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On the Topology of Cut Complexes of Graphs

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Abstract. For a positive integer k and a finite graph G, we define the *k-cut complex* $\Delta_k(G)$ to be the pure simplicial complex in which the complement of each face contains a set of k vertices inducing a disconnected subgraph of G. This generalises a simplicial complex considered by John Eagon and Victor Reiner (1998), who use $\Delta_2(G)$ to reformulate and extend a famous theorem of Ralf Fröberg (1990) relating certain Stanley-Reisner ideals to chordal graphs. In particular their combined results imply that $\Delta_2(G)$ is shellable if and only if G is a chordal graph.

We investigate $\Delta_k(G)$ with this inspiration, using techniques from algebraic and combinatorial topology. We describe the effect of various graph operations on the cut complex, consider its shellability, and determine the homotopy type and Betti numbers of $\Delta_k(G)$ for various families of graphs. When the homotopy type is a wedge of spheres, we also determine the group representation on the rational homology, notably in the case of complete multipartite graphs.

Keywords: Graph complex, chordal graph, shellability, homotopy, homology representation

1 Introduction

In recent years, there has been much interest in the topology of simplicial complexes associated with graphs. A comprehensive reference is Jonsson's book [9].

A *graph complex* is a simplicial complex associated to a finite graph G. In this paper we introduce a new family of graph complexes which we call *cut complexes*. We consider only simple graphs. Our work is motivated by a famous theorem of Ralf Fröberg (1990) connecting commutative algebra and graph theory through topology. We investigate our new complexes in the spirit of Fröberg's theorem, relating topological properties of the cut complex to structural properties of the graph.

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For a field $\mathbb{K}$ and a finite simplicial complex Δ with vertex set $[n] = \{1, 2, \ldots, n\}$, the Stanley-Reisner ideal of Δ is the ideal I_Δ of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ generated by monomials $x_{i_1} \cdots x_{i_k}$, where $\{i_1, \ldots, i_k\}$ runs over the inclusion-minimal subsets of $[n]$ which are NOT faces of Δ. The Stanley-Reisner ring $\mathbb{K}[\Delta]$ is the quotient of the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ by the ideal I_Δ.

For a graph G, the *clique complex* $\Delta(G)$ is the simplicial complex whose simplices are subsets of vertices of G, in which every pair of vertices is connected by an edge of G. Fröberg [7] characterised ideals generated by monomials which have a *linear resolution*, by first reducing to the case of square-free monomial ideals. The ideal I_Δ is generated by quadratic square-free monomials precisely when the simplicial complex Δ is $\Delta(G)$ for some graph G (see [6, Proposition 8]). Hence Fröberg's theorem can be stated as follows:

Theorem 1. *(Fröberg) [[7], [6, p. 274]] A Stanley-Reisner ideal I_Δ generated by quadratic square-free monomials has a linear resolution if and only if Δ is the clique complex $\Delta(G)$ of a chordal graph G.*

Define the combinatorial *Alexander dual* of a simplicial complex Δ [5, p.188] on n vertices to be

$$\Delta^\vee := \{F \subset [n] : [n] \setminus F \notin \Delta\}.$$

The ith homology of Δ and the $(n-i-3)$th cohomology of $\Delta^\vee$ are isomorphic by Alexander duality in the sphere $\mathbb{S}^{n-2}$.

For a graph G, write $\Delta_2(G)$ for the Alexander dual $\Delta(G)^\vee$ of the clique complex $\Delta(G)$. The facets of $\Delta_2(G)$ are the complements of independent sets of size 2 in G. Eagon and Reiner's [6, Proposition 8] reformulation of Fröberg's theorem includes the following equivalences.

Theorem 2. *The graph G is chordal $\iff$ $\Delta_2(G)$ is shellable $\iff$ $\Delta_2(G)$ is vertex decomposable.*

Inspired by this theorem, we introduce the following generalisation of the simplicial complex $\Delta_2(G)$. Let $k \geq 1$. Define a complex whose facets are complements of sets F of size k in G such that the induced subgraph of G on the vertex set F is disconnected; we call this the k-cut complex of G, and denote it by $\Delta_k(G)$. A different generalisation, the total k-cut complex $\Delta_k^t(G)$, is treated in [1]. The two notions coincide for $k = 2$.

