Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #81, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

On linear intervals in the alt ν-Tamari lattices

Cesar Ceballos*[1], and Clément Chenevière†[2,3]

[1] *Institute of Geometry, TU Graz, Graz, Austria*
[2] *Institut de Recherche Mathématique Avancée, Université de Strasbourg, Strasbourg, France*
[3] *Ruhr-Universität Bochum, Bochum, Germany*

Abstract. Given a lattice path ν, the ν-Tamari lattice and the ν-Dyck lattice are two natural examples of partial order structures on the set of lattice paths that lie weakly above ν. In this paper, we introduce a more general family of lattices, called alt ν-Tamari lattices, which contains these two examples as particular cases. Unexpectedly, we show that all these lattices have the same number of linear intervals.

Résumé. Étant donné un chemin discret ν, le treillis ν-Tamari et le treillis ν-Dyck sont deux ordres partiels très naturels sur l'ensemble des chemins qui restent toujours au-dessus de ν. Dans cet article, nous définissons une famille plus générale de treillis, appelés les treillis alt ν-Tamari, qui contient ces deux ordres partiels comme cas particuliers. Nous montrons que, de façon surprenante, tous ces treillis possèdent le même nombre d'intervalles linéaires.

Keywords: Tamari lattice, Dyck lattice, linear intervals.

1 Introduction

The classical Tamari lattice is a partial order on Catalan objects which has inspired a vast amount of research in various mathematical fields [11]. One direction of research which has received a lot of attention in recent years regards its number of intervals [7], which is conjectured to be equal to the dimension of the alternating component in the study of trivariate diagonal harmonics [10]. Motivated by this intriguing connection, Bergeron introduced a generalization of the Tamari lattice called the m-Tamari lattice, and conjectured that its number of intervals again coincides with the dimension of the alternating component in higher trivariate diagonal harmonics [1]. A formula for their enumeration and connections to representation theory can be found in [2, 3]. A further generalization of the Tamari lattice, which includes the m-Tamari lattice, is the ν-Tamari lattice introduced by Préville-Ratelle and Viennot [12]. These lattices are indexed by a lattice path ν, and their number of intervals is connected to the enumeration of non-separable planar maps as shown in [9].

*cesar.ceballos@tugraz.at. Supported by the Austrian Science Fund FWF, grants P 33278 and I 5788.
†ccheneviere@unistra.fr. Supported by the French National Research Agency, grants ANR-19-CE48-0011 and ANR-21-CE48-0020.

Inspired by the enumeration of intervals in the classical Tamari lattice and its generalizations, and guided by computer experimentation, Chapoton proposed to study the enumeration of the simpler class of linear intervals (intervals which are chains). This led to the work of the second author in [8], where he provides an explicit simple formula for the number of linear intervals in the classical Tamari lattice, and shows that their enumeration coincides with the enumeration of linear intervals in the Dyck lattice. The Dyck lattice, sometimes called the Stanley lattice, is perhaps the most natural poset on Dyck paths, defined by $P \leq Q$ if Q is weakly above P. In [8], the author also defines a new family of posets called alt Tamari posets, which contain the Tamari lattice and the Dyck lattice as particular cases. He shows that all alt Tamari posets have the same number of linear intervals of any given length.

In this paper, we generalize the results in [8] by introducing a new family of posets called alt ν-Tamari posets. We show that they are lattices, and that they all have the same number of linear intervals of any given length. Figure 1 and Figure 2 illustrate the three different alt ν-Tamari lattices for $\nu = ENEENN$. In each case, the number of linear intervals of length k is given by ℓ_k where $\ell = (\ell_0, \ell_1, \ell_2, \ell_3) = (16, 24, 16, 3)$. For instance, 16 represents the trivial intervals of length 0, which are just the elements of each poset; there are 24 linear intervals of length 1, which correspond to the cover relations (edges in the figures); there are 16 linear intervals of length 2, and 3 linear intervals of length 3. The fact that these numbers coincide is somewhat surprising, since the posets look quite different. As a warm up exercise, the reader is invited to find the 3 linear intervals of length 3 in each of the figures.

