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Double Dimers and Super Ptolemy Relations

Gregg Musiker*[1], Nick Ovenhouse†[2], and Sylvester W. Zhang‡[1]

[1] *School of Mathematics, University of Minnesota, Minneapolis, MN, USA*
[2] *Department of Mathematics, Yale University, New Haven, CT, USA*

Abstract. Ptolemy's theorem relates the lengths of the diagonals and sides of a quadrilateral inscribed in a circle, and this is the inspiration for the mutation relation in a cluster algebra associated to a triangulated surface. A super-symmetric version of the Ptolemy relation was introduced recently by Penner and Zeitlin, involving anti-commuting variables. Previous work of the first author and Schiffler gave a formula for cluster variables in terms of perfect matchings of some planar graph. Motivated by this, we investigate certain algebraic expressions, obtained via iterating the super Ptolemy relation, that may be given as a sum over double dimer covers of this graph.

Keywords: cluster algebras, super algebras, snake graphs, dimer models

1 Introduction

Ptolemy's theorem is a well-known identity between the lengths of the diagonals and sides of a quadrilateral inscribed in a circle, as exemplified in Figure 1.

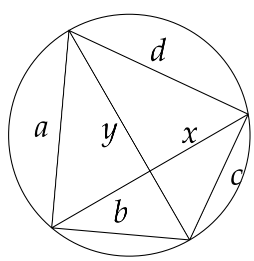


Figure 1: A quadrilateral inscribed in a circle. Ptolemy's relation says $xy = ac + bd$.

In a way, the Ptolemy relation is like a "skein relation", in the sense that it realizes a product of crossing edges by a sum of products of non-crossing ones. More generally, if a polygon is inscribed in a circle, then using the Ptolemy relation iteratively, one may express a product of any number of crossing diagonals as an expression involving only products of non-crossing ones. For example, consider the inscribed pentagon in Figure 2, and the product of the lengths of the three diagonals shown.

*musiker@umn.edu. Gregg Musiker was partially supported by Grants DMS-1745638, DMS-1854162.
†nicholas.ovenhouse@yale.edu. Nick Ovenhouse was partially supported by Grant DMS-1745638.
‡swzhang@umn.edu. Sylvester Zhang was partially supported by Grant DMS-1745638.

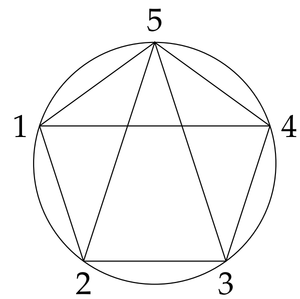


Figure 2: A pentagon inscribed in a circle, and three distinguished diagonals.

Let x_{ij} be the length of the diagonal between vertices labeled i and j (including the boundary sides $x_{i,i+1}$). Then using the Ptolemy relation twice, we can re-write the product of the three diagonals shown in Figure 2 as follows.

$$\begin{aligned} x_{14}x_{25}x_{35} &= (x_{12}x_{45} + x_{15}x_{24})\, x_{35} \\ &= x_{12}x_{45}x_{35} + x_{15}x_{24}x_{35} \\ &= x_{12}x_{45}x_{35} + x_{15}\,(x_{23}x_{45} + x_{34}x_{25}) \\ &= x_{12}x_{45}x_{35} + x_{15}x_{23}x_{45} + x_{15}x_{34}x_{25} \end{aligned}$$

Note that the diagonals $(2,5)$ and $(3,5)$ give a triangulation of the pentagon. By dividing both sides of the equation above by $x_{25}x_{35}$, we obtain

$$x_{14} = \frac{1}{x_{25}x_{35}}\,(x_{12}x_{45}x_{35} + x_{15}x_{23}x_{45} + x_{15}x_{34}x_{25})$$

In particular, this expresses x_{14} as a subtraction-free expression in terms of the lengths of the edges in the triangulation. This is an example of the following fact.

Theorem 1. *[10] Let T be a fixed triangulation of a polygon inscribed in a circle. The length of any diagonal not in T is expressible as a Laurent polynomial with positive integer coefficients in terms of the lengths of the diagonals in T.*

2 Perfect Matching Formula for Diagonal Lengths

In [6], the first author and Schiffler showed that when one uses the Ptolemy relation to re-write the length of a diagonal in terms of lengths from a triangulation, the resulting Laurent polynomial expression is a sum over perfect matchings of some planar graph. We will recall this result below, after some definitions.

