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Generalized Pitman–Stanley flow polytopes

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Abstract. The Pitman–Stanley polytope is well-studied and related to plane partitions with entries 0 and 1. We study a generalization of this polytope related to plane partitions with entries $0, 1, \ldots, m$. We show that this generalization is realized as a flow polytope of a grid graph. We give multiple characterizations of its vertices and give formulas for the number of vertices and faces. We also study formulas for its number of lattice points and volume.

Résumé. Le polytope de Pitman–Stanley a été très étudié et a des connexions aux partitions de plan avec entrées 0 et 1. Nous étudions une généralisation de ce polytope avec des connexions aux partitions de plan avec entrées $0, 1, \ldots, m$. Nous démontrons que cette généralisation peut être réalisée comme un polytope de flot d'un graphe en forme de grille. Nous donnons plusieurs caractérisations de ses sommets ainsi que des formules pour le nombre de sommets et de faces. Nous étudions également des formules pour le nombre de points à coordonnées entières et le volume.

Keywords: Pitman-Stanley polytope, flow polytope, plane partitions

1 Introduction

The eponymous Pitman-Stanley polytope introduced in [11] is a well-studied polytope in geometric, algebraic, and enumerative combinatorics. This polytope is defined as follows. For a positive integer n and vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{N}^n$, let

$$\mathrm{PS}_n(\mathbf{a}, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} \mid b_1 + \cdots + b_i \leq x_1 + \cdots + x_i \leq a_1 + \cdots + a_i \text{ for } i = 1, \ldots, n\},$$

i.e., $\mathrm{PS}_n(\mathbf{a}, \mathbf{b})$ consists of the nonnegative vectors $\mathbf{x}$ in $\mathbb{R}^n$ that are between vectors $\mathbf{a}$ and $\mathbf{b}$ in *dominance order*. (Recall that, for $\mathbf{v}, \mathbf{w} \in \mathbb{N}^n$, we say that $\mathbf{v}$ *dominates* $\mathbf{w}$ if $\sum_{j=1}^i v_j \geq \sum_{j=1}^i w_j$ for every $i = 1, \ldots, n$.)

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Postnikov showed that $\mathrm{PS}_n(\mathbf{a}) := \mathrm{PS}_n(\mathbf{a}, \mathbf{0})$ is a *generalized permutohedron* [10], and Baldoni and Vergne [1] realized $\mathrm{PS}_n(\mathbf{a})$ as a flow polytope of a graph (see Figure 1a).

Important results of [11] include that the polytope $\mathrm{PS}_n(\mathbf{a})$ is combinatorially equivalent to a product of simplices. We denote by Δ_d the d-dimensional simplex. Given $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_1 > 0$, its *signature* $\mathrm{sgn}(\mathbf{a}) := (c_1, \ldots, c_k)$ is a sequence of nonnegative integers such that $\chi(\mathbf{a}) := (1, 0^{c_1-1}, 1, 0^{c_2-1}, \ldots, 1, 0^{c_k-1})$ where 1 and 0 represent a positive a_i and zero a_i respectively. Then the polytope $\mathrm{PS}_n(\mathbf{a})$ is combinatorially equivalent to the product of simplices $\Delta_{c_1} \times \cdots \times \Delta_{c_k}$. In particular, if each $a_i > 0$, then $\mathrm{PS}_n(\mathbf{a})$ is combinatorially equivalent to an n-cube.

The volume of the polytope $\mathrm{PS}_n(\mathbf{a})$ is of interest in probability [11, §2] and it is known that $\operatorname{vol}\mathrm{PS}_n(\mathbf{1}) = (n+1)^{n-1}$, the number of *parking functions* of size n [11, Thm. 11]. Recall that a *plane partition* of skew shape λ/μ is an array π of nonnegative integers of shape λ/μ that is weakly decreasing in rows and columns. Pitman and Stanley also showed that the lattice points of $\mathrm{PS}_n(\mathbf{a}, \mathbf{b})$ are in correspondence with plane partitions of skew shape $\theta(\mathbf{a}, \mathbf{b}) := (a_1 + \cdots + a_n, \ldots, a_1 + a_2, a_1)/(b_1 + \cdots + b_n, \ldots, b_1 + b_2, b_1)$ with entries 0 and 1 [11, Thm. 12].

When $\mathbf{b} = \mathbf{0}$, note that $\theta(\mathbf{a}, \mathbf{b})$ is a straight shape and the number of lattice points of $\mathrm{PS}_n(\mathbf{1})$ is given by the *Catalan number* $C_{n+1} = \frac{1}{n+2}\binom{2n+2}{n+1}$. Pitman and Stanley also mentioned in [11, Sec. 5] how their polytope can be generalized so that lattice points correspond to plane partitions of the same shape with entries $0, 1, \ldots, m$ for a fixed positive integer m. The *mth generalized Pitman–Stanley polytope* is defined as follows.

$$\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b}) := \{(x_{ij})_{1 \le i \le n, 1 \le j \le m} \in \mathbb{R}^{nm}_{\ge 0} \quad | \quad b_1 + \cdots + b_i \le x_{1m} + x_{2m} + \ldots + x_{im} \le \cdots$$
$$\cdots \le x_{11} + x_{21} + \ldots + x_{i1} \le a_1 + \cdots + a_i \text{ for } i = 1, \ldots, n\},$$

i.e., $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ consists of nonnegative matrices (x_{ij}) in $\mathbb{R}^{nm}$ whose columns form *a multichain* of length m between $\mathbf{a}$ and $\mathbf{b}$ in dominance order. Note that $\mathrm{PS}_n(\mathbf{a}, \mathbf{b}) = \mathrm{PS}_n^1(\mathbf{a}, \mathbf{b})$.