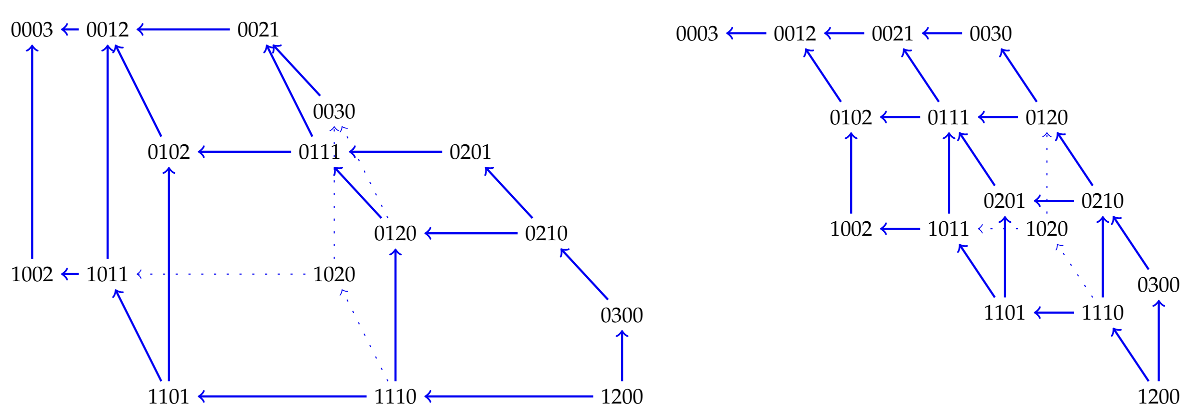

Figure 1: The ν-Tamari lattice and ν-Dyck lattice for $\nu = ENEENN$. They are the alt ν-Tamari lattices $\mathrm{Tam}_\nu(\delta)$ for $\delta = (2, 0, 0)$ and $\delta = (0, 0, 0)$, respectively.

As the figures suggest, the alt ν-Tamari lattices possess a rich underlying geometric structure, which seems to be realizable as a polytopal complex in some Euclidean space.

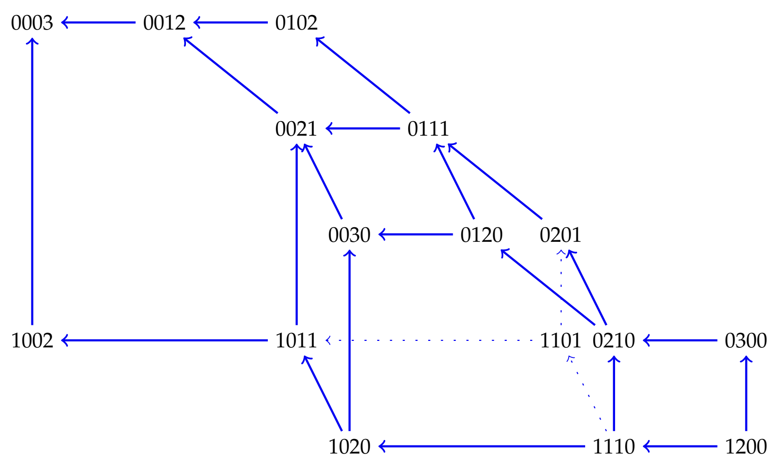

Figure 2: The alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ for $\nu = ENEENN$ and $\delta = (1,0,0)$.

This was shown to be true for ν-Tamari lattices in [5], where polytopal complex realizations induced by some arrangements of tropical hyperplanes are provided. Similar geometric realizations in this general context will be presented in forthcoming work.

2 The ν-Dyck lattice

Let ν be a lattice path on the plane consisting of a finite number of north and east unit steps. We may represent a path ν as a word in the letters E and N for east and north steps respectively. We may as well represent ν as a sequence of non negative integers $(\nu_0, \nu_1, \ldots, \nu_k)$, where $k \in \mathbb{N}$ is the number of north steps of ν, ν_0 is the number of initial east steps, and $\nu_i \geq 0$ is the number of consecutive east steps immediately following the i-th north step of ν. For instance, the path $ENEENNENEEE$ would correspond to the sequence $(1,2,0,1,3)$, while $ENEENN$ corresponds to (1,2,0,0).

A *ν-path* μ is a lattice path using north and east steps, with the same endpoints as ν, that is weakly above ν. Alternatively, $\mu = (\mu_0, \ldots, \mu_k)$ is a ν-path if and only if $\sum_{i=0}^{j} \mu_i \leq \sum_{i=0}^{j} \nu_i$ for all $0 \leq j \leq k$. The elements of the posets in Figure 1 and Figure 2 are labelled by ν-paths using this representation, were we omit the commas and parentheses for simplicity. For instance, the label 1200 is the minimal path $(1,2,0,0)$, which corresponds to $\nu = ENEENN$.

Definition 1. *The* ν-Dyck lattice Dyck_ν *is the poset on ν-paths where $P \leq Q$ if Q is weakly above P.*

2.1 Left and right intervals in the ν-Dyck lattice

We focus on the special class of linear intervals in a poset. An interval $[P, Q]$ is *linear* if it is totally ordered, or equivalently if it is a chain of the form $P = P_0 < P_1 < \cdots < P_\ell = Q$. The *length* of such a linear interval is defined to be ℓ. A linear interval of length zero (containing only one element) is said to be *trivial*. The non trivial linear intervals of the ν-Dyck lattice can be easily characterized into two different classes.

