Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #82, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

RSK for **3**-free posets

Kyle Celano[1*]

[1]*Department of Mathematics, University of Miami, Miami, FL*

Abstract. A long-standing open problem is to find an RSK-like correspondence between permutations and pairs of tableaux coming from Gasharov's decomposition of Stanley's chromatic symmetric functions into Schur functions. In this, we present such a correspondence RSK_P for incomparability graphs of **3**-free posets P that moreover preserve the descent and inversion statistics. We then extend RSK_P to bijections from proper colorings and multicolorings providing new combinatorial proofs for the Schur expansions of Gasharov for the chromatic symmetric function, of Shareshian–Wachs for the chromatic quasisymmetric function, and of Hwang for the multichromatic quasisymmetric function, and its refinement to equivalence classes of acyclic orientations in the case that P is **3**-free.

Keywords: symmetric functions, RSK algorithms, chromatic symmetric functions

1 Introduction

The famous Robinson-Schensted-Knuth RSK correspondence is a bijection

$$\mathfrak{S}_n \to \bigsqcup_{\lambda \vdash n} SYT_\lambda \times SYT_\lambda \tag{1.1}$$

where $\mathfrak{S}_n$ is the set of permutations on $[n] := \{1, \ldots, n\}$ and SYT_λ is the set of standard Young tableaux of shape λ. This is a *row-insertion* algorithm taking each letter w_i of a permutation $w = w_1 \cdots w_n \in \mathfrak{S}_n$ and inserting into a pair (T_{i-1}, R_{i-1}) of size $i-1$ by a *bumping* procedure in T_{i-1} and a *recording* procedure in R_{i-1} to make a pair (T_i, R_i) of size i. The bijection takes w to (T_n, R_n) (see [12, Chapter 7]).

A general form of RSK is a bijection

$$\mathcal{P}^n \to \bigsqcup_{\lambda \vdash n} SSYT_\lambda \times SSYT_\lambda \tag{1.2}$$

where $SSYT_\lambda$ is the set of semistandard Young tableaux of shape λ and where $\mathcal{P}^n$ is the set of size n generalized permutations, which are multisets $\{\binom{u_i}{v_i}\}_{i=1}^n$ of biletters $\binom{u_i}{v_i}$

*kac265@miami.edu.

where u_i and v_i are positive integers. These can be written as two-line arrays so that if

$$\binom{u}{v} = \begin{pmatrix} u_1 & u_2 & \dots & u_n \\ v_1 & v_2 & \dots & v_n \end{pmatrix}$$

then u is weakly increasing and if $u_i = u_j$ and $i < j$, then $v_i < v_j$. This general *RSK* inserts the *bottom* word v into the first tableau and records the procedure with *top* word u.

If we restrict to when the top word u is $12 \cdots n$, we get a bijection from the set $\mathbb{P}^n$ of length n words on the alphabet $\mathbb{P}$ of positive integers

$$\mathbb{P}^n \to \bigsqcup_{\lambda \vdash n} SSYT_\lambda \times SYT_\lambda. \tag{1.3}$$

If we restrict to when the bottom word v is a permutation in $\mathfrak{S}_n$, we get another bijection

$$\{(u,v) \in \mathcal{P}^n \mid v \in \mathfrak{S}_n\} \to \bigsqcup_{\lambda \vdash n} SYT_\lambda \times SSYT_\lambda. \tag{1.4}$$

The bijection (1.3) provides a combinatorial proof of the decomposition of the complete homogeneous symmetric function h_{1^n} in terms of Schur functions

$$h_{1^n} = \sum_{\lambda \vdash n} f^\lambda s_\lambda$$

and the bijection (1.1) provides a bijective proof of

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

where f^λ is the number of standard Young tableaux of shape $\lambda \vdash n$.

Gasharov [5] obtains the following decomposition of Stanley's chromatic symmetric function [11] for the incomparability graph $\mathrm{Inc}(P)$ of a $(\mathbf{3}+\mathbf{1})$-free poset P,

$$X_{\mathrm{Inc}(P)}(\mathbf{x}, t) = \sum_{\lambda \vdash n} f_\lambda^P s_\lambda \tag{1.5}$$

where f_λ^P counts a certain class of Young tableaux $\mathcal{T}_{P,\lambda}$ called *P-tableaux* of shape λ.[1] This specializes to

$$n! = \sum_{\lambda \vdash n} f_P^\lambda f^\lambda. \tag{1.6}$$

Gasharov obtains (1.5) by applying Jacobi–Trudi and then a sign-reversing involution. A long-standing open problem then is to find an *RSK-like* bijection proving (1.5) and (1.6) directly.

[1] Definitions to be given in the next section

Progress has been made on this problem for certain classes of $(\mathbf{3}+\mathbf{1})$-free posets by Sundquist–Wagner–West and Chow [13, 4] for a large class of subposets and Kim–Pylyavskyy [8] for a slightly smaller one.