Definition 1. A *snake graph* is a planar graph consisting of a sequence of square faces, where each square is attached either above or to the right of the previous one.

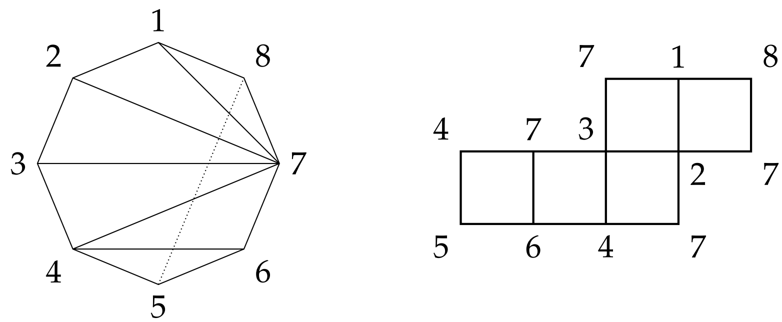


Figure 3: The snake graph G_γ for $\gamma = (5, 8)$.

Remark 1. In terms of the language of Young diagrams, a snake graph is the same thing as a *border strip* skew Young diagram. That is, a skew Young diagram λ/μ which contains no 2×2 block of boxes.

To each diagonal in a triangulated polygon, we will associate a particular snake graph, with some edge labeling.

Definition 2. Let T be a triangulation of an inscribed polygon and $\gamma = (a, b)$ an arc not in T. Let $t_1, t_2, \cdots, t_k$ be the diagonals in T which cross γ. For each t_i, let S_{t_i} be the quadrilateral in T containing t_i as its diagonal. For each pair of adjacent diagonals, their corresponding tiles $S_{t_i}, S_{t_{i+1}}$ share a common edge. Connect all $S_{t_1}, \cdots, S_{t_k}$ by identifying the common edges, and call the resulting graph the *snake graph* of γ, denoted G_γ.

Remark 2. Note that in order to attach the tiles as described in Definition 2, the even-numbered tiles (the second, fourth, etc.) must have opposite orientation from the corresponding quadrilateral in T.

Example 1. Figure 3 shows an example of the snake graph associated to a diagonal in an octagon. The vertices of the snake graph are labeled by the corresponding vertices in the polygon. Although edge labels are not shown in the figure, assume each edge is labeled by x_{ij}, the length of the corresponding diagonal.

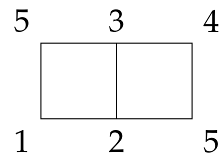
Definition 3. Let G be a graph. A *perfect matching* of G (also called a *dimer cover*) is a subset M of the edges of G, such that each vertex is incident to exactly one edge of M. We will write $D(G)$ for the set of dimer covers of G. If G has an edge-weighting, then $\mathrm{wt}(M)$ is defined as the product of the edge weights in M.

Theorem 2. [6, Theorem 3.1] *Let T be a triangulation of an inscribed polygon, and let γ be a diagonal not in T. Then the length x_γ is given by*

$$x_\gamma = \frac{1}{x_{i_1} x_{i_2} \cdots x_{i_k}} \sum_{M \in D(G_\gamma)} \mathrm{wt}(M)$$

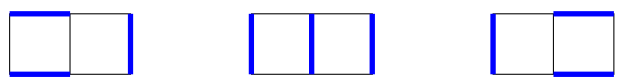
where $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ are the lengths of the diagonals which γ crosses.

Example 2. Consider the triangulated pentagon shown in Figure 2, with triangulation consisting of the diagonals $(2,5)$ and $(3,5)$. The snake graph of $\gamma = (1,4)$ is shown below.



This graph has three dimer covers, corresponding to the three terms in the expression given earlier:

$$x_{14} = \frac{1}{x_{25}x_{35}} \left(x_{12}x_{45}x_{35} + x_{15}x_{23}x_{45} + x_{15}x_{34}x_{25} \right)$$



3 The Super Ptolemy Relation

In this section, we will review a simplified version of the setup from [9]. Informally, in addition to variables x_{ij} representing diagonal lengths, we will also consider some extra non-commuting variables.