Definition 2. *An interval $[P, Q]$ in Dyck_ν is a* left interval *if Q is obtained from P by transforming a subpath $E^\ell N$ into NE^ℓ for some $\ell \geq 1$. It is a* right interval *if Q is obtained from P by transforming a subpath EN^ℓ into $N^\ell E$ for some $\ell \geq 1$.*

Proposition 3. *The left and right intervals in the previous definition are linear intervals of length ℓ. Moreover, all non trivial linear intervals in Dyck_ν are either left or right intervals. Only covering relations are both left and right (when $\ell = 1$).*

Corollary 4. *Left intervals of length ℓ in Dyck_ν are in bijection with ν-paths marked at a north step preceded by ℓ east steps. Right intervals of length ℓ in Dyck_ν are in bijection with ν-paths marked at an east step followed by ℓ north steps.*

3 The ν-Tamari lattice

The ν-Tamari lattices are a generalization of the Tamari lattice. They were defined in terms of ν-paths by Préville-Ratelle and Viennot in [12]. An alternative description in terms of ν-trees was presented in [6].

3.1 On ν-paths

For a lattice point p on a ν-path μ, define its ν-altitude $\mathrm{alt}_\nu(p)$ to be the maximum number of horizontal steps that can be added to the right of p without crossing ν. Given a valley EN of μ, let p be the lattice point between the east and north steps. Let q be the next lattice point of μ such that $\mathrm{alt}_\nu(q) = \mathrm{alt}_\nu(p)$, and $\mu_{[p,q]}$ be the subpath of μ that starts at p and ends at q. Let μ' be the path obtained from μ by switching $\mu_{[p,q]}$ with the east step E that precedes it. The ν-rotation of μ at the valley p is defined to be $\mu \lessdot_\nu \mu'$. An example is illustrated in Figure 3.

Definition 5. *The ν-Tamari poset Tam_ν is the reflexive transitive closure of ν-rotations on ν-paths.*

Theorem 6 ([12])**.** *The ν-Tamari poset is a lattice. Its covering relations are exactly ν-rotations.*

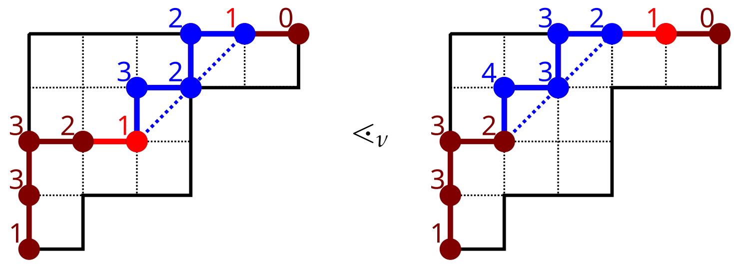

Figure 3: The rotation operation of a ν-path. Each node is labelled with its ν-altitude.

Another approach to define the ν-Tamari lattice is to introduce the ν-elevation of a subpath as the difference of ν-altitude between its ending point and its starting point. We thus write $\mathrm{elev}_\nu(E) = -1$ for an east step E and $\mathrm{elev}_\nu(N_i) = \nu_i$ if N_i is the i-th north step of a ν-path μ. For any subpath A of μ, we then have $\mathrm{elev}_\nu(A) = \sum_{a \in A} \mathrm{elev}_\nu(a)$ as the sum of the ν-elevation of the steps of A.

The ν-excursion of a north step N of a ν-path μ is defined as the shortest subpath A of μ that starts with this N and such that $\mathrm{elev}_\nu(A) = 0$. It follows from the definition of the ν-excursion that exchanging the east step E of a valley with the ν-excursion that follows it is exactly a covering relation in Tam_ν.

This alternative description is useful to generalize the ν-Tamari lattice in Section 4.

3.2 On ν-trees

One can also define a poset on ν-trees which is isomorphic to the ν-Tamari lattice.

We denote by F_ν the Ferrers diagram that lies weakly above ν in the smallest rectangle containing ν. Let L_ν denote the set of lattice points inside F_ν. We say that two points $p, q \in L_\nu$ are ν-incompatible if p is strictly southwest or strictly northeast of q, and the smallest rectangle containing p and q lies entirely in F_ν. Otherwise, p and q are said to be ν-compatible. A *ν-tree* is a maximal collection of pairwise ν-compatible elements in L_ν. In particular, the vertex at the top-left corner of F_ν is ν-compatible with everyone else, and belongs to every ν-tree. Connecting two consecutive elements in the same row or column allows us to visualize ν-trees as classical rooted binary trees [6]. The vertex at top-left corner of F_ν is always the *root*. An example of a ν-tree and the rotation operation which we now describe is shown in Figure 4.

Let T be a ν-tree and $p, r \in T$ be two elements which do not lie in the same row or same column. We denote by $p \square r$ the smallest rectangle containing p and r, and write $p \llcorner r$ (resp. $p \urcorner r$) for the lower left corner (resp. upper right corner) of $p \square r$.

Let $p, q, r \in T$ be such that $q = p \llcorner r$ and no other elements besides p, q, r lie in $p \square r$. The *ν-rotation* of T at q is defined as the set $T' = \big(T \setminus \{q\}\big) \cup \{q'\}$, where $q' = p \urcorner r$. As proven in [6, Lemma 2.10], the rotation of a ν-tree is also a ν-tree.