This generalization is the object of this extended abstract whose full version will appear in [2, 3]. We show that the polytope $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ is also a flow polytope of a *grid graph* $G(n, m)$ denoted as $\mathcal{F}_{G(n,m)}(\mathbf{a}, \mathbf{b})$. See both Figures 1a and 1b for $m = 1$ and Figure 1c for general m. This allows us to use the theory of flow polytopes (e.g., [1, 5, 9]) to study the faces, lattice points, and volume of $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$. In Lemma 2.5, we give a bijection between the lattice points of $\mathcal{F}_{G(n,m)}(\mathbf{a}, \mathbf{b})$ and plane partitions of skew shape $\theta(\mathbf{a}, \mathbf{b})$ with entries $0, 1, \ldots, m$. Note that Liu–Mészáros–St. Dizier [7, 8] have realized *Gelfand–Tsetlin polytopes*, whose lattice points correspond to (skew) semistandard Young tableaux with bounded entries, as flow polytopes. These tableaux and plane partitions of skew shape with bounded entries are related but not equivalent (e.g., see [12, §7.10, 7.22]) and this extends to the associated polytopes.

Vertices and faces: Theorem 3.2 gives a characterization of the vertices of $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ as flows of $\mathcal{F}_{G(n,m)}(\mathbf{a}, \mathbf{b})$ whose support is a subgraph of $G(n, m)$ that is a forest. This

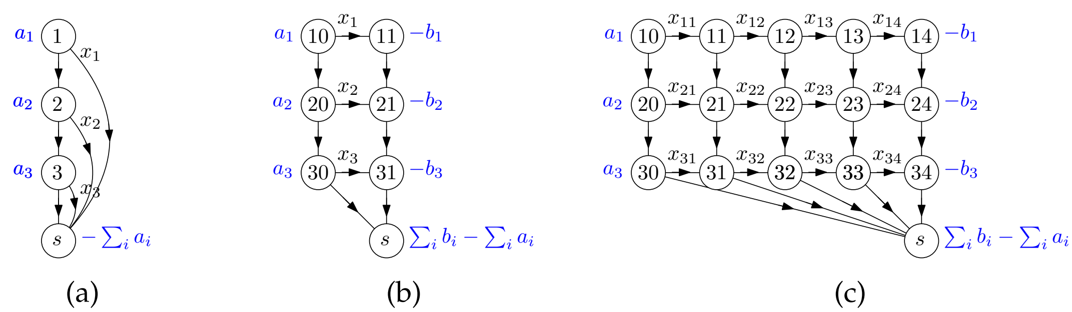


Figure 1: The first graph realizes $\mathrm{PS}_n^1(\mathbf{a}, \mathbf{0})$ for $n = 3$ as shown by Baldoni and Vergne [1]. The second and third graphs are $G(n, 1)$ and $G(n, m)$ which realize as flow polytopes the original and generalized Pitman–Stanley polytopes $\mathrm{PS}_n^1(\mathbf{a}, \mathbf{b})$ and $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$, respectively, for $n = 3$, $m = 1$ and $n = 3$, $m = 4$, respectively.

result follows from a general result of flow polytopes by Gallo and Sodini [4]. This characterization gives an explicit characterization of the flows on the grid corresponding to vertices in terms of *unsplittable flows* (see Corollary 3.5).

Since the lattice points of $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ are in bijection with plane partitions, in Theorem 3.7 we characterize the plane partitions corresponding to vertices, which we call *vertex plane partitions* (see Definition 3.6).

After giving a flow and plane partition interpretation of the vertices of $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$, we turn to enumerating them. Let $v^{(n,m)}(\mathbf{a})$ be the number of vertices of the polytope $\mathrm{PS}_n^m(\mathbf{a}) := \mathrm{PS}_n^m(\mathbf{a}, \mathbf{0})$. We show that the number $v^{(n,m)}(\mathbf{a})$ is a polynomial in m and show that this number has the following nonnegative expansion in the basis $\binom{m+1}{k}$ in terms of plane partitions of staircase shape $\delta_n = (n, n-1, \ldots, 1)$. Let $p_{n,k}$ be the number of vertex plane partitions of staircase shape δ_n with the set of entries equal to $\{0, 1, \ldots, k-1\}$.

Theorem 1.1. *For $\mathbf{a} \in \mathbb{N}^n$ such that $\chi(\mathbf{a}) = \mathbf{1}$, we have that $v^{(n,m)}(\mathbf{a}) = \sum_{k=1}^{\binom{n+1}{2}} p_{n,k}\binom{m+1}{k}$. In particular, $p_{n,\binom{n+1}{2}}$ equals the number of shifted* Standard Young Tableaux *(SYT) of staircase shape $(n, n-1, \ldots, 1)$ [14, A003121].*

Moreover, in Theorem 4.4, we give a recurrence for the number of unsplittable flows of $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ by fixing the flows on the first column. As a corollary, we obtain a recurrence for the number $v^{(n,m)}(\mathbf{a})$ of vertices in terms of a generalization of dominance order called 1-*domination* (see Definition 4.2).

Corollary 1.2. *Let $\mathbf{a} \in \mathbb{N}^n$, then $v^{(n,m)}(\mathbf{a}) = \sum_{\mathbf{j}} v^{(n,m-1)}(\mathbf{j})$, where the sum is over $\mathbf{j} \in \{0,1\}^n$ such that $\chi(\mathbf{a})$ 1-dominates $\mathbf{j}$.*

This recurrence allows us to compute generating functions for $\sum_{m \geq 0} v^{(n,m)}(\mathbf{a}) x^m$, for fixed n. Lastly, we give in Theorem 5.5 a recurrence for the number of d-faces of $\mathrm{PS}_n^m(\mathbf{a})$ similar to Corollary 1.2.

Lattice points, volume: Since lattice points of $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ correspond to plane partitions with bounded entries, their number is given by a determinant formula of Kreweras [6].