Definition 4. A *super algebra* is an associative algebra $A = A_0 \oplus A_1$ with a $\mathbb{Z}_2$-grading. That is, $A_i A_j \subseteq A_{i+j}$. A super algebra is called *super-commutative* (or often just *commutative*) if $xy = (-1)^{ij} yx$ whenever $x \in A_i$ and $y \in A_j$. In other words, elements of A_0, called *even*, are central (they commute with everything), and elements of A_1, called *odd*, anti-commute with each other.

Given a triangulation T of an inscribed polygon, and a choice of orientation of each of its edges, we define a commutative superalgebra A_T as follows. Recall that x_{ij} denotes the length of diagonal (i,j). If the triangulation T has edges $e_1, \cdots, e_{2n-3}$, let $C^\infty(x_{e_1}, \cdots, x_{e_{2n-3}})$ be the commutative algebra of all smooth functions in the x_{e_i} variables corresponding to these lengths. Also let E be the exterior algebra on the $(n-2)$-dimensional space with basis vectors θ_{ijk}, corresponding to triangles in T with vertices i, j, k. Then we define A_T to be

$$A_T := C^\infty(x_{e_1}, \ldots, x_{e_{2n-3}}) \otimes E$$

By definition, the generators of A_T correspond to edges and triangles from a triangulation T. Penner and Zeitlin defined a super Ptolemy transformation [9], which can be iterated to produce new elements x_{ij} and θ_{ijk} of A_T corresponding to diagonals and triangles which are not in T. We will describe this transformation now.

When two triangulations are related by a flip, as in Figure 4, one can define new elements of A_T by the following "*super Ptolemy relations*" [9, Theorem B]:

$$ef = ac + bd + \sqrt{abcd}\,\sigma\theta \qquad \sigma' = \frac{\sqrt{bd}\,\sigma - \sqrt{ac}\,\theta}{\sqrt{ac+bd}} \qquad \theta' = \frac{\sqrt{bd}\,\theta + \sqrt{ac}\,\sigma}{\sqrt{ac+bd}} \tag{3.1}$$

A flip alters the orientation of one of the edges, as depicted in Figure 4. In particular, the order of multiplying the odd variables σ and θ are dictated by the orientation of the edge being flipped, as in Figure 4.

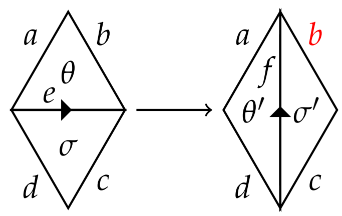


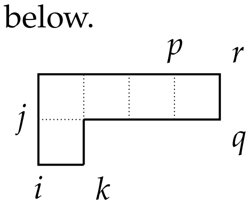
Figure 4: Super Ptolemy relation. The orientation of the edge b is changed by the flip.

4 Double Dimer Covers

Similar to the ordinary Ptolemy relation, one may iteratively apply the super Ptolemy relation to write x_{ij} in terms of the original generators, corresponding to a fixed triangulation T. In [4], the authors showed that this can be calculated using the same snake graph, but summing over double dimer covers, which we define as follows.

Definition 5. Let G be a graph. A *double dimer cover* is a multi-set M of edges in G, such that every vertex of G is contained in exactly two elements of M. In other words, a double dimer cover is the superposition of two dimer covers. Denote $DD(G)$ the set of all double dimer covers of G.

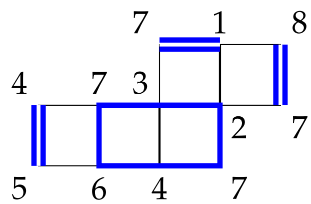
We will now define the weight of a double dimer cover of a snake graph G_γ. Observe that every double dimer cover consists of doubled edges and cycles. For each cycle we associate a product of two odd variables to it. Let c be a cycle, with i, j, k being the three vertices of on the bottom-left corner of c and p, q, r being the three vertices on the top-right corner of c, as illustrated below.



Associate a product of two odd variables to c by $\nu(c) := \theta_{ijk}\theta_{pqr}$, which are the odd variables corresponds to the triangles (i,j,k) and (p,q,r). We then define the weight of a double dimer cover $M \in DD(G_\gamma)$ to be

$$\mathrm{wt}(M) = \prod_{e \in M} \sqrt{x_e} \prod_{c \text{ a cycle in } M} \nu(c) \tag{4.1}$$

Example 3. Consider the following double dimer cover of the snake graph in Figure 3.