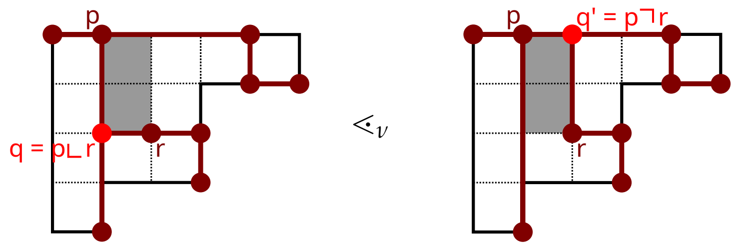

Figure 4: The rotation operation of a ν-tree.

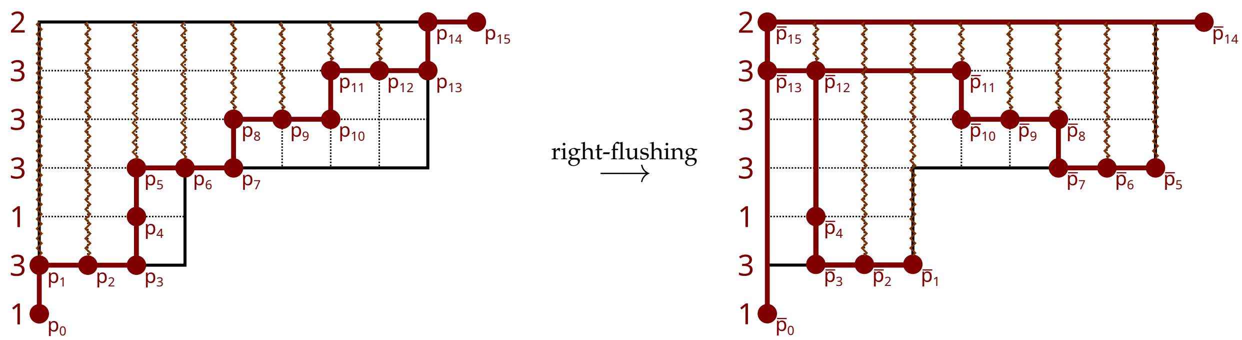

Figure 5: Right flushing bijection from ν-paths to ν-trees.

Definition 7. *The* rotation poset of ν-trees *is the reflexive transitive closure of ν-rotations.*

Theorem 8 ([6])**.** *The ν-Tamari lattice is isomorphic to the rotation poset of ν-trees.*

A bijection between these two posets is given by the *right flushing bijection* introduced in [6]. This bijection maps a ν-path $\mu = (\mu_0, \ldots, \mu_k)$ to the unique ν-tree with $\mu_i + 1$ nodes at height i. This tree can be recursively obtained by adding $\mu_i + 1$ nodes at height i from bottom to top, from right to left, avoiding forbidden positions. The forbidden positions are those above a node that is not the left most node in a row (these come from the initial points of the east steps in the path μ). In Figure 5, the forbidden positions are the ones that belong to the wiggly lines. Note that the order of the nodes per row is reversed.

The inverse of the right flushing bijection is called the *left flushing bijection*, and can be described similarly, adding points from left to right, from bottom to top, avoiding the forbidden position given by the wiggly lines.

3.3 Left and right intervals in the ν-Tamari lattice

The description of the ν-Tamari lattice on ν-trees gives an easy description of its linear intervals.

Definition 9. *An interval $[T, T']$ in the ν-Tamari lattice on ν-trees is a* left interval *if T' is obtained from T by applying ℓ rotations at nodes $q_0, \ldots, q_{\ell-1}$ which are (maybe not directly)* *consecutive in the same row, from left to right, for example $\overline{p}_{13}, \overline{p}_{12}$ in Figure 5. It is a* right interval *if T' is obtained from T by applying ℓ rotations at nodes $q_0, \ldots, q_{\ell-1}$ which are (maybe not directly) consecutive in the same column, from bottom to top, for example $\overline{p}_3, \overline{p}_4$ in Figure 5.*

Proposition 10. *The left and right intervals in the previous definition are linear intervals of length ℓ. Moreover, all non trivial linear intervals in Tam_ν are either left or right intervals.*

Remark 11. *The left flushing of a left interval on the rotation lattice of ν-trees produces a left interval $[P, Q]$ of ν-paths in Tam_ν, where P is of the form AE^kBC with B some ν-excursion and Q is of the form ABE^kC. In other words, P is a ν-path with a valley preceded by k east steps.*

The left flushing of a right interval on the rotation lattice of ν-trees produces a right interval $[P, Q]$ of ν-paths in Tam_ν, where P is of the form $AEB_1 \ldots B_kC$ with $B_1, \ldots, B_k$ being k consecutive ν-excursions, and Q is of the form $AB_1 \ldots B_kEC$.

4 The alt ν-Tamari lattice

Given a fixed path ν, the ν-Dyck lattice and the ν-Tamari lattice are two posets defined on ν-paths with quite similar covering relations. In both cases, a covering relation consists of swapping the east step of a valley with a subpath that follows it. We can in fact define a whole family of posets that are described in a similar way, and we call them the alt ν-Tamari posets. We also prove that they are lattices.