Corollary 1.3. *The number of lattice points of* $\mathrm{PS}_n^m(\boldsymbol{a},\boldsymbol{b})$ *has the following formula*

$$\#(\mathrm{PS}_n^m(\boldsymbol{a},\boldsymbol{b})\cap\mathbb{Z}^{mn}) \;=\; \det\left[\binom{a_1+\cdots+a_{n-i+1}-b_1-\cdots-b_{n-j+1}+m}{i-j+m}\right]_{i,j=1}^{n}.$$

By replacing a_i and b_j by $t\cdot a_i$ and $t\cdot b_j$ this gives a formula for the Ehrhart polynomial of $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ whose leading term in t gives the volume of the polytope. The case $\mathbf{a}=\mathbf{1}$ and $\mathbf{b}=\mathbf{0}$ is particularly nice for both lattice points and volume [11, Thm. 14].

Corollary 1.4 ([11])**.** *The polytope* $\mathrm{PS}_n^m(\mathbf{1})$ *has the following number of lattice points and volume*

$$\#(\mathrm{PS}_n^m(\mathbf{1})\cap\mathbb{Z}^{mn}) = \prod_{1\le i<j\le n+1}\frac{2m+i+j-1}{i+j-1},\quad \mathrm{vol}\,\mathrm{PS}_n^m(\mathbf{1}) = SYT(n^m)\cdot\prod_{i=1}^{m} i!(n+i)^{n-m-1+i},$$

where $SYT(n^m)$ is the number of SYT of shape $n\times m$.

Pitman and Stanley asked in [11] for a combinatorial proof of the volume formula above. We give such a proof for the case $m=2$ and give a bijection between SYT of shape $n\times m$ and the pieces of a subdivision of $\mathrm{PS}_n^m(\mathbf{1})$ coming from the *Lidskii formula* for flow polytopes [1, 9]. See [3] for details.

2 $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ as a flow polytope of a grid graph

Given a connected acyclic directed graph $G=(V,E)$ and a netflow vector $\mathbf{a}\in\mathbb{Z}^{|V|}$, let $\mathcal{F}_G(\mathbf{a})$ denote the *flow polytope* of graph G with *net flow* $\mathbf{a}$, i.e., the set of flows on G with net flow $\mathbf{a}$ (see, e.g., [9] for a detailed definition).

In [1, Example 16], Baldoni–Vergne showed that $\mathrm{PS}_n(\mathbf{a})$ is *integrally equivalent* to a flow polytope. We extend this result to $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$. The underlying directed graph is the following grid graph with a sink (see Figure 1c).

Definition 2.1. *For positive integers n and m, let $G(n,m)$ be a directed graph on the vertex set $V=\{(i,j)\,|\,1\le i\le n,\ 0\le j\le m\}\cup\{s\}$ and the following directed edges:*

- *$((i,j),(i,j+1))$ and $((i,j),(i+1,j))$ for $i\in\{1,2,\ldots,n-1\}$ and $j\in\{0,1,\ldots,m-1\}$.*
- *$((n,j),(n,j+1))$ and $((n,j),s)$ where $j\in\{1,2,\ldots,m-1\}$, and $((n,m),s)$.*
- *$((i,m),(i+1,m))$ where $i\in\{1,2,\ldots,n-1\}$.*

Definition 2.2. *For $\mathbf{a}=(a_1,a_2,\ldots,a_n)$ and $\mathbf{b}=(b_1,b_2,\ldots,b_n)$ in $\mathbb{R}^n_{\ge 0}$, and the graph $G(n,m)$ as defined above, denote by $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$ the flow polytope on $G(n,m)$ where:*

- *vertices* $(1,0),(2,0),(3,0),\ldots,(n,0)$ *have respective net flows* $a_1,a_2,\ldots,a_n$,
- *vertices* $(1,m),(2,m),(3,m),\ldots,(n,m)$ *have respective net flows* $-b_1,-b_2,\ldots,-b_n$,
- *the sink vertex* s *has net flow* $-\sum_{i=1}^n a_i + \sum_{i=1}^n b_i$, *and all other vertices have net flow* 0.

Example 2.3. *Flow polytope* $\mathcal{F}_{G(3,5)}(\mathbf{a},\mathbf{b})$ *is illustrated in Figure 1c.*

Theorem 2.4. *The polytope* $\mathrm{PS}_n^m(\boldsymbol{a},\boldsymbol{b})$ *is integrally equivalent to the flow polytope* $\mathcal{F}_{G(n,m)}(\boldsymbol{a},\boldsymbol{b})$.

Proof sketch. Let $\Phi : \mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b}) \to \mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ be defined as follows:

$$x_{i,j} = f((i,j-1),(i,j)), \quad \text{for } i \in \{1,2,\ldots,n\} \text{ and } j \in \{1,2,\ldots,m\},$$

that is, $x_{i,j}$ is the flow of the horizontal edge $((i,j-1),(i,j))$. See Figure 1c. One can check that Φ gives the desired integral equivalence. □

We denote integral equivalence of polytopes by $\equiv$.

Lemma 2.5. *There is a bijection between integral flows for* $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b}) \equiv \mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$ *and plane partitions of shape* $\theta(\mathbf{a},\mathbf{b})$ *with entries at most* m.