We examine the topology of the cut complex $\Delta_k(G)$ and consider how it is affected by properties of the graph G (Section 3). For example, in analogy with Fröberg's theorem, Proposition 21 asserts that graph chordality implies shellability of the 3-cut complex Δ_3. We show that for many common families of graphs (Section 5), the homotopy type of $\Delta_k(G)$ is a wedge of spheres in a single dimension. Often there is a simplicial group action on the cut complex $\Delta_k(G)$, which in turn acts on the rational homology. We determine this homology representation, notably in the case of complete multipartite graphs [2].

2 Definitions

General references for simplicial complexes, shellability and topology are [3], [8] and [15], and [17] for graph theory. All graphs in this paper are simple (no loops and no multiple edges).

Definition 3. A simplicial complex Δ on a set A is a collection of subsets of A

$$\sigma \in \Delta \text{ and } \tau \subseteq \sigma \Rightarrow \tau \in \Delta.$$

The elements of Δ are called its *faces* or *simplices*. If the collection of subsets is empty, i.e. Δ has no faces, we call Δ the void complex. Otherwise Δ always contains the empty set as a face.

The *dimension of a face* σ, $\dim(\sigma)$, is one less than its cardinality; thus the dimension of the empty face is (-1), and the 0-dimensional faces are the *vertices* of Δ. A *d-face* or *d-simplex* is a face of dimension d. The maximal faces of Δ are called its *facets*, and the maximum dimension of a facet is the *dimension* $\dim(\Delta)$ of the nonvoid simplicial complex Δ. We write $\Delta = \langle \mathcal{F} \rangle$ for the simplicial complex Δ whose set of facets is $\mathcal{F}$.

In this paper all simplicial complexes will be *finite*, that is, the vertex set is finite.

A (nonvoid) simplicial complex is *pure* if all its facets have the same dimension, which is then the dimension of the complex.

The *join* of two simplicial complexes Δ_1 and Δ_2 with disjoint vertex sets is the complex

$$\Delta_1 * \Delta_2 = \{\sigma \cup \tau : \sigma \in \Delta_1, \tau \in \Delta_2\}.$$

Thus the join $\Delta_1 * \Delta_2$ contains Δ_1 and Δ_2 as subcomplexes.

The *cone* over Δ and the *suspension* of Δ are the complexes

$$\text{cone}(\Delta) = \Delta * \Gamma_1, \ \text{susp}(\Delta) = \Delta * \Gamma_2 = \Delta * \{u\} \cup \Delta * \{v\},$$

where Γ_1 is the 0-dimensional simplicial complex with one vertex, and Γ_2 is the 0-dimensional complex with two vertices u, v.

Definition 4 ([3, Sec. 11.2])**.** An ordering $F_1, F_2, \ldots, F_t$ of the facets of a simplicial complex Δ is a *shelling* if, for every j with $1 < j \le t$, $\left(\bigcup_{i=1}^{j-1} \langle F_i \rangle\right) \cap \langle F_j \rangle$ is a simplicial complex whose facets all have cardinality $|F_j| - 1$, where $\langle F_i \rangle$ is the simplex generated by the face F_i. The simplicial complex Δ is *shellable* if it admits a shelling order of its facets.

Remark 5. By convention, the void complex is shellable. The complex whose only face is the empty set is vacuously shellable. The complex with a unique nonempty facet is (also vacuously) shellable, and contractible.

See Figures 1(b) and 2(b) for examples of shellable and nonshellable complexes.

In combinatorial topology, shellability is an important tool for determining the homotopy type of simplicial complexes, thanks to the following theorem of Björner.