4.1 On ν-paths

Let $\nu = (\nu_0, \ldots, \nu_k)$ be a fixed path. We say that $\delta = (\delta_1, \ldots, \delta_k) \in \mathbb{N}^k$ is an increment vector with respect to ν if $\delta_i \leq \nu_i$ for all $1 \leq i \leq k$.

Similarly as the ν-elevation, we introduce a notion of δ-elevation of a subpath and declare that the δ-elevation $\mathrm{elev}_\delta(N_i)$ or $\delta(N_i)$ of the i-th north step N_i of any ν-path μ is equal to δ_i and that $\delta(E) = -1$ for an east step E. Thus, for any subpath A of μ, its δ-elevation is $\delta(A) = \sum_{a \in A} \delta(a)$.

The δ-excursion of a north step N of a ν-path μ is defined as the smallest subpath A of μ that starts with this N and such that $\delta(A) = 0$. Let EN be a valley of μ and A be the δ-excursion of N. Let μ' be the path obtained from μ by switching the step E and the subpath A. We say that $\mu \lessdot_\delta \mu'$ is a δ-rotation.

Definition 12. *Let δ be an increment vector with respect to ν. The* alt ν-Tamari poset $\mathrm{Tam}_\nu(\delta)$ *is the reflexive transitive closure of δ-rotations on the set of ν-paths.*

Remark 13. *For a fixed path ν, there are two extreme choices of increment vector δ. If $\delta_i = \nu_i$ for all $1 \leq i \leq k$, the alt ν-Tamari lattice coincides with the ν-Tamari lattice. If $\delta_i = 0$ for all $1 \leq i \leq k$, the alt ν-Tamari lattice coincides with the ν-Dyck lattice.*

For a general increment vector δ with respect to ν, it is not a priori clear that $\mathrm{Tam}_\nu(\delta)$ is a lattice. This is a consequence of the following results.

Theorem 14. *Let $\check{\nu}_0 = \sum_{i=0}^k \nu_i - \sum_{i=1}^k \delta_i$ and $\check{\nu}_i = \delta_i \leq \nu_i$. Then $\check{\nu} = (\check{\nu}_0, \check{\nu}_1, \ldots, \check{\nu}_k)$ is a path below ν whose endpoints are the same as ν. Let 1^ν be the ν-path with all north steps at the beginning. The following hold:*

1. *The alt ν-Tamari poset $\mathrm{Tam}_\nu(\delta)$ is the restriction of $\mathrm{Tam}_{\check{\nu}}$ to the subset of paths above ν.*
2. *The restriction of $\mathrm{Tam}_{\check{\nu}}$ to the subset of paths above ν is the interval $[\nu, 1^\nu]$ in $\mathrm{Tam}_{\check{\nu}}$.*
3. *The alt ν-Tamari poset $\mathrm{Tam}_\nu(\delta)$ is a lattice.*

Proof. Firstly, we have $\sum_{i=0}^j \check{\nu}_i - \sum_{i=0}^j \nu_i = \sum_{i=j+1}^k (\nu_i - \delta_i) \geq 0$, where equality holds for $j = k$. Then, δ-rotations of a ν-path μ coincide with $\check{\nu}$-rotations of μ.

Secondly, the condition on δ ensures that any ν-rotation of a ν-path μ can be achieved as a sequence of δ-rotations. Thus, all ν-paths are indeed above ν in $\mathrm{Tam}_{\check{\nu}}$, thus $\mathrm{Tam}_\nu(\delta)$ is indeed an interval in a lattice, and it is therefore a lattice. □

4.2 On (δ, ν)-trees

The alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ is the interval $[\nu, 1^\nu]$ in $\mathrm{Tam}_{\check{\nu}}$. So, it can be described as the rotation lattice of $\check{\nu}$-trees that are above the $\check{\nu}$-tree T_ν corresponding to ν in $\mathrm{Tam}_{\check{\nu}}$. These trees can be described as maximal collections of pairwise compatible elements in a shape $F_{\delta,\nu}$ which we will now describe. This point of view will be essential to show that all alt ν-Tamari lattices have the same number of linear intervals of any length.

Let δ, ν and $\check{\nu}$ as in Theorem 14. Let $F_{\check{\nu}}$ be the Ferrers diagram that lies weakly above $\check{\nu}$. We consider the lattice path $\hat{\nu}$ that starts at the lowest right corner of $F_{\check{\nu}}$ (the point with coordinates $(\check{\nu}_0, 0)$) which consists of the sequence of west and north steps

$$W^{\nu_0} N W^{\gamma_1} N W^{\gamma_2} \ldots N W^{\gamma_k}, \qquad \text{for } \gamma_i = \nu_i - \delta_i.$$

We define $F_{\delta,\nu}$ to be the subset of $F_{\check{\nu}}$ consisting of the boxes that are not below $\hat{\nu}$, and denote by $L_{\delta,\nu}$ its set of lattice points. A (δ, ν)*-tree* is a maximal collection of pairwise $\check{\nu}$-compatible elements in $L_{\delta,\nu}$. An example is illustrated on the right of Figure 6.