Its weight is $x_{45}x_{17}x_{78}\sqrt{x_{67}x_{46}x_{47}x_{27}x_{23}x_{37}}\,\theta_{467}\theta_{237}$.

In [4], we proved the following result, generalizing Theorem 2.

Theorem 3. [4, Theorem 6.1] *Let T be a triangulation T, and let $\gamma = (i,j)$ be a diagonal not in T. Then x_{ij} is given by*

$$x_\gamma = x_{ij} = \frac{1}{x_{i_1}x_{i_2}\cdots x_{i_k}} \sum_{M \in DD(G_\gamma)} \pm \mathrm{wt}(M)$$

where x_{i_1}, x_{i_2}, …, x_{i_k} are the lengths of the diagonals which γ crosses. In particular, all x_γ's are Laurent polynomials in $\sqrt{x_{ij}}$'s and polynomials in θ_{ijk}'s.[1]

There is a stronger version of Theorem 3, which says there there is a special orientation on T, called the *default orientation*, and a total ordering on the odd variables, called the *positive order* (See Section 2.2 of [4] for details), with the property that all terms in the sum in Theorem 3 have positive sign when the odd variables in $\mathrm{wt}(M)$ are multiplied in the positive order.

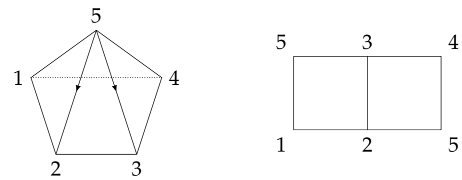
In [3, Proposition 1.4] , the authors showed that every choice of orientation is equivalent to a default orientation, after negating some of the odd θ_{ijk} variables. This allows one to explicitly determine the signs in Theorem 3 for an arbitrary choice of orientation.

Remark 3. For a snake graph G_γ, there is a special family of double dimer covers which only uses doubled edges and no cycles. The weights of these double dimer covers are exactly the same as that of the (single) dimer cover using the same set of edges (but only once). Therefore the set of double dimer covers of G_γ naturally contains the set of (single) dimer covers as a weight-preserving subset.

[1] In [4, Theorem 6.1 part (b)], formulas for some odd variables are given, however, they are not Laurent polynomials.

We now give an example of Theorem 3.

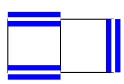
Example 4. Consider the same pentagon as in Figure 2 and its snake graph as in Example 2.



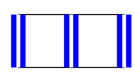
The super-length x_{14} is given by

$$x_{14} = \frac{1}{x_{25}x_{35}}\Big(x_{12}x_{45}x_{35} + x_{15}x_{23}x_{45} + x_{15}x_{34}x_{25} + x_{15}\sqrt{x_{23}x_{25}x_{45}x_{34}}\,\theta_{235}\theta_{345}$$
$$+\, x_{45}\sqrt{x_{15}x_{12}x_{23}x_{35}}\,\theta_{125}\theta_{235} + \sqrt{x_{15}x_{12}x_{25}x_{45}x_{34}x_{35}}\,\theta_{125}\theta_{345}\Big)$$

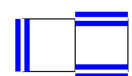
The terms in parentheses are exactly the weighted sum of the six double dimer covers shown below.



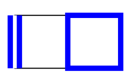
$x_{12}x_{35}x_{45}$



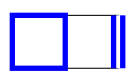
$x_{15}x_{23}x_{45}$



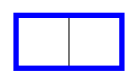
$x_{15}x_{25}x_{34}$



$x_{15}\sqrt{x_{23}x_{25}x_{45}x_{34}}\theta_{235}\theta_{345}$



$x_{45}\sqrt{x_{15}x_{12}x_{23}x_{35}}\theta_{125}\theta_{235}$



$\sqrt{x_{15}x_{12}x_{25}x_{45}x_{34}x_{35}}\theta_{125}\theta_{345}$

5 Partial Order on Double Dimer Covers

In [7], it was noted (Theorem 5.2) that the set of dimer covers of a snake graph (and hence also the set of terms in the Laurent polynomial expression of x_{ij}) has a partial order making it into a distributive lattice. We will recall this partial order, and then see that the set of double dimer covers also has a natural partial order.