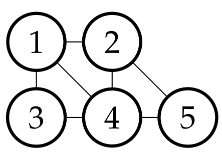
Theorem 6 ([3, (9.19) and Sec. 11]). *A pure shellable simplicial complex of dimension d has the homotopy type of a wedge of spheres, all of dimension d (it may be contractible).*

Definition 7. Let $G = (V, E)$ be a graph on $|V| = n$ vertices. If S is a subset of the vertex set V, write $G[S]$ to denote the *induced subgraph* of G whose vertex set is S.

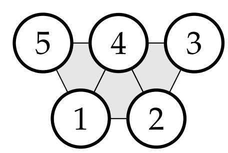
Definition 8. Let $G = (V, E)$ be a graph on $|V| = n$ vertices, and let $k \geq 2$. Define the *k-cut complex* of the graph G to be the $(n-k-1)$-dimensional simplicial complex

$$\Delta_k(G) := \langle F \subseteq V, |F| = n-k \mid G[V \setminus F] \text{ is disconnected} \rangle.$$

The facets of the cut complex $\Delta_k(G)$ are the vertex subsets of G of size $(n-k)$ whose removal disconnects the graph G. Thus σ is a face of the cut complex $\Delta_k(G)$ if and only if its complement $V \setminus \sigma$ contains a subset S of size k such that the induced subgraph $G[S]$ is disconnected. Note the inclusion $\Delta_{k+1}(G) \subseteq \Delta_k(G)$ for $k \geq 2$, and the fact that the vertices of $\Delta_k(G)$ are a subset of the vertices of the graph G. See Figures 1 and 2 .

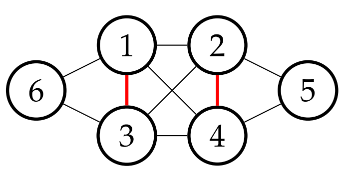


(a) The graph G is disconnected by removing any of the following 3-sets: $\{1,2,4\}, \{1,4,5\}$ and $\{2,3,4\}$

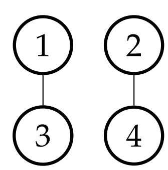


(b) $\Delta_2(G) = \langle 234, 145, 124 \rangle$

Figure 1: (Shellable) 2-cut complex of graph G



(a) G is disconnected by removing one of the 2-sets $\{1,3\}$, $\{2,4\}$



(b) $\Delta_4(G) = \langle 13, 24 \rangle$

Figure 2: (Nonshellable) 4-cut complex of graph G

Example 9. Let G be a graph on n vertices. We record some easy facts about cut complexes.

1. $\Delta_k(G)$ is void if $k = 1$ or $k > n$.

2. $\Delta_n(G)$ is $\begin{cases} \text{the void complex,} & \text{if } G \text{ is connected,} \\ \text{the } (-1)\text{-dimensional complex } \{\emptyset\}, & \text{otherwise.} \end{cases}$

3. $\Delta_k(G)$ is void for $n-k \leq r-1$ if G is r-connected, since at least r vertices must be removed to disconnect the graph.

4. If G is the complete graph K_n, then $\Delta_k(G)$ is void for all $k \geq 1$.

5. If $G = E_n$ is the edgeless graph on n vertices, then for $2 \leq k \leq n-1$, $\Delta_k(G)$ is the $(n-k-1)$-skeleton of an $(n-1)$-dimensional simplex, hence shellable [4].

In view of Item (1.) above, we will assume $\mathbf{n} \geq \mathbf{k} \geq \mathbf{2}$ for the cut complex $\Delta_k(G)$.

In order to determine the homotopy type of a simplicial complex Δ, and especially a group representation on the rational homology, it is often useful to work with the face lattice $\mathcal{L}(\Delta)$ of Δ. In many cases the automorphism group of the graph G acts simplicially on the cut complex $\Delta_k(G)$.