Proposition 15. *The (δ, ν)-trees are exactly the $\check{\nu}$-trees that are above T_ν in $\mathrm{Tam}_{\check{\nu}}$, where T_ν is the $\check{\nu}$-tree corresponding to ν under the right flushing bijection with respect to $\check{\nu}$.*

Definition 16. *The* rotation poset of (δ, ν)-trees *is the reflexive transitive closure of $\check{\nu}$-rotations on (δ, ν)-trees.*

Theorem 17. *The alt ν-Tamari lattice $\mathrm{Tam}_\nu(\delta)$ is isomorphic to the rotation poset of (δ, ν)-trees.*

Proposition 18. *For each ν-path $\mu = (\mu_0, \mu_1, \dots, \mu_k)$, there is a unique (δ, ν)-tree T that has exactly $\mu_i + 1$ nodes at height i, for $0 \le i \le k$.*

We call the bijection in this proposition the (δ, ν) *right flushing bijection* from ν-paths to (δ, ν)-trees. It can be described in exactly the same way as the right flushing bijection from Section 3.2: we recursively add $\mu_i + 1$ nodes to the tree inside the shape $F_{\delta,\nu}$ from right to left, from bottom to top, while avoiding the forbidden positions above a node which is not the left most node in a row. Figure 6 shows the image of the path $\mu = (1,0,1,1,3,2,1,2)$ for $\delta^{max} = (1,0,2,2,0,3,0)$ (left) and for $\delta = (0,0,1,2,0,1)$ (right), where the base path is $\nu = (3,1,0,2,2,0,3,0)$.

4.3 Left and right intervals in the alt ν-Tamari lattice

Since $\mathrm{Tam}_\nu(\delta)$ is an interval in $\mathrm{Tam}_{\hat{\nu}}$, its linear intervals are linear intervals in $\mathrm{Tam}_{\hat{\nu}}$.

Proposition 19. *The non trivial linear intervals in* $\mathrm{Tam}_\nu(\delta)$ *are either left or right intervals $[T, T']$ where T and T' are both (δ, ν)-trees.*

We identify a left interval $[T, T']$ of length ℓ in $\mathrm{Tam}_\nu(\delta)$ with the bottom tree T marked at the nodes $q_0, q_1, \dots, q_\ell$, where T' is obtained by consecutively rotating $q_0, \dots, q_{\ell-1}$ in T. Here, $q_0, \dots, q_\ell$ are consecutive initial nodes (left most nodes) of T in some row different than the top. A $\llcorner$ *-marked (δ, ν)-tree* is a tuple $(T, q_0, \dots, q_\ell)$ obtained this way.

For simplicity in the figures, we represent a left interval by marking an $\llcorner$ (*horizontal L*) in T, which connects the nodes $q_0, \dots, q_\ell$ to each other and the node q_0 to the node directly above it. Figure 6 shows two trees with six different marked $\llcorner$'s. They correspond to six left intervals of lengths 1,1,1,1,2, and 3.

Similarly, we identify a right interval $[T, T']$ of length ℓ in $\mathrm{Tam}_\nu(\delta)$ with the top tree T' marked at the nodes $q'_0, q'_1, \dots, q'_\ell$, where T is obtained by consecutively rotating down $q'_0, \dots, q'_{\ell-1}$ in T'. Here, $q'_0, \dots, q'_\ell$ are consecutive final nodes (top most nodes) of T' in some column different than the left most. In addition, the points q'_i have the condition that they are not the bottom lattice point of a north step in the path $\hat{\nu}$; otherwise, a down rotation could not be performed at the node q'_{i+1}. A $\urcorner$ *-marked (δ, ν)-tree* is a tuple $(T', q'_0, \dots, q'_\ell)$ obtained this way.

For simplicity in the figures, we represent a right interval by marking an $\urcorner$ (*vertical L*) in T', which connects the nodes $q'_0, \dots, q'_\ell$ to each other and the node q'_0 to the node directly to its left. Figure 7 shows two trees with four different marked $\urcorner$'s. They correspond to four right intervals of lengths 1,1,1, and 3.

Proposition 20. *The following hold:*

1. *Left intervals in* $\mathrm{Tam}_\nu(\delta)$ *are in bijection with* $\llcorner$ *-marked (δ, ν)-trees $(T, q_0, \dots, q_\ell)$.*

2. *Right intervals in* $\mathrm{Tam}_\nu(\delta)$ *are in bijection with* $\urcorner$ *-marked (δ, ν)-trees $(T', q'_0, \dots, q'_\ell)$.*

In both cases, the length of the interval is equal to ℓ.