Proof. Let $\pi = (\pi_{ij})$ be a plane partition of shape $\theta(\mathbf{a},\mathbf{b})$ with entries at most m. Consider a column j with (non-empty) entries $\pi_{f_j,j},\pi_{f_j+1,j},\ldots,\pi_{i_j,j}$ where f_j depends on $\mathbf{b}$. (In the case when $\mathbf{b}=\mathbf{0}$, then $f_j = 1$ for every column.) This corresponds to the *trajectory* of one unit of flow in the graph $G(n,m)$ that starts at vertex $(n+1-i_j,0)$, goes to $(n+1-i_j,\pi_{i_j,j})$, then to $(n+1-i_j+1,\pi_{i_j,j})$, then to $(n+1-i_j+1,\pi_{i_j-1,j})$, then to $(n+1-i_j+2,\pi_{i_j-1,j})$, and so on. At the end, two cases happen: if $f_j = 1$, it keeps going until it reaches $(n,\pi_{1,j})$ from where it goes to the sink s, or if $f_j \geq 2$, until it reaches $(n+2-f_j,\pi_{f_j,j})$ from where it goes to the vertex $(n+2-f_j,m)$ with negative netflow $-b_{n+2-f_j}$ if $f \geq 2$. In other words, the entries of a column of π give the columns of $G(n,m)$ where a unit of flow starting at vertex $(n+1-i_j,0)$ goes down.

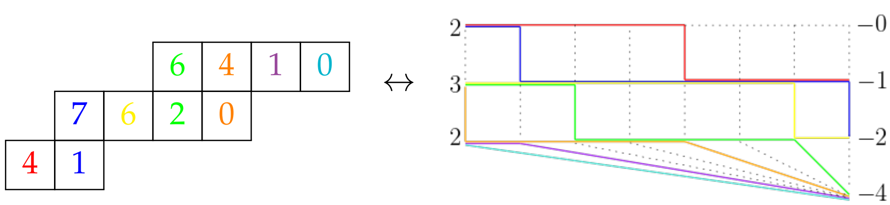


Figure 2: Constructing an integral flow on $G(3,7)$ from a skew plane partition.

For the inverse map see the long version of the paper [2]. See Figure 2 for an example. □

Figure 2 shows the *trajectory decomposition* of an integer $(\mathbf{a},\mathbf{b})$-flow.
As a corollary one can count the lattice points of $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ with a determinant.

Proof of Corollary 1.3. By Lemma 2.5, the number of lattice points of $\mathrm{PS}_n^m(\mathbf{a},\mathbf{b})$ equals the number of plane partitions of shape $\theta(\mathbf{a},\mathbf{b})$ with entries at most m. The result then follows by the following result of Kreweras [6, §2.3.7]: the number of plane partitions of shape λ/μ with entries at most m equals $\det\left[\binom{\lambda_i-\mu_j+m}{i-j+m}\right]_{i,j=1}^{\ell}$. □

3 Flow and plane partition characterization of vertices

The dimension of the flow polytope $\mathcal{F}_G(\mathbf{a})$ is $\beta_1(G) := |E| - |V| + 1$ [1, §1.1]. Hille studied flow polytopes and their faces in [5] in the context of *quivers*. The following notion of his is important to understand the faces of $\mathcal{F}_G(\mathbf{a})$.

Definition 3.1 (**a**-valid[1])**.** *Given a flow polytope $\mathcal{F}_G(\mathbf{a})$ with underlying graph G, a subgraph H of G is* **a***-valid if the edges of H are the support of some* **a***-flow on G.*

The following characterization of vertices of $\mathcal{F}_G(\mathbf{a})$ appeared in earlier work of Gallo and Sodini [4, Thm. 3.1].

Theorem 3.2 ([4])**.** *Vertices of $\mathcal{F}_G(\mathbf{a})$ are in correspondence with* **a***-valid subforests of G.*

We can use Theorem 3.2 to give a local criterion (at each vertex of G) for whether or not a flow is a 0-cell of the corresponding flow polytope, as follows.

Definition 3.3. *For an integer $(\mathbf{a},\mathbf{b})$-flow on $G(n,m)$, the trajectories of two units of flow split (resp. merge) if*

- *at a vertex where the two trajectories meet, they leave (resp. enter) through different edges,*
- *or if at a vertex with negative (resp. positive) net flow where the two trajectories meet, exactly one leaves (resp. enters) through an edge.*

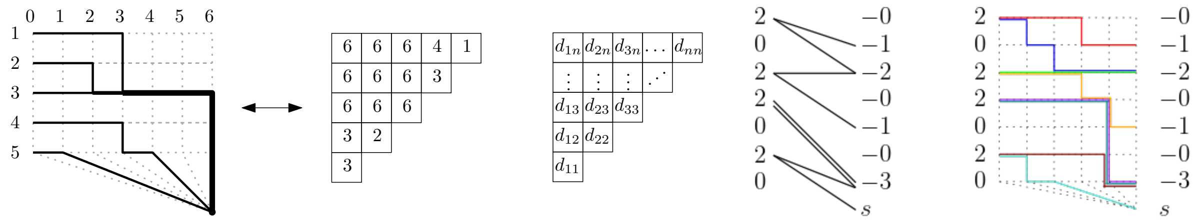
An integer $(\mathbf{a},\mathbf{b})$-flow $\mathbf{x}$ in $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$ is an unsplittable flow if no two of its trajectories split.

Example 3.4. *Figure 3a (left) is an example of an unsplittable flow.*

From the proof of Lemma 2.5, recall that any integral flow for $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$ can be seen as non-crossing trajectories of units of flow. Through this interpretation, we see that the ith trajectory from the top ends at the same vertex in any integral flow for $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$. Note that this does not depend on m. See Figure 3c. The next result gives a characterization of the vertices of $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$ in terms of unsplittable flows. See Figure 3d for an example.

Corollary 3.5. *Consider an integral flow for $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$ with $\mathbf{a},\mathbf{b} \in \mathbb{N}^n$. That flow corresponds to a vertex of $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$ if and only if the following conditions are satisfied.*

[1] In Hille [5], these are called *regular* flows.