Clearly the symmetric group $\mathfrak{S}_n$ acts on the cut complex $\Delta_k(E_n)$ of the edgeless graph E_n. The face lattice $\mathcal{L}(\Delta_k(E_n))$ is a rank-selected subposet of the Boolean lattice B_n of subsets of an n-element set, consisting of subsets of size at most $n-k$. Using the fact that this is shellable (see e.g. [15]) in conjunction with a well-known result of Solomon [12], Observation (5.) above gives us our first nontrivial result for the homotopy type and equivariant homology of a cut complex:

Proposition 10. *Let E_n be the edgeless graph on n vertices. If $k \geq n$, the cut complex $\Delta_k(E_n)$ is void. If $2 \leq k \leq n-1$, the cut complex is shellable and homotopy equivalent to a wedge of $\binom{n-1}{k-1}$ spheres in dimension $n-k-1$:*

$$\Delta_k(E_n) \simeq \bigvee_{\binom{n-1}{k-1}} \mathbb{S}^{n-k-1}.$$

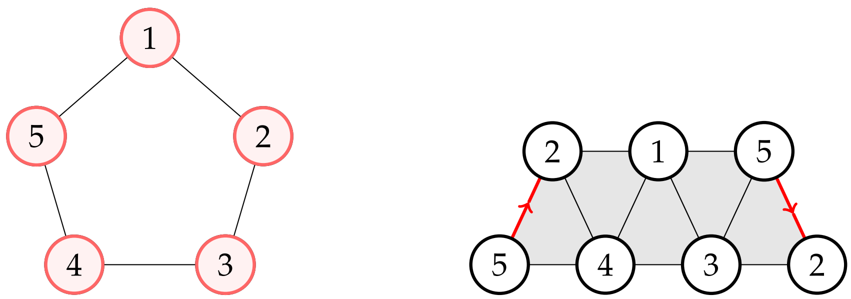
The $\mathfrak{S}_n$-representation on the unique nonvanishing homology of $\Delta_k(E_n)$ is the irreducible module indexed by the partition $(k, 1^{n-k})$ of n.

Definition 11 ([17])**.** A graph is *chordal* if it has no induced cycle of size greater than 3.

Figure 1 shows a chordal graph and its shellable 2-cut complex, whereas Figure 3 is an example of a nonchordal graph, with nonshellable 2-cut complex. These examples also illustrate Fröberg's Theorem 2.

3 Constructive Theorems

In this section we consider the effect of some common graph operations on the cut complex. We begin with what may be viewed as a universal property of cut complexes.

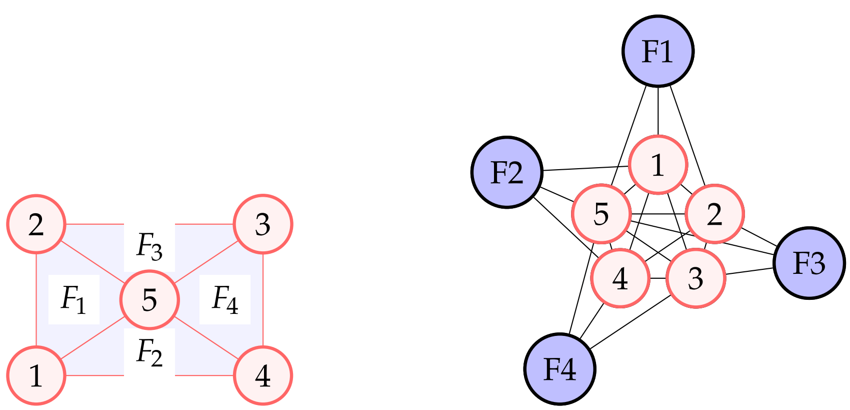


(a) The cycle C_5 (b) $\Delta_2(C_5) = \langle 245, 124, 134, 135, 235 \rangle$

Figure 3: The 2-cut complex for C_5 is a Möbius strip

Theorem 12. *Let Δ be any pure simplicial complex. There exists some k and some chordal graph G such that Δ is equal to the cut complex $\Delta_k(G)$.*

The construction starts with a clique whose vertices correspond to the vertex set of the complex Δ. For each facet of Δ, add a vertex which is connected to every vertex of that facet. If Δ has n vertices, t facets and dimension d, the resulting graph G has $(n+t)$ vertices, and $\Delta_{n+t-(d+1)}(G) = \Delta$. Figure 4 illustrates the procedure.