5 Bijections between linear intervals

As we have seen in Proposition 18, a (δ, ν)-tree T is completely characterized by its *row vector* $r = (r_0, r_1 \dots, r_k)$, where r_i is the number of nodes of T at height i. The *horizontal flushing bijection* $\varphi_{\delta,\delta'}$ is the bijection from (δ, ν)-trees to (δ', ν)-trees that preserves the number of nodes at each height (preserving the row vector). This naturally extends to a map $\widetilde{\varphi}_{\delta,\delta'}$ from $\mapsto$-marked (δ, ν)-trees to $\mapsto$-marked (δ', ν)-trees, defined by

$$\widetilde{\varphi}_{\delta,\delta'}(T, q_0, \dots, q_\ell) = (\varphi_{\delta,\delta'}(T), \widetilde{q}_0, \dots, \widetilde{q}_\ell),$$

where $\widetilde{q}_i$ is the node of $\varphi_{\delta,\delta'}(T)$ corresponding to q_i in T under the bijection $\varphi_{\delta,\delta'}$. An example is illustrated in Figure 6.

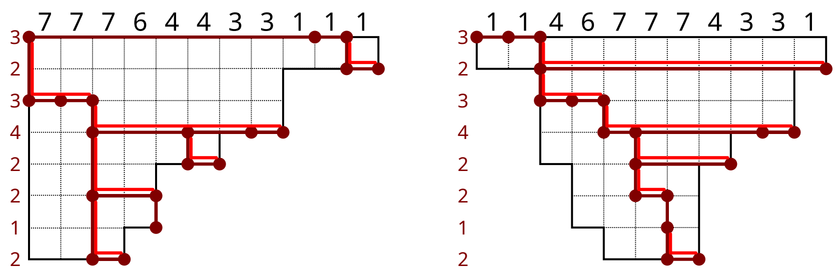

Figure 6: Bijection between left intervals for δ^{max} and $\delta = (0, 0, 1, 2, 0, 1, 0)$.

Proposition 21. *The map $\widetilde{\varphi}_{\delta,\delta'}$ is a bijection from $\mapsto$-marked (δ, ν)-trees to $\mapsto$-marked (δ', ν)-trees.*

Theorem 22. *For a fixed ν, all alt ν-Tamari lattices have the same number of left intervals of length ℓ.*

In order to define a bijection between right intervals of two alt ν-Tamari lattices we need a more subtle idea. For this, we use a bijection that "preserves" a modified column vector of (δ, ν)-trees.

Let δ, ν and $\check{\nu}$ as in Theorem 14, and $m = \nu_0 + \cdots + \nu_k$. The paths ν and $\check{\nu}$ start at the origin $(0, 0)$ and end at the coordinate (m, k). Given a (δ, ν)-tree T, the *column vector* of T is the vector $c(T) = (\bar{c}_0, \bar{c}_1, \dots, \bar{c}_m)$ such that $\bar{c}_i$ counts the number of nodes in T with x-coordinate equal to i.

The (δ, ν)-tree T is completely determined by $c(T)$. It can be reconstructed via the *down flushing bijection*, which recursively adds points to T in $F_{\delta,\nu}$ from right to left, from bottom to top, avoiding forbidden positions appearing on the left of previously added nodes that are not the top node in their column (the wiggle forbidden lines are horizontal in this case).

We call a lattice point in $F_{\delta,\nu}$ *irrelevant* if it is the bottom point of a north step in the path $\hat{\nu}$ (recall this is the left boundary path of the shape $F_{\delta,\nu}$). All other lattice points are called *relevant*. Given a (δ, ν)-tree T, the *δ-column vector* of T is the vector $c_\delta(T) = (c_1, \dots, c_m)$ such that c_i counts the number of relevant nodes in T with x-coordinate equal to i. Since the only relevant point with x-coordinate equal to zero is the top left corner (corresponding to the root of every tree), we note that $c_0 = 1$ for every tree, and is omitted in the vector $c_\delta(T)$. In Figure 7, the irrelevant points are drawn green, and the relevant points brown; the δ-column vectors of the two examples in this figure are $(1, 4, 1, 2, 2, 2, 1, 1, 1, 1, 2, 1)$ (left) and $(1, 2, 2, 2, 2, 1, 4, 1, 1, 1, 1, 1)$ (right). Note that one is a permutation of the other.

Proposition 23. *A (δ, ν)-tree T is completely determined by its δ-column vector $c_\delta(T)$.*

Let $b_\delta = (b_1, \dots, b_m)$ be such that b_i counts the number of boxes in column i in the diagram $F_{\delta,\nu}$. Note that the vector $b_{\delta'}$ is just a permutation of the vector b_δ. We define the permutation $\sigma_{\delta,\delta'}$ as the permutation determined by $b_{\delta'} = \sigma_{\delta,\delta'} \circ b_\delta$, which sends the j-th appearance of a number b in b_δ to the j-th appearance of b in $b_{\delta'}$.

Proposition 24. *Let T be a (δ, ν)-tree and δ' another increment vector with respect to ν. There exists a unique T' be a (δ', ν)-tree T' such that $c_{\delta'}(T') = \sigma_{\delta,\delta'} \circ c_\delta(T)$, and the irrelevant nodes in T and T' have the same heights.*

We define the *vertical flushing bijection* $\psi_{\delta,\delta'}$ as the bijection between (δ, ν)-trees and (δ', ν)-trees that preserves the heights of the irrelevant nodes and that transforms the δ-column vector according to the permutation $\sigma_{\delta,\delta'}$. That is,

$$\psi_{\delta,\delta'}(T) = T', \text{ where } c_{\delta'}(T') = \sigma_{\delta,\delta'} \circ c_\delta(T).$$

An example is illustrated in Figure 7.

