

$$\big|\,4\ \ 9\,\big|\,2\ \ 6\,\big|\,3\ \ 1\ \ 8\ \ 7 \qquad \longmapsto \qquad \big|\,9\ \ 4\,\big|\,6\ \ 2\,\big|\,8\ \ 3\ \ 1\ \ 7$$

So, $\mathsf{arm}_5(w) = 94628317$. Note also that in the word $5w = 549263187$, we have inversions $5 > \mathbf{4}, 5 > \mathbf{2}, 5 > \mathbf{3}, 5 > \mathbf{1}$ which are not present in w. However, when we perform arm_5 on w, we obtain new inversions $9 > \mathbf{4}, 6 > \mathbf{2}, 8 > \mathbf{3}, 8 > \mathbf{1}$ and no others, so the smaller entries in the inversions of $5w$ and of $\mathsf{arm}_5(w)$ are the same.

After shuffling around the first row of $\mathsf{bump}(S)$ to correct for the lost row inversions, we must appropriately reorder the first column, for which we need the following map.

Definition 6. *For a fixed $v \in \mathbb{N}$, consider the set of words*

$$\mathcal{L}_v = \{w_1 w_2 \cdots w_m : m \in \mathbb{N}, w_1 < v\}.$$

For any $w = w_1 w_2 \cdots w_m \in \mathcal{L}_v$, define a new word $\mathsf{leg}_v(w)$ as follows: let $c_1, c_2, \ldots, c_k$ denote the indices for which $w_{c_i} > v$. If there are no such indices, then define $\mathsf{leg}_v(w) = w$. Otherwise,

1. *Draw a vertical bar immediately to the right of w_{c_i} if either $w_{c_i+1} < v$, or if $i = k$ and $c_k = m$.*

2. *Within each newly created block which contains at least one of the w_{c_i}'s, move the rightmost entry which is $< v$ to the end of the block, immediately left of the (rightmost) vertical bar.*

Define the resulting word to be $\mathsf{leg}_v(w)$.

Note that leg_v is in a sense "dual" to arm_u, in that arm_u shuffles larger entries to the left, and leg_v shuffles smaller entries to the right. Again, for a bit of context, we wish to perform leg_v on the first column of a filling after performing bump, and then $\mathsf{arm}_{S_{1,1}}$, on some $S \in SF_<(\mu)$, when a new entry is introduced into row 1, column 1. Denoting the resulting filling by T, we will have $v = T_{1,1} < T_{2,1}$ by construction, so that $T_{2,1}T_{3,1} \cdots T_{\ell+1,1} \in \mathcal{L}_v$, and leg_v then shuffles smaller numbers further up in the first column to negate the column inversions created by v.

Example 4. *Let $w = 48731926 \in \mathcal{L}_5$. To compute $leg_5(w)$ we draw vertical bars to the right of $7, 9$, and 6, and within each block shuffle the rightmost number < 5 to the end:*

$$4\ \ 8\ \ 7\,\big|\,3\ \ 1\ \ 9\,\big|\,2\ \ 6\,\big| \quad \longmapsto \quad 8\ \ 7\ \ 4\,\big|\,3\ \ 9\ \ 1\,\big|\,6\ \ 2\,\big|$$

So, $\mathsf{leg}_5(48731926) = 87439162$. *Since in practice we read the entries in the first column from bottom to top, the* coinversions *in the resulting word will correspond to column inversions. With this in mind, note that word $5w$ has coinversions $5 < \mathbf{8}, 5 < \mathbf{7}, 5 < \mathbf{9}, 5 < \mathbf{6}$ which are not present in w. However, when we perform* leg_5 *on w, we* remove *the coinversions $4 < \mathbf{8}, 4 < \mathbf{7}, 1 < \mathbf{9}, 2 < \mathbf{6}$ and no others, so the larger entries in the coinversions of $5w$ and of* $\mathsf{leg}_5(w)$ *agree.*