There are exactly two dimer covers of a snake graph which contain only boundary edges. Define the *minimal dimer cover* M_0 of a snake graph to the dimer cover, using only boundary edges, which contains the bottom edge of the first square. An example is shown in Figure 5.

The partial order on the set of dimer covers is defined as follows. Number the squares of the snake graph $1, 2, \ldots, N$ from bottom-left to top-right. For a dimer cover M, consider the double dimer cover $M \cup M_0$. Define a subset $s(M) \subseteq [N]$ as the set of squares which are surrounded by cycles of $M \cup M_0$. The partial order is then defined as $M \leq M'$ if $s(M) \subseteq s(M')$. An example is shown in Figure 6. Denote $P_D(G)$ the poset of dimer covers on G.

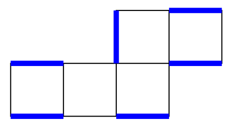


Figure 5: The minimal dimer cover of a snake graph.

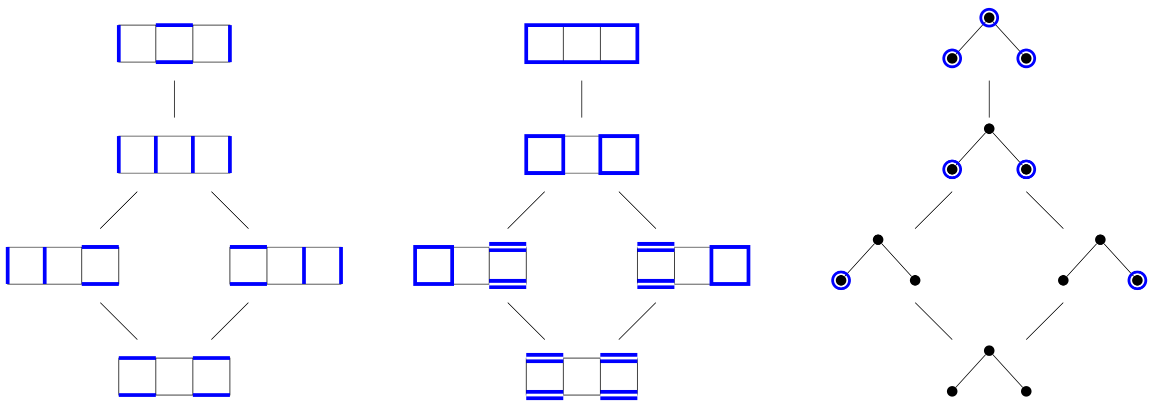
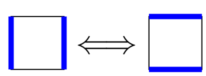


Figure 6: Left: The poset of dimer covers of a snake graph. Middle: the corresponding double dimer covers, after superimposing the minimal dimer cover. Right: Lattice of order ideals of the fence poset, where a blue circle indicate elements in the order ideal.

Alternatively, we can construct the $P_D(G_\gamma)$ inductively by "flipping" the dimer covers on tiles of the snake graph. If a tile of snake graph is covered by two dimer edges, then one can perform a flip on that tile as follows to obtain a new dimer cover.



Let M be a dimer cover of G_γ and M' is obtained from M via a flip, then either M' is covered by M or M is covered by M'. If we require the poset to have a unique minimum element, then the above rule and the choice of minimal element (the minimal dimer cover M_0) are sufficient to determine the entire poset.

Moreover, for a snake graph G, the poset $P_D(G)$ is a distributive lattice and can be realized as the poset of order ideals of the *fence poset* associated to G , denoted F_G. See the right image in Figure 6. We refer the reader to Section 9 of [4] for more details about fence posets.

In the case of double dimer covers, there exists a similar natural partial order structure. We first define the *minimal double dimer cover* $\tilde{M}_0$ of a snake graph G_γ to be the super position of two identical copies of the minimal dimer covers of G_γ. In other words, the minimal double dimer cover looks exactly the same as the minimal single dimer cover but with doubled edges instead.