(a) (b) (c) (d)

Figure 3: (a) an unsplittable flow and the corresponding vertex plane partition, (b) plane partition encoding where flows corresponding to vertices descend, (c) Starting and ending vertices for all trajectories for any flow in $\mathcal{F}_{G(5,m)}((2,0,2,2,0,2,0),(0,1,2,0,1,0,3))$ for any m, and (d) example of a vertex flow from Corollary 3.5.

(i) The trajectories of any two units of flow starting at the same vertex with positive net flow do not merge.

(ii) The trajectories of any two units of flow ending at the same vertex with negative net flow (including sink s) do not split.

(iii) The trajectories of any two units of flow starting and ending at different vertices can split at most once and merge at most once.

In particular, if $\mathbf{b} = \mathbf{0}$*, flows corresponding to vertices of* $\mathcal{F}_{G(n,m)}(\mathbf{a})$ *are unsplittable flows.*

Proof. First note that if any of (i), (ii) or (iii) does not hold for some integral flow for $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$, then the flow contains a cycle. By Theorem 3.2, this implies that this flow does not correspond to a vertex of $\mathcal{F}_{G(n,m)}(\mathbf{a},\mathbf{b})$. Contrapositively, this means that if a flow is a vertex, then (i), (ii), (iii) must all hold.

For the converse, suppose that (i), (ii), (iii) all hold for some integral flow and that this flow contains a cycle between the trajectories of two units of flow. Note that any cycle contains both a merge and a split. Therefore, if the trajectories of those two units started or ended at the same vertex of $G(n,m)$, we would have a contradiction to (i) and (ii) respectively. Finally, note that if the two units started and ended at different vertices, there must be at least one merge before the cycle and a split after the cycle, which means that in total, there are at least two merges and two splits, a contradiction to (iii). Therefore, if (i), (ii), (iii) hold for some integral flow, then the flow contains no cycles, and by Theorem 3.2, it corresponds to a vertex. □

Next, we give a plane partition characterization for the vertices.

Definition 3.6. *Let n and m be nonnegative integers. A plane partition $\pi = (\pi_{ij})$ of staircase shape $\delta_n = (n, n-1, \ldots, 1)$ with entries at most m is a vertex plane partition if (i) either $\pi_{i,j} < \pi_{i+1,j-1}$ or $\pi_{i,j} = \pi_{i-1,j}$, and (ii) if $\pi_{i,j} = \pi_{i,j+1}$ then $\pi_{i-1,j} = \pi_{i-1,j+1}$.*

Theorem 3.7. *Let n and m be nonnegative integers, then the vertices of* $\mathrm{PS}_n^m(\mathbf{a}) \equiv \mathcal{F}_{G(n,m)}(\mathbf{a})$ *with* $\chi(\mathbf{a}) = \mathbf{1}$ *are in correspondence with the vertex plane partitions.*

Proof. As was explained in Lemma 2.5, any lattice point $\mathbf{x}$ in $\mathcal{F}_{G(n,m)}(\mathbf{a})$ corresponds to some integer flow in $G(n,m)$. Moreover, we know that if an integral flow is a vertex of $\mathcal{F}_{G(n,m)}(\mathbf{a})$, then no two trajectories split. Let d_{tr} be the index of the column in $G(n,m)$ where the flow starting at $(t,0)$ (for $0 \leq t \leq n$) descends from row r to row $r+1$. Since $\chi(\mathbf{a}) = \mathbf{1}$, we get a plane partition of shape $(n, n-1, \ldots, 1)$ recording the numbers d_{tr} (see Figure 3b). We already know that $d_{tr} \geq d_{t+1,r}$, and $d_{t,r} \geq d_{t,r+1}$ for any flow. For an unsplittable flow, we know that if the unit flows starting respectively at $(t,0)$ and $(t',0)$ where $t < t'$ join, then they never split again. Moreover, if in row r of $G(n,m)$, those two flows haven't joined yet, then the flow for t' must descend to row $r+1$ before the flow for t descends *into* row r, i.e., $d_{t',r} < d_{t,r-1}$. If on the other hand those two flows have already joined in or by row r, then they will descend together in row r, and they will also descend together in all subsequent rows, i.e., $d_{t,r'} = d_{t',r'}$ for all $r \leq r' \leq n$. Thus we obtain the conditions for a vertex plane partition $\pi = (\pi_{ij})$, $\pi_{ij} = d_{j,n+1-i}$ (see Figure 3b). This argument can be generalized for the skew case, $\mathbf{b} \neq \mathbf{0}$ [2]. □

Example 3.8. *Figure 3a illustrates the bijection between an unsplittable flow of the polytope* $\mathrm{PS}_5^6(\mathbf{1}) \equiv \mathcal{F}_{G(5,6)}(\mathbf{1})$ *and a vertex plane partition.*

4 Enumeration of number of vertices

Let $v^{(n,m)}(\mathbf{a},\mathbf{b})$ be the number of unsplittable flows of $PS_n^m(\mathbf{a},\mathbf{b})$, and let $v^{(n,m)}(\mathbf{a}) := v^{(n,m)}(\mathbf{a},\mathbf{0})$, which by Corollary 3.5 gives the number of vertices of $\mathrm{PS}_n^m(\mathbf{a})$. In this section, we give expansions and recursions for $v^{(n,m)}(\mathbf{a},\mathbf{b})$.

4.1 Nonnegative combinations of binomial coefficients

We show that $v^{(n,m)}(\mathbf{a})$ is a polynomial in m by giving a nonnegative expansion in terms of the polynomial basis $\binom{m+1}{k}$.