Remark 1. *Note the similarity of* arm_u *and* leg_v *to the maps γ_x introduced by Foata [6], which he used to bijectively prove that the major index and inversion number are equidistributed over $\mathfrak{S}_n$. We point out that these maps are not, in fact equivalent; one immediate difference is the manner in which the respective maps factor a word w into subwords. In Example 3, for instance, Foata's map γ_5 would factor w as $|49|26|3|187$ and cyclicly rotate the entries within each block. When performing* arm_5 *on w, however, the factorization is different and the manner in which we reorder each block is as well.*

By composing these maps, we obtain our desired bijection $\theta : SF_<(\mu) \to SF_<(\rho)$:

Definition 7. *Given $S \in SF_<(\mu)$, let $u = S_{1,1}$, and let v be the leftmost entry in the first row of S which is $> u$. Then define*

$$\theta(S) = \mathsf{leg}_v \circ \mathsf{arm}_u \circ \mathsf{bump}(S),$$

where arm_u *acts only on the first row of* $\mathsf{bump}(S)$ *— that is, we replace the first row of* $\mathsf{bump}(S)$ *with* $\mathsf{arm}_u(S_{1,2}S_{1,3}\cdots S_{1,a})$ *— and* leg_v *acts on the first column of* $\mathsf{arm}_u(\mathsf{bump}(S))$*, strictly above the first row, so that these entries are replaced by* $\mathsf{leg}_v(S_{1,1}S_{2,1}\cdots S_{\ell,1})$*, entered from bottom to top.*

Since $S_{1,a} > u$ and $u < v$ for any $S \in SF_<(\mu)$, each step in the composition is defined; thus θ is well-defined. Furthermore we have $\theta(S)_{\ell+1,1} < v = \theta(S)_{1,1}$ by construction, so that $\theta(S) \in SF_<(\rho)$.

Example 5. *Let* $\mu = (5, 1^4)$ *and* $\rho = (4, 1^5)$*, and let*

$$S = \begin{ytableau} 9 \\ 1 \\ 7 \\ 4 \\ 5 & 6 & 3 & 2 & 8 \end{ytableau} \in SF_<(\mu).$$

Then in the language of the above definition, we have $u = 5, v = 6$*, so that*

$$\theta(S) = \mathsf{leg}_6 \circ \mathsf{arm}_5 \circ \mathsf{bump}(S) = \mathsf{leg}_6 \circ \mathsf{arm}_5 \left(\begin{ytableau} 9 \\ 1 \\ 7 \\ 4 \\ 5 \\ 6 & 3 & 2 & 8 \end{ytableau} \right)$$

$$= \mathsf{leg}_6 \left(\begin{ytableau} 9 \\ 1 \\ 7 \\ 4 \\ 5 \\ 6 & 8 & 3 & 2 \end{ytableau} \right) = \begin{ytableau} 1 \\ 9 \\ 4 \\ 7 \\ 5 \\ 6 & 8 & 3 & 2 \end{ytableau} \in SF_<(\rho).$$

Note that $\varphi_S = \varphi_{\theta(S)} = x_7^2 x_9^4 y_2^3 y_3^2$.

Finally, we have:

Theorem 4. *For partitions* $\mu = (a, 1^{\ell-1})$ *and* $\rho = (a-1, 1^{\ell})$*, the map* $\theta : SF_<(\mu) \to SF_<(\rho)$ *defined above satisfies* $\varphi_S = \varphi_{\theta(S)}$ *for every* $S \in SF_<(\mu)$*. Furthermore* θ *is a bijection, and* $\dim(\mathcal{H}_\mu \cap \mathcal{H}_\rho) \geq \frac{n!}{2}$.