For a snake graph G, denote $P_{DD}(G)$ the poset of double dimer covers of G. We can

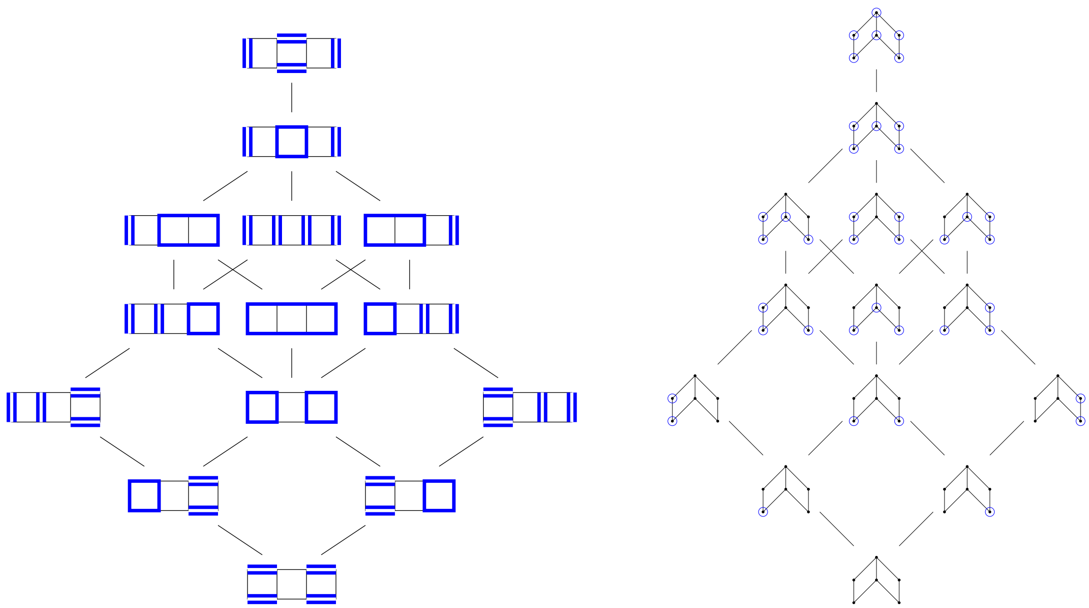
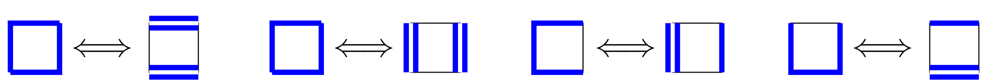


Figure 7: Left: the partial order on double dimer covers of a snake graph. Right: the poset of order ideals of the corresponding doubled fence poset.

then define $P_{DD}(G)$ in the same way by performing flips (of a pair of single edges) on the tiles of G. Note we are allowed to flip a pair of singled edges inside some doubled edges. The possible local flips are summarized as follows.



Similar to the single dimer case, these flips corresponds to covering relations in the poset. Therefore starting with the minimal double dimer cover $\tilde{M}_0$ and applying flips inductively generates the entire partial order.[2] See Figure 7 for example.

Similar to the single dimer case, the poset $P_{DD}(G)$ is also a distributive lattice. In this case, it can be realized as the poset of order ideals on the *doubled fence poset* that is the product of the fence poset associated to G and a two element chain: $\{0,1\} \times F_G$. See Figure 7 for example.

6 Special Cases: Zig-zag and Straight Snake Graphs

We conclude with a discussion on two special cases of snake graphs: the zig-zag and the straight snake graphs.

[2]Note that we can define $P_{DD}(G)$ in a way similar to the first definition we gave for $P_D(G)$, by super imposing every double dimer cover with the minimal double dimer cover. However it is more complicated to illustrate in the case of double dimer covers, therefore we omit this definition.

Zig-zag Snakes Graphs. Zig-zag snake graphs corresponds to the longest arc in a fan triangulation, i.e. the triangulation where all triangles share a common vertex. Let Z_n be the zig-zag snake graph with n tiles. Then the number of double dimer covers of Z_n is the binomial coefficient $\binom{n+2}{2}$. Moreover, there is a q-analogue of this statement, given in the following theorem, a special case of a more general result from work in progress of the authors and R. Schiffler [2].