Proof of Theorem 1.1. We know from Theorem 3.7 that $v^{(n,m)}(\mathbf{a})$ counts the number of vertex plane partitions. We partition these according to how many different values they use. Consider all the vertex plane partitions using k values. You can further partition these by the k values among $\{0,1,\ldots,m\}$ that they use. There are $\binom{m+1}{k}$ such classes, and each class contains the same number of vertex plane partitions. Since δ_n has $\binom{n+1}{2}$ cells, there can be at most that many different labels within a plane partition, and so the result holds. For the bijection between plane partitions counted by $p_{n,\binom{n+1}{2}}$ and shifted SYT of staircase shape δ_n, see Figure 4 and the long version of [2]. □

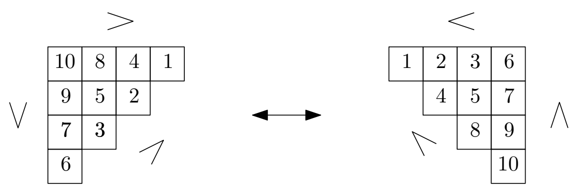


Figure 4: Example of correspondence between vertex plane partitions with distinct entries and shifted SYT of staircase shape.

4.2 Recurrences by fixing the flow of a column of $G(n, m)$

For some flow $\mathbf{x}$ on $G(n, m)$ and a column c, for $0 \leq c \leq m-1$, of horizontal arcs $((i, c),(i, c+1))$ for $1 \leq i \leq n$ in $G(n, m)$, let $\mathbf{x}_{\bullet c} := (x_{1c}, x_{2c}, \ldots, x_{nc})$ be the vector recording the flow on column c. Further, we let $v^{(n,m)}(\mathbf{a}, \mathbf{b}, c, \mathbf{u})$ for $0 \leq c \leq m-1$ be the number of vertices $\mathbf{x}$ in $PS_n^m(\mathbf{a}, \mathbf{b})$ such that $\mathbf{x}_{\bullet c} = \mathbf{u}$. Note that, for $\mathbf{a} \in \mathbb{N}^n$, by partitioning all different possible flows $\mathbf{x}$ on $G(n, m)$ by their flow $\mathbf{x}_{\bullet c}$ for some column $0 \leq c \leq m-1$, we have that $v^{(n,m)}(\mathbf{a}, \mathbf{b}) = \sum_{\mathbf{u} \in \mathbb{N}^n} v^{(n,m)}(\mathbf{a}, \mathbf{b}, c, \mathbf{u})$. Indeed, as explained earlier, if $\mathbf{a} \in \mathbb{N}^n$, then unsplittable flows in $\mathrm{PS}_n^m(\mathbf{a}, \mathbf{b})$ are integral. Furthermore, since flow can only go down and right, we can immediately note that $v^{(n,m)}(\mathbf{a}, \mathbf{b}, c, \mathbf{u}) = 0$ if $\mathbf{a}$ does not dominate $\mathbf{u}$. So instead of summing over all vectors in $\mathbb{N}^n$, we already know that

$$v^{(n,m)}(\mathbf{a}, \mathbf{b}) = \sum_{\substack{\mathbf{u} \in \mathbb{N}^n: \\ \mathbf{a} \text{ dominates } \mathbf{u}}} v^{(n,m)}(\mathbf{a}, \mathbf{b}, c, \mathbf{u}).$$

Knowing where $\mathbf{a}$ is 0 and where it is positive, which is captured by $\chi(\mathbf{a})$, allows to further make this sum smaller.

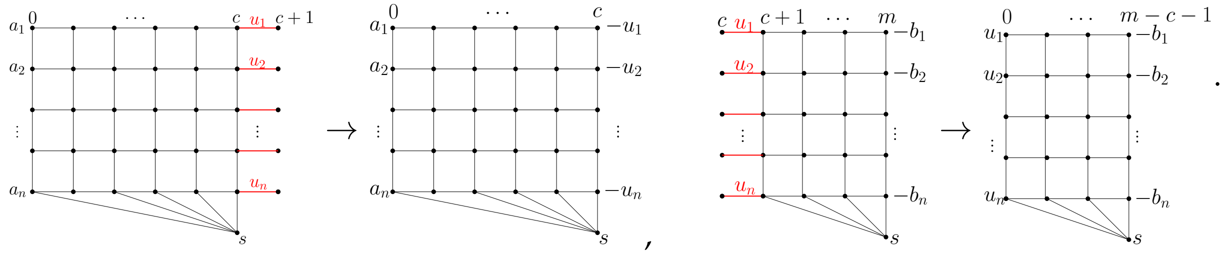
Note that, because of unsplittability, for $\mathbf{a}, \mathbf{a}' \in \mathbb{N}^n$, if $\chi(\mathbf{a}) = \chi(\mathbf{a}')$, then we have that $v^{(n,m)}(\mathbf{a}) = v^{(n,m)}(\mathbf{a}')$.

In this section, we refine our understanding of when $v^{(n,m)}(\mathbf{a}, \mathbf{b}, c, \mathbf{u}) = 0$ to give a recursion to calculate $v^{(n,m)}(\mathbf{a}, \mathbf{b})$ based on the following: we can calculate $v^{(n,m)}(\mathbf{a}, \mathbf{b}, c, \mathbf{u})$ by calculating the product of the number of flows on two graphs obtained after deleting column c of horizontal arcs in $G(n, m)$.

Lemma 4.1. *Fix positive integers n, m and let $1 \leq c \leq m-2$. Then*

$$v^{(n,m)}(\mathbf{a}, \mathbf{b}) = \sum_{\substack{\mathbf{u} \in \mathbb{N}^n: \\ \mathbf{a} \text{ dominates } \mathbf{u} \text{ and} \\ \chi(\mathbf{a}) \text{ dominates } \chi(\mathbf{u})}} v^{(n,c)}(\mathbf{a}, \mathbf{u}) \cdot v^{(n,m-c-1)}(\mathbf{u}, \mathbf{b}).$$

Proof. We observe that the number of unsplittable flows counted by $v^{(n,m)}(\mathbf{a}, \mathbf{b}, c, \mathbf{u})$ is the product of the number of unsplittable flows on the following two graphs:



□

For the recursions, we need to understand unsplittable flows. This will also allow us to characterize what $\mathbf{x}_{\bullet c+1}$ can be if one is given $\mathbf{x}_{\bullet c}$. The patterns of zeros will end up playing an important role through the following definition.