Proof. (Sketch) By construction, we have

$$\{r : (t, r) \in \mathsf{rowInv}(S) \text{ for some } t\} = \{r' : (t', r') \in \mathsf{rowInv}(\theta(S)) \text{ for some } t'\}$$

and

$$\{d : (d, c) \in \mathsf{colInv}(S) \text{ for some } c\} = \{d' : (d', c') \in \mathsf{colInv}(\theta(S)) \text{ for some } c'\},$$

and these sets uniquely determine φ_S (resp. $\varphi_{\theta(S)}$, so $\varphi_S = \varphi_{\theta(S)}$.

We furthermore have

$$\theta(S) = \theta(T) \Rightarrow \varphi_{\theta(S)} = \varphi_{\theta(T)} \Rightarrow \varphi_S = \varphi_T \Rightarrow S = T,$$

so θ is injective and therefore also bijective, thus

$$\{\varphi_S : S \in SF_<(\mu)\} \subset \mathcal{H}_\mu \cap \mathcal{H}_\rho.$$

□

The reverse assertion, that $\dim(\mathcal{H}_\mu \cap \mathcal{H}_\rho) \leq \frac{n!}{2}$, follows from a comparatively clunkier argument.

Theorem 5. *For $\mu = (a, 1^{\ell-1}), \rho = (a-1, 1^\ell)$, we have $\dim(\mathcal{H}_\mu \cap \mathcal{H}_\rho) \leq \frac{n!}{2}$.*

Proof. (Sketch) We show that $\varphi_S \notin \mathcal{H}_\rho$ for any $S \in SF(\mu) - SF_<(\mu)$ by demonstrating that from any such polynomial φ_S, one can infer that there must be at least a distinct entries in the first row of S, i.e. that φ_S can not be achieved from any filling in $SF(\rho)$ (resp. $\varphi_T \notin \mathcal{H}_\mu$ for any $T \in SF(\rho) - SF_<(\rho)$ by an analogous argument). □

Combining Theorems 4 and 5 proves this special case of the $\frac{n!}{k}$ conjecture.

Remark 2. *Garsia and Haiman noted in [7] that one may define the Garsia-Haiman module as*

$$D_\mu = span\{f(\tfrac{\partial}{\partial x_1}, \ldots, \tfrac{\partial}{\partial x_n}; \tfrac{\partial}{\partial y_1}, \ldots, \tfrac{\partial}{\partial y_n})\Delta_\mu : f \in \mathbb{Q}[X, Y]\},$$

the span of all partial derivatives of all orders of Δ_μ, and an isomorphism $\mathcal{H}_\mu \xrightarrow{\sim} sD_\mu$ is given by $f(X, Y) \mapsto f(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y})\Delta_\mu$.

It is worth remarking that the "science fiction" heuristics in [5], which establish remarkable connections between Garsia-Haiman modules and the Kostka-Macdonald coefficients $\widetilde{K}_{\lambda\mu}(q,t)$, are formulated in terms of D_μ, rather than $\mathcal{H}_\mu$. The restriction of the Artin basis to $\mathcal{H}_\mu \cap \mathcal{H}_\rho$ does not immediately yield a basis for $D_\mu \cap D_\rho$ in the sense that

$$\{\varphi_S(\tfrac{\partial}{\partial X}, \tfrac{\partial}{\partial Y})\Delta_\mu : S \in SF_<(\mu)\} \neq \{\varphi_T(\tfrac{\partial}{\partial X}, \tfrac{\partial}{\partial Y})\Delta_\rho : T \in SF_<(\rho)\}.$$

In fact, experimental evidence suggests that these sets are disjoint.

As the above isomorphism is bidegree-complementing, the optimistic (some would say naïve) conjecture would be that the complement $SF(\mu) - SF_<(\mu)$ (resp. ρ) indexes a basis for $D_\mu \cap D_\rho$. However, it is not true in general that $\varphi_S(\frac{\partial}{\partial X}, \frac{\partial}{\partial Y})\Delta_\mu \in D_\rho$ for arbitrary $S \in SF(\mu) - SF_<(\mu)$, so this basis is not particularly well-equipped to make this paradigm shift.

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