Theorem 4. *For a double dimer cover $M \in P_{DD}(Z_n)$, let $\mathrm{rk}(M)$ denote its rank in the poset. Then we have*

$$\sum_{M \in P_{DD}(Z_n)} q^{\mathrm{rk}(M)} = \binom{n+2}{2}_q$$

In fact, the poset $P_{DD}(Z_n)$ is isomorphic to the poset of lattice paths on the $2 \times n$ board, whose rank generating function is well known to be the q-binomial coefficient. See Figure 8 for an illustration.

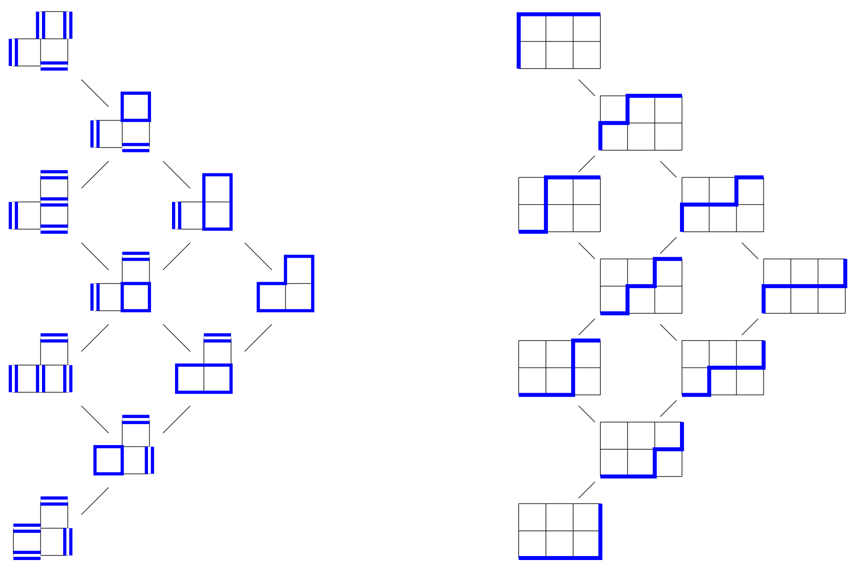


Figure 8: Left: The poset of double dimer covers on a zig-zag snake graph. Right: The poset of lattice path on $2 \times n$ grid.

Straight Snakes Graphs and Super Fibonacci Numbers. Another interesting family of snake graphs are the straight snake graphs. Let L_n denote the straight snake graph that is a horizontal row of n tiles. The number of dimer covers of L_n gives a nice combinatorial interpretation of Fibonacci numbers. More precisely, $|D(L_n)| = f_{n+1}$ where $f_0 = f_1 = 1$ and $f_{n+1} = f_n + f_{n-1}$. In the case of double dimer covers, this gives rise to a generalization which we call *super Fibonacci numbers*.

Definition 6. Consider the family of double dimer covers $\mathbb{D}(L_n)$ on L_n such that every $M \in \mathbb{D}(L_n)$ has at most one cycle, and that cycle has odd length. Further, define $\mathbb{D}^0(L_n)$ to be the set of double dimers without cycles and $\mathbb{D}^1(L_n) := \mathbb{D}(L_n) \setminus \mathbb{D}^0(L_n)$. Denote $y_n = |\mathbb{D}^1(L_{n-1})|$ and recall that $|\mathbb{D}^0(L_{n-1})| = |D(L_{n-1})| = f_n$. Finally, define the *n-th super Fibonacci number* to be

$$F_n := |\mathbb{D}^0(L_{n-1})| + |\mathbb{D}^1(L_{n-1})|\varepsilon = f_n + y_n\varepsilon$$

where $\varepsilon^2 = 0$.

Super Fibonacci numbers live in the super algebra $\mathbb{Z} \otimes \bigwedge(\varepsilon)$ with one odd generator, i.e. the algebra of dual numbers.

Remark 4. The reason why we forbid multiple cycles and cycles of even length is related to the hyperbolic geometry of the annulus, which we do not explain here. See Section 11 of [4] for more details. In fact, in [5] we showed that F_n's can be interpreted as certain "super-lengths" on a marked annulus.

The F_n's satisfy several identities, generalizing those of the classical Fibonacci numbers. To better state these identities, we make a simple adjustment to the definition of super Fibonacci numbers as follows. Define

$$\tilde{y}_n := \begin{cases} y_n & \text{if } n \text{ is odd.} \\ y_n - 1 & \text{if } n \text{ is even.} \end{cases}$$

and $\mathcal{F}_n := f_n + \tilde{y}_n\varepsilon$. In terms of double dimer covers, $\mathcal{F}_n$ corresponds to disallowing the double dimer cover that is the longest cycle.