This map naturally extends to a map $\widetilde{\psi}_{\delta,\delta'}$ from $\daleth$-marked (δ, ν)-trees to $\daleth$-marked (δ', ν)-trees, defined by

$$\widetilde{\psi}_{\delta,\delta'}(T', q'_0, \dots, q'_\ell) = (\psi_{\delta,\delta'}(T'), \widetilde{q}'_0, \dots, \widetilde{q}'_\ell),$$

where $\widetilde{q}'_i$ is the node of $\psi_{\delta,\delta'}(T')$ corresponding to q'_i in T' under the bijection $\psi_{\delta,\delta'}$. An example is illustrated in Figure 7.

Proposition 25. *The map $\widetilde{\psi}_{\delta,\delta'}$ is a bijection from $\daleth$-marked (δ, ν)-trees to $\daleth$-marked (δ', ν)-trees.*

Theorem 26. *For a fixed ν, all alt ν-Tamari lattices have the same number of right intervals of length ℓ.*

Acknowledgements

This project started during discussions at the SLC 88 conference in Strobl in 2022, and we are grateful for the pleasant and motivating atmosphere during the conference. We are specially thankful to Wenjie Fang for interesting discussions that helped us realizing that the alt ν-Tamari posets are intervals in some $\check{\nu}$-Tamari lattice, and to Henri Mühle for allowing us to use Figures 3, 4 and 5, which are reproduced from [4].

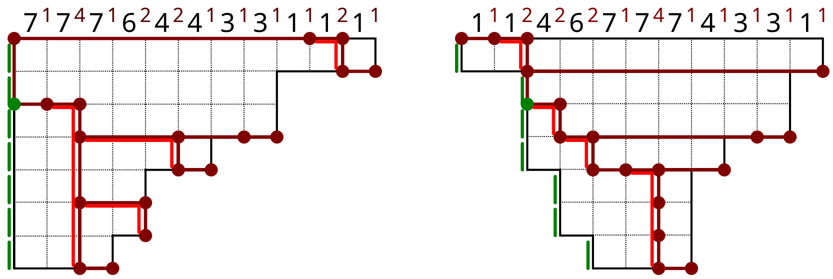

Figure 7: Bijection between right intervals for δ^{max} and $\delta = (0, 0, 1, 2, 0, 1, 0)$.

References

[1] F. Bergeron and L.-F. Préville-Ratelle. “Higher trivariate diagonal harmonics via generalized Tamari posets”. *J. Comb.* **3**.3 (2012), pp. 317–341. DOI.

[2] M. Bousquet-Mélou, G. Chapuy, and L.-F. Préville-Ratelle. “The representation of the symmetric group on m-Tamari intervals”. *Adv. Math.* **247** (2013), pp. 309–342. DOI.

[3] M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle. “The number of intervals in the m-Tamari lattices”. *Electron. J. Combin.* **18**.2 (2011), Paper 31, 26. DOI.

[4] C. Ceballos and H. Mühle. “F- and H-triangles for ν-associahedra”. *Comb. Theory* **2**.2 (2022), Paper No. 3, 26.

[5] C. Ceballos, A. Padrol, and C. Sarmiento. “Geometry of ν-Tamari lattices in types A and B”. *Trans. Amer. Math. Soc.* **371**.4 (2019), pp. 2575–2622. DOI.

[6] C. Ceballos, A. Padrol, and C. Sarmiento. “The ν-Tamari lattice via ν-trees, ν-bracket vectors, and subword complexes”. *Electron. J. Combin.* **27**.1 (2020), Paper No. 1.14, 31.

[7] F. Chapoton. “Sur le nombre d’intervalles dans les treillis de Tamari”. *Sém. Lothar. Combin.* **55** (2005/07), Art. B55f, 18.

[8] C. Chenevière. “Linear intervals in the Tamari and the Dyck lattices and in the alt-Tamari posets”. 2022. arXiv:2209.00418.

[9] W. Fang and L.-F. Préville-Ratelle. “The enumeration of generalized Tamari intervals”. *European J. Combin.* **61** (2017), pp. 69–84. DOI.

[10] M. D. Haiman. “Conjectures on the quotient ring by diagonal invariants”. *J. Algebraic Combin.* **3**.1 (1994), pp. 17–76. DOI.

[11] F. Müller-Hoissen, J. M. Pallo, and J. Stasheff, eds. *Associahedra, Tamari lattices and related structures*. Vol. 299. Progress in Mathematics. Birkhäuser/Springer, Basel, 2012, pp. xx+433. DOI.

[12] L.-F. Préville-Ratelle and X. Viennot. “The enumeration of generalized Tamari intervals”. *Trans. Amer. Math. Soc.* **369**.7 (2017), pp. 5219–5239. DOI.