(a) A pure simplicial complex Δ (b) A graph G such that $\Delta = \Delta_6(G)$

Figure 4: The construction of Theorem 12

Let $G = (V, E)$ be a graph. Recall that we assume $k \geq 2$ for the cut complex $\Delta_k(G)$. The link [3, Sec. 9.9] of a face σ of a simplicial complex Δ is defined to be the simplicial complex

$$\mathrm{lk}_\Delta\, \sigma = \{\tau \in \Delta \mid \sigma \cap \tau = \emptyset, \text{ and } \sigma \cup \tau \in \Delta\}.$$

Lemma 1. *Let $v \in V$. Then $\Delta_k(G \setminus v) = \mathrm{lk}_{\Delta_k(G)}\{v\}$, if v is a vertex of $\Delta_k(G)$. Otherwise $\Delta_k(G \setminus v)$ is the void complex.*

By applying this lemma repeatedly and using the fact that links preserve shellability [3, Sec. 11.2], we obtain the following important corollary.

Corollary 13. *Let $W \subseteq V$. If the k-cut complex $\Delta_k(G)$ is shellable, so is $\Delta_k(G \setminus W)$. Equivalently, if $\Delta_k(G)$ is shellable for a graph G, then $\Delta_k(H)$ is shellable for every induced subgraph H of G.*

Definition 14. If G_1, G_2 are graphs, their disjoint union is the graph $G_1 + G_2$ having vertex set equal to the union of the vertex sets of G_1 and G_2, and edge set equal to the union of the edge sets of G_1 and G_2.

Theorem 15. $\Delta_k(G_1 + G_2)$ *is shellable if and only if* $\Delta_k(G_1)$ *and* $\Delta_k(G_2)$ *are shellable.*

For instance, from Example 9, this theorem tells us that the k-cut complex of the disjoint union of two complete graphs is always shellable.

Definition 16. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ on disjoint vertex sets, their *join* $G_1 * G_2$ is their disjoint union with the set of all edges between V_1 and V_2 included.

Theorem 17. $\Delta_k(G_1 * G_2)$ *is shellable if and only if between* $\Delta_k(G_1)$ *and* $\Delta_k(G_2)$*, one is shellable and the other is the void complex.*

The homotopy type of the cut complex of the join of two graphs bears a precise relationship to the homotopy types of the cut complexes of the individual graphs. Passing to the face lattices gives an alternative proof of the homotopy equivalence, and also shows that it is group-equivariant, a fact we will need in Section 5, where we will apply the full force of this theorem to complete bipartite graphs, which are joins of edgeless graphs.

Theorem 18. *Let $k \geq 1$, let G_i be a graph with vertex set $V_i, i = 1, 2$, where $V_1 \cap V_2 = \emptyset$. Assume $\Delta_k(G_i)$ is not void for at least one $i = 1, 2$.*

If only one cut complex is void, then $\Delta_k(G_1 * G_2)$ is contractible.

Now assume both cut complexes $\Delta_k(G_i), i = 1, 2$, are nonvoid. Then there is a homotopy equivalence

$$\operatorname{susp}(\Delta_k(G_1) * \Delta_k(G_2)) \simeq \Delta_k(G_1 * G_2). \tag{3.1}$$

Moreover, there is a group-equivariant poset map from the product $\mathcal{L}(\Delta_k(G_1)) \times \mathcal{L}(\Delta_k(G_2))$ of face lattices, to the face lattice of the simplicial complex $\Delta_k(G_1 * G_2)$ which induces a group-equivariant homotopy equivalence of the respective order complexes. This in turn gives a group-equivariant $(H_1 \times H_2)$-homotopy equivalence $\operatorname{susp}(\Delta_k(G_1) * \Delta_k(G_2)) \simeq_{H_1 \times H_2} \Delta_k(G_1 * G_2)$, where H_i is a group acting simplicially on the cut complex $\Delta_k(G_i)$.