Definition 4.2. *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{N}^n$ *be such that* $\mathbf{a}$ *dominates* $\mathbf{b}$*, and* $\chi(\mathbf{a}) =: \mathbf{v}$ *dominates* $\chi(\mathbf{b}) =: \mathbf{w}$*. If* $(v_{z+2}, v_{z+3}, \ldots, v_n)$ *dominates* $(w_{z+2}, w_{z+3}, \ldots, w_n)$ *for every* $z \in [n]$ *such that* $b_z = 0$*, then we say that* $\mathbf{a}$ 1-dominates $\mathbf{b}$*. Note that* 1*-dominance does not yield a poset.*

Example 4.3. *The vector* $(3, 2, 0, 0, 2, 3, 4)$ 1*-dominates* $(3, 0, 1, 0, 1, 0, 6)$.

Theorem 4.4. *Let* $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$*. Then*

$$v^{(n,m)}(\mathbf{a}, \mathbf{b}) = \sum_{\substack{\mathbf{j} \in \{0,1\}^n: \\ \chi(\mathbf{a}) \text{ 1-dominates } \mathbf{j}}} v^{(n,m-1)}(\mathbf{w_j}, \mathbf{b}),$$

where $\mathbf{w_j}$ *is the vector defined as follows. Let* i_0 *be the smallest index such that* $j_{i_0} = 1$*. Then* $w_{\mathbf{j},i_0} = \sum_{i=1}^{i_0} a_i$*. Further, for any* $i_1 < i_2$ *such that* $j_{i_1} = j_{i_2} = 1$*, and* $j_i = 0$ *for all* $i_1 < i < i_2$*,* $\mathbf{w}_{\mathbf{j},i_2} = \sum_{i=i_1+1}^{i_2} a_i$*. All other components of* $\mathbf{w_j}$ *are zero. Note that* $\chi(\mathbf{w_j}) = \mathbf{j}$*.*

In particular when $\mathbf{b} = \mathbf{0}$, we get the recurrence in Corollary 1.2 for the number of vertices of $\mathrm{PS}_n^m(\mathbf{a})$.

Remark 4.5. *This result allows to compute* $v^{(n,m)}(\mathbf{a}, \mathbf{b})$ *via a* transfer-matrix *[13, Sec. 4.7] approach: adding entries of powers of an adjacency matrix of a digraph with vertices* $\mathbf{j} \in \{0,1\}^n$ *and arcs* $(\mathbf{j}, \mathbf{k})$ *if* $\mathbf{j}$ 1*-dominates* $\mathbf{k}$*. This allows us to find generating functions like*

$$\sum_{m \geq 0} v^{5,m}(\mathbf{1}) x^m = \frac{1 + 16x + 4x^2 + 48x^3 + 62x^4 + 20x^5 + 88x^6 + 14x^7 + 37x^8 - 8x^9 + 4x^{10}}{(1-x)^{16}}.$$

5 Enumeration of d-faces of $\mathrm{PS}_n^m(\mathbf{a})$

In this section, we use a similar recurrence to the one in Theorem 4.4, i.e., fixing the flow in the first column of $G(n, m)$, to give a recurrence for the number of higher-dimensional faces of $\mathrm{PS}_n^m(\mathbf{a}) = \mathcal{F}_{G(n,m)}(\mathbf{a})$. We assume that $\mathbf{b} = \mathbf{0}$.

First, we need the following description of the faces of flow polytopes $\mathcal{F}_G(\mathbf{a})$ due to Hille [5, part of Thm. 3.2] for the case $a_i \geq 0$. A graph G is $\mathbf{a}$-*valid* if it admits an $\mathbf{a}$-flow.

Theorem 5.1 ([5])**.** *Let $\mathcal{F}_G(\boldsymbol{a})$ be a flow polytope and $d \geq 0$ be a nonnegative integer. Then the d-dimensional faces of $\mathcal{F}_G(\boldsymbol{a})$ are of the form $\mathcal{F}_H(\boldsymbol{a})$ where H is an $\mathbf{a}$-valid connected subgraph of G such that $\beta_1(H) = |E(H)| - |V(H)| + 1 = d$, after forgetting the directions of edges of H.*

Let $f_d^{(n,m)}(\mathbf{a})$ be the number of d-dimensional faces of $PS_n^m(\mathbf{a})$. For $\mathbf{u} \in \{0,1\}^n$, let $f_d^{(n,m)}(\mathbf{a}, \mathbf{u})$ be the number of d-dimensional faces of $PS_n^m(\mathbf{a})$ corresponding to $\mathbf{a}$-valid subgraphs of $G(n,m)$ with support $\mathbf{u}$ in the mth column.

Lemma 5.2. *Fix positive integers n, m, and let $1 \leq c \leq m-2$. Then:*

$$f_d^{(n,m)}(\mathbf{a}) = \sum_{\mathbf{u}\in\{0,1\}^n} \sum_{i=0}^{d} f_i^{(n,c)}\left(\chi(\mathbf{a}), \mathbf{u}\right) \cdot f_{d-i}^{(n,m-c-1)}(\mathbf{u}) \tag{5.1}$$

Proof sketch. The proof follows the same approach as Lemma 4.1 but using Theorem 5.1. □

In the next theorem, we obtain a more explicit recurrence by fixing $c = 1$ in the previous lemma. We need the following notation and omit the proof.