In this extended abstract[2], we first note that proper colorings can be represented as generalized permutations whose bottom word is a permutation and hence (1.5) is equivalent to existence of a bijection analogous to that in (1.4). We then construct a relatively simple *RSK*-like bijection when P is **3**-free (or, length at most 1): this row-insertion algorithm inserts bottom letters of a generalized permutation representing the coloring and records the procedure with the top word, just as in *RSK*. Although the previous algorithms apply to much broader classes of posets, ours is much simpler.

Also, for **3**-free natural unit interval orders, our *RSK* gives a bijective proof of the Shareshian–Wachs [9, 10] refinement of (1.5) for chromatic quasisymmetric functions. Hwang [7] further generalizes (1.5) to multichromatic quasisymmetric functions, which involve multicolorings of a graph. By viewing these multicolorings as generalized permutations, we extend our *RSK* to a bijective map from multicolorings to pairs of semistandard P-tableaux and semistandard Young tableaux, thereby attaining an analogue to a dual form of (1.2). This provides a bijective proof of Hwang's generalization in the case of **3**-free natural unit interval orders. Moreover, we can restrict our *RSK* to colorings of Hwang's equivalence classes of acyclic orientations, which provides a bijective proof of the Schur expansion for his pieces of the chromatic quasisymmetric function in the case of **3**-free natural unit interval orders.

This extended abstract is organized as follows: In Section 2, we provide the relevant definitions and background for chromatic (quasi)symmetric functions and P-tableaux. In Section 3, we describe the *RSK* for 3-free posets on permutations (Definition 3.1, Theorem 3.4) and then show how to extend *RSK* to obtain bijective proofs of the Schur expansions of Gasharov, Shareshian–Wachs (Theorem 3.10) and Hwang (Theorems 3.13 and 3.14).

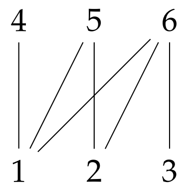
2 Chromatic Symmetric Functions and P-tableaux

Stanley's *chromatic symmetric function* [11] is defined for a graph $G = ([n], E)$ to be

$$X_G(\mathbf{x}) = \sum_{\kappa \in \mathcal{K}(G)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

where $\mathcal{K}(G)$ is the set of proper colorings of G. The incomparability graph of a poset P, denoted $\mathrm{Inc}(P)$, is the graph whose vertices are the elements of P and whose edges correspond to pairs of incomparable elements of P.

[2]See [2] for the full paper



3	1	2	4
6	5		

Figure 1: A poset P and a P-tableaux

A *P-tableau*[3] is a Young tableau of size $|P| = n$ filled with the elements of P such that

- each element of P appears exactly once,
- if x is immediately to the left of y, then $x \not>_P y$, and
- if x is immediately above of y, then $x <_P y$.

See Figure 1 for an example.

Let $\mathcal{T}_{P,\lambda}$ be the set of P-tableaux of shape λ and $f_P^\lambda = |\mathcal{T}_{P,\lambda}|$. Recall that a poset is *$(\mathbf{a}+\mathbf{b})$-free* if it contains no induced subposet that is the disjoint union of an a-chain and a b-chain. For a partition λ, let λ^* denote its conjugate. Then, Gasharov showed the following by applying the Jacobi–Trudi identity and then a sign-reversing involution.

Theorem 2.1 ([5, Theorem 4])**.** *For a $(\mathbf{3}+\mathbf{1})$-free poset P, $X_{\mathrm{Inc}(P)}(\mathbf{x}) = \sum_{\lambda \vdash n} f_P^\lambda s_{\lambda^*}$.*

A *row-strict* Young tableau is one where the rows strictly increase and the columns weakly increase. Let RST_λ be the set of row-strict tableaux of shape λ. If $R \in RST_\lambda$, then its conjugate $R^* \in SSYT_{\lambda^*}$. Hence, $s_{\lambda^*} = \sum_{R \in RST_\lambda} x^{m_1(R)} x^{m_2(R)} \cdots$ where $m_i(R)$ is the number of times i appears in R. We identify the set of permutations $\mathfrak{S}_n$ with the subset $\{\kappa_w\}_{w \in \mathfrak{S}_n} \subset \mathcal{K}(G)$ given by $\kappa_w(i) = w^{-1}(i)$ for $w \in \mathfrak{S}_n$ (why we do this is will be apparent later, see Remark 2.4). Thus, our problem of finding an appropriate RSK that would provide direct bijective proofs of Theorem 2.1 and (1.6) is as follows.

Problem 2.2. *For a $(\mathbf{3}+\mathbf{1})$-free poset P with $G = \mathrm{Inc}(P)$, find an RSK-like bijection*

$$RSK_P : \mathcal{K}(G) \to \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times