Theorem 5. *The 'simpler' super Fibonacci numbers $\mathcal{F}_n$ satisfy the recurrence*

$$\mathcal{F}_0 = 1 - \varepsilon, \quad \mathcal{F}_1 = 1, \quad \mathcal{F}_{n+1} = (1+\varepsilon)\mathcal{F}_n + \mathcal{F}_{n-1},$$

and its generating function is

$$\sum_n \mathcal{F}_n x^n = \frac{1-\varepsilon}{1-(1+\varepsilon)x - x^2}$$

The ratio of Fibonacci numbers has a nice continued fraction expansion:

$$\frac{f_n}{f_{n+1}} = [\underbrace{1, \cdots, 1}_{n \text{ times}}] = 1 + \frac{1}{1 + \frac{1}{1+\cdots}}$$

There is a similar identity for ratios of the $\mathcal{F}_n$'s.

Theorem 6.
$$\frac{\mathcal{F}_n}{\mathcal{F}_{n+1}} = [\underbrace{1+\varepsilon, \cdots, 1+\varepsilon}_{n \text{ times}}] = 1+\varepsilon+\frac{1}{1+\varepsilon+\frac{1}{1+\varepsilon+\frac{1}{\cdots}}}$$

In addition, by analysing the generating function, we are able to write the super Fibonacci numbers in terms of the golden ratio:

Theorem 7. *Let* $\Phi_\pm = \frac{1\pm\sqrt{5}}{2} + \left(\frac{1\pm\frac{1}{\sqrt{5}}}{2}\right)\varepsilon = \frac{1\pm\sqrt{5}}{2}\left(1 \pm \frac{1}{\sqrt{5}}\varepsilon\right)$. *Then we have*
$$\mathcal{F}_n = \frac{1}{\sqrt{5}}\left(1-\frac{6}{5}\varepsilon\right)\left(\Phi_+^{n+1} - \Phi_-^{n+1}\right)$$

Remark 5. Note that y_n is the OEIS sequence A054454 as well as the third column of the triangular array A054453. The super Fibonacci numbers also appeared in [8] under the name of "shadow sequences". An analogue of Theorem 6 has also been obtained in [1] for shadow Fibonacci numbers as an example of super continued fractions.

References

[1] C. H. Conley and V. Ovsienko. "Shadows of rationals and irrationals: supersymmetric continued fractions and the super modular group". 2022. arXiv:2209.10426.

[2] G. Musiker, N. Ovenhouse, R. Schiffler, and S. W. Zhang. "Higher Dimer Covers on Snake Graphs". In preparation. 2023.

[3] G. Musiker, N. Ovenhouse, and S. W. Zhang. "An Expansion Formula for Decorated Super-Teichmüller Spaces". *Symmetry, Integrability and Geometry: Methods and Applications* **17** (2021).

[4] G. Musiker, N. Ovenhouse, and S. W. Zhang. "Double dimer covers on snake graphs from super cluster expansions". *Journal of Algebra* **608** (2022), pp. 325–381.

[5] G. Musiker, N. Ovenhouse, and S. W. Zhang. "Matrix Formulae for Decorated Super Teichmüller Spaces". 2022. arXiv:2208.13664.

[6] G. Musiker and R. Schiffler. "Cluster expansion formulas and perfect matchings". *Journal of Algebraic Combinatorics* **32**.2 (2010), pp. 187–209.

[7] G. Musiker, R. Schiffler, and L. Williams. "Bases for cluster algebras from surfaces". *Compositio Mathematica* **149**.2 (2013), pp. 217–263.

[8] V. Ovsienko. "Shadow Sequences of Integers: From Fibonacci to Markov and Back". *The Mathematical Intelligencer* (2022), pp. 1–5.

[9] R. Penner and A. M. Zeitlin. "Decorated super-Teichmüller space". *Journal of Differential Geometry* **111**.3 (2019), pp. 527–566.

[10] R. Schiffler. "A cluster expansion formula (A_n case)". *The Electronic Journal of Combinatorics* **15**.R64 (2008), p. 1.