Definition 5.3. *Let $\mathbf{j}$ and $\mathbf{a}$ be vectors of the same length such that $\mathbf{a}$ has signature $\mathrm{sgn}(\mathbf{a}) = (b_1 - 1, b_2 - 1, \ldots, b_\ell - 1)$. Then the **block structure of $\mathbf{j}$ according to $\mathbf{a}$**, denoted $\mathrm{sgn}_{\mathbf{a}}(\mathbf{j})$, is the tuple $(B_1, ..., B_\ell)$ where B_1 consists of the first b_1 elements of $\mathbf{j}$, B_2 consists of the next b_2 elements of $\mathbf{j}$ and so on.*

Example 5.4. *Pick $\mathbf{a}$ with $\chi(\mathbf{a}) = (1,1,0,1,0,0,1,0,1)^T$ and let $\mathbf{j} = (1,0,0,0,1,1,1,0,0)^T$. Then the signature of $\mathbf{a}$ is $\mathrm{sgn}(\mathbf{a}) = (0,1,2,1,0)$. Hence the block structure of $\mathbf{j}$ according to $\mathbf{a}$ is the tuple $((1),(0,0),(0,1,1),(1,0),(0))$.*

Theorem 5.5. *Let n and m be positive integers, $\mathbf{a} \in \mathbb{N}^n$, and $d \in \{0,1,\ldots,nm\}$ then for $m > 1$:*

$$f_d^{(n,m)}(\mathbf{a}) = \sum_{\mathbf{j}\in\{0,1\}^n} \sum_{k=0}^{d} \binom{\beta_{\mathbf{j},\mathbf{a}}}{k} f_{d-k-\gamma_{\mathbf{j},\mathbf{a}}}^{(n,m-1)}(\mathbf{j}),$$

where $\beta_{\mathbf{j},\mathbf{a}}$ is the number of nonzero blocks of $\mathrm{sgn}_{\mathbf{a}}(\mathbf{j})$, $\gamma_{\mathbf{j},\mathbf{a}} := b_{\mathbf{j}} - \beta_{\mathbf{j},\mathbf{a}}$, and $b_{\mathbf{j}}$ is the total number of 1's appearing in $\mathbf{j}$. For $m = 1$, we have $f_d^{(n,1)}(\mathbf{j}) = [x^{d+k}] \prod_{i=1}^{k}((x+1)^{c_i+1} - 1)$ where $(c_1, \ldots, c_k) := \mathrm{sgn}(\mathbf{j})$.

Corollary 5.6. *For $\mathbf{a} \in \mathbb{N}^n$ with $\chi(\mathbf{a}) = \mathbf{1}$, then $f_d^{(n,m)}(\mathbf{a}) = \sum_{\mathbf{j}\in\{0,1\}^n} \sum_{k=0}^{d} \binom{b_{\mathbf{j}}}{k} f_{d-k}^{(n,m-1)}(\mathbf{j})$, where $b_{\mathbf{j}}$ is the number of ones appearing in $\mathbf{j}$.*

Example 5.7. *For $m = n = 2$, $f_d^{(2,2)}(1,1) = \left(f_d^{(2,1)}(1,1) + 2 \cdot f_{d-1}^{(2,1)}(1,1) + f_{d-2}^{(2,1)}(1,1)\right) + \left(f_d^{(2,1)}(1,0) + f_{d-1}^{(2,1)}(1,0)\right) + \left(f_d^{(2,1)}(0,1) + f_{d-1}^{(2,1)}(0,1)\right) + f_d^{(2,1)}(0,0)$. One can use this to obtain the f-vector $(1,10,21,18,7,1)$ of $\mathrm{PS}_2^2(1,1)$.*

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We were inspired by [7, 8] to extend a link between skew plane partitions with bounded parts and flow polytopes.

References

[1] W. Baldoni and M. Vergne. "Kostant partitions functions and flow polytopes". *Transform. Groups* **13**.3-4 (2008), pp. 447–469.

[2] W. T. Dugan, M. Hegarty, A. H. Morales, and A. Raymond. "Pitman–Stanley flow polytopes: vertices and faces". In preparation.

[3] W. T. Dugan, M. Hegarty, A. H. Morales, and A. Raymond. "Pitman–Stanley flow polytopes: volume and lattice points". In preparation.

[4] G. Gallo and C. Sodini. "Extreme points and adjacency relationship in the flow polytope". *Calcolo* **15**.3 (1978), pp. 277–288. DOI.

[5] L. Hille. "Quivers, cones and polytopes". *Linear Algebra and its Applications* **365** (2003). Special Issue on Linear Algebra Methods in Representation Theory, pp. 215–237. DOI.

[6] G. Kreweras. "Sur une classe de problèmes de dénombrement liés au treillis des partitions des entiers". *undefined* (1965). Link.

[7] R. I. Liu, K. Mészáros, and A. St. Dizier. "Gelfand-Tsetlin polytopes: a story of flow and order polytopes". *SIAM J. Discrete Math.* **33**.4 (2019), pp. 2394–2415. DOI.

[8] R. I. Liu, K. Mészáros, and A. St. Dizier. "Schubert polynomials as projections of Minkowski sums of Gelfand-Tsetlin polytopes". *Combin. Theory* **2**.3 (2022).

[9] K. Mészáros and A. H. Morales. "Volumes and Ehrhart polynomials of flow polytopes". *Math. Z.* **293**.3-4 (2019), pp. 1369–1401. DOI.

[10] A. Postnikov. "Permutohedra, Associahedra, and Beyond". *International Mathematics Research Notices* **2009**.6 (Jan. 2009), pp. 1026–1106.

[11] R. P. Stanley and J. Pitman. "A polytope related to empirical distributions, plane trees, parking functions, and the associahedron". *Discrete Comput. Geom.* **27**.4 (2002), pp. 603–634.

[12] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Vol. 62. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999, pp. xii+581.

[13] R. P. Stanley. *Enumerative combinatorics. Volume 1*. Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, pp. xiv+626.

[14] *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation Inc. Link.