Hence we have the following isomorphism of $(H_1 \times H_2)$-modules in rational homology:

$$\tilde{H}_d(\Delta_k(G_1 * G_2)) \cong \bigoplus_{p+q=d-2} \tilde{H}_p(\Delta_k(G_1)) \otimes \tilde{H}_q(\Delta_k(G_2)).$$

The proof of Equation (3.1) uses gluing homeomorphisms and homotopy equivalences from algebraic topology [8, Ex. 0.14], [1, Prop. 2.7]. To establish the group-equivariant homotopy equivalence of face lattices, we use an equivariant Quillen fibre lemma [11], [16]. The Künneth theorem then gives the isomorphism in homology.

Definition 19. Given graphs G_1 and G_2 on disjoint vertex sets, a *wedge product* of G_1 and G_2, $G_1 \vee G_2$, is formed by taking one vertex from G_1 and one vertex from G_2, and identifying them. Note that a wedge product of two graphs is not unique in general.

For example, any tree on n vertices can be obtained as a wedge product of a smaller tree on $n-1$ vertices with the complete graph K_2.

Theorem 20. *Let $k \geq 2$, and let G_1, G_2 be graphs. Then $\Delta_k(G_1 \vee G_2)$ is shellable if and only if $\Delta_k(G_1)$ and $\Delta_k(G_2)$ are shellable.*

The cycle graph gives a counterexample (see Theorem 27) to the converse of the next result, offered in the spirit of Fröberg's Theorem 2.

Proposition 21. *If G is chordal, then $\Delta_3(G)$ is shellable.*

An example of a chordal graph whose 4-cut complex is not shellable appears in Figure 2.

Corollary 13 says that if $\Delta_k(H)$ is *not* shellable for some induced subgraph H of G, then $\Delta_k(G)$ is not shellable. It is thus natural to seek a description of the *minimal* graphs whose k-cut complex fails to be shellable, for fixed k. We will call such a graph a *minimal forbidden subgraph* for k-cut complex shellability.

Proposition 22. *Every minimal forbidden subgraph for 3-cut complex shellability is at least 3-connected.*

4 Reduced Euler characteristics and Betti numbers

Our proofs of the preceding results determine the homotopy type, but do not give Betti numbers. In this section we present a method for determining the Betti numbers of a cut complex, under certain favourable conditions. Recall [13] that the reduced Euler characteristic of a simplicial complex Δ is the Möbius number $\mu(\mathcal{L}(\Delta))$ of its face lattice $\mathcal{L}(\Delta)$. The idea is to describe the face poset of the cut complex as a subposet of the truncated Boolean lattice, and then use poset topology techniques as in [13, Lemma 3.16.4], [14, Proposition 1.3] to determine the Möbius number.

We say a subset A of the vertex set $V(G)$ of a graph G is *connected* (resp. disconnected) if the induced subgraph $G[A]$ is connected (resp. disconnected).

Let G be a graph with vertex set $V(G)$ of size n. Let $P(n,k)$ be the truncated Boolean lattice $B_n^{\leq n-k}$ of subsets of $[n]$ of size at most $(n-k)$. Clearly the face poset of $\Delta_k(G)$

is a subposet of $P(n,k)$. Let A^c denote the complement of a subset A of $[n]$, and define $\mathcal{Z}_k(G)$ to be the subset $\{A^c \in P(n,k) : |A| = k, A \text{ induces a connected subgraph of } G\}$. Thus $\mathcal{Z}_k(G)$ is the set of maximal elements of $P(n,k)$ which are not facets of the cut complex $\Delta_k(G)$. Also the number of facets of $\Delta_k(G)$ is $\binom{n}{k} - |\mathcal{Z}_k(G)|$.

Theorem 23. *We have the following.*

1. *The face poset of $\Delta_k(G)$ is a subposet of $P(n,k) \setminus \mathcal{Z}_k(G)$.*
2. *The following are equivalent:*

   (i) The face poset of $\Delta_k(G)$ coincides with $P(n,k) \setminus \mathcal{Z}_k(G)$.

   (ii) Let $A^c \in \mathcal{Z}_k(G)$, $X \subset A^c$, $|X| = (n-k-1)$. Then X^c contains a disconnected set of size k.

   (iii) Let $A \subset V(G)$ be connected of size k. For every $x \notin A$ there is a $y \in A$ such that $(A \setminus \{y\}) \cup \{x\}$ is disconnected.
3. *If any condition in Part (2.) holds, the reduced Euler characteristic $\mu(\Delta_k(G))$ of $\Delta_k(G)$ is given by*

$$(-1)^{n-k-1}\mu(\Delta_k(G)) = \binom{n-1}{k-1} - |\mathcal{Z}_k(G)| = |\{F : F \text{ is a facet of } \Delta_k(G)\}| - \binom{n-1}{k}.$$

   Furthermore, in this case the nonzero homology of $\Delta_k(G)$ is torsion-free and occurs in at most two dimensions, $n-k-1$ and $n-k-2$.

4. *If any condition in Part (2.) holds and $\Delta_k(G)$ is shellable, then it is homotopy equivalent to a wedge of $|\mu(\Delta_k(G))|$ spheres in dimension $n-k-1$. It is contractible if $\mu(\Delta_k(G)) = 0$.*

The technical conditions in Part (2.) above, as well as variations, hold for several common families of graphs, including trees, cycles and grid graphs.

5 Families of Graphs

We now examine specific classes of graphs. Theorem 23 can be applied to compute Betti numbers for cut complexes of forests and cycles, which satisfy the conditions in Part (2.) of Theorem 23.

Trees

Recall that for every k, the k-cut complex of the complete graph K_2 is the void complex (Example 9). Since trees may be recursively constructed by taking a wedge with K_2, and forests are disjoint unions of trees, Theorems 15 and 17 give:

Proposition 24. *If G is a forest, $\Delta_k(G)$ is shellable for all $k \geq 2$.*

For any (finite) graph G, let $\tau_k(G)$ denote the number of subtrees of G with k vertices. We have $\tau_2(\mathcal{F}_n) = n - c$, where c is the number of connected components of the forest $\mathcal{F}_n$. If the tree is a path P_n, then $\tau_k(P_n) = (n-k+1)$. From Theorem 23, we obtain:

Theorem 25. *Let $\mathcal{F}_n$ be a forest on n vertices. Then the shellable cut complex $\Delta_k(\mathcal{F}_n)$ is homotopy equivalent to a wedge of $\left(\binom{n-1}{k-1} - \tau_k(\mathcal{F}_n)\right)$ spheres in dimension $(n-k-1)$ if this number is nonzero, and contractible otherwise.*

In particular, for a tree T_n on n vertices, $\Delta_k(T_n)$ is contractible if $k = 2$, and has the homotopy type of a wedge of $\binom{n-1}{k-1} - \tau_k(T_n)$ spheres of dimension $(n-k-1)$ if $k \geq 3$.

Cycles

For the cycle graph C_n, we may assume $n \geq 4$ and $2 \leq k \leq n-2$. If $n = 4$, the one-dimensional cut complex $\Delta_2(C_4)$ has two facets $\{1,3\}$ and $\{2,4\}$, and is homotopy equivalent to the 0-sphere $\mathbb{S}^0$; it is thus not shellable.

Recall from Figure 3 that the 2-dimensional cut complex $\Delta_2(C_5)$ is in fact a Möbius strip, and hence it is homotopy equivalent to the one-sphere $\mathbb{S}^1$. Using a discrete Morse matching [9], it was shown more generally in [1] that:

Proposition 26 ([1, Theorem 3.9])**.** *Let C_n be a cycle graph, $n \in \mathbb{N}$, and $n \geq 5$. Then the $(n-3)$-dimensional cut complex $\Delta_2(C_n)$ is homotopy equivalent to the sphere $\mathbb{S}^{n-4}$.*

We show that for $k \geq 3$, the cut complex of the cycle graph is shellable. The Betti number follows by verifying the conditions in Theorem 23.

Theorem 27. *For all $n > k > 2$, the $(n-k-1)$-dimensional cut complex $\Delta_k(C_n)$ is shellable. It has the homotopy type of a wedge of $\binom{n-1}{k-1} - n$ spheres of dimension $(n-k-1)$.*

Bipartite graphs

Let E_n denote the edgeless graph on n vertices. The complete bipartite graph $K_{m,n}$ is the join of two edgeless graphs E_m and E_n. We use the structural result of Theorem 18, completely determining both the homotopy type and homology representation for the cut complexes of all complete multipartite graphs. For lack of space we state our results only for the complete bipartite case.

Let V_λ denote the irreducible module of the symmetric group $\mathfrak{S}_n$ indexed by the integer partition λ of n. The group $\mathfrak{S}_m \times \mathfrak{S}_n$ (resp. the wreath product $\mathfrak{S}_2[\mathfrak{S}_n]$) induces a representation on the rational homology of $\Delta_k(K_{m,n}), m \neq n$, (resp. $m = n$).

From Theorem 17, Theorem 18 and Proposition 10, we have, since $K_{m,n} = E_m * E_n$:

Theorem 28. *Let $1 \leq m \leq n$ and $2 \leq k$. Then $\Delta_k(K_{m,n})$ is shellable if and only if $m < k$. Furthermore, if $m < k \leq n$, then $\Delta_k(K_{m,n})$ is contractible, and if $k > n$, the cut complex is void and hence shellable.*

If $k \leq m \leq n$, the $(m+n-k-1)$-dimensional complex $\Delta_k(K_{m,n})$ is homotopy equivalent to a wedge of $\binom{m-1}{k-1}\binom{n-1}{k-1}$ spheres of dimension $m+n-2k$.

The representation of $\mathfrak{S}_m \times \mathfrak{S}_n$ on the rational homology is the irreducible $V_{(k,1^{m-k})} \otimes V_{(k,1^{n-k})}$ indexed by the pair of integer partitions $((k,1^{m-k}),(k,1^{n-k}))$.

If $m = n$, the representation of $\mathfrak{S}_2[\mathfrak{S}_n]$ is $\Phi_{\mathfrak{S}_2}[V_{(k,1^{n-k})}]$, where $\Phi_{\mathfrak{S}_2}$ equals sgn^{n-k-1}, and sgn is the sign representation of $\mathfrak{S}_2$.

The last statement uses [14, Proposition 2.3] to compute the homology representation for a product of posets.

The homotopy type and homology representation determined in Theorem 28 bear a striking resemblance to that of a complex studied by Linusson, Shareshian and Welker [10, Theorem 1.4, Conjecture 1.15], a connection we propose to explore further.

Grid Graphs

The m by n grid graph $G(m,n)$ is the Cartesian product of two paths, $P_m \times P_n$. Here we describe our most significant results on grid graphs. The first result is proved by a discrete Morse matching [9].

Theorem 29. *[1, Theorem 3.10] For $n, m \geq 2$, the $(mn-3)$-dimensional cut complex $\Delta_2(G(m,n))$ has the homotopy type of a wedge of $(m-1)(n-1)$ spheres of dimension $mn-4$.*

Complexes associated to grid graphs appear to be difficult to analyse. We obtain shellability of $\Delta_3(G(m,n))$ below as an unexpected corollary of Proposition 22. The Betti number is the result of a generalisation of Theorem 23, exploiting the fact that grid graphs are triangle-free.

Theorem 30. *Let $m, n \geq 2$. The 3-cut complex $\Delta_3(G(m,n))$ is shellable and has the homotopy type of a wedge of $\binom{mn-1}{2} - 5mn + 5(m+n) - 3$ spheres in dimension $m+n-4$.*

For $k = 4, 6$, we use poset topology tools to show that the homology of $\Delta_k(G(m,n))$ is torsion-free and concentrated in the top two dimensions, and give formulas for the reduced Euler characteristic.

We conjecture that $\Delta_k(G(m,n))$ is shellable for $mn-3 \geq k \geq 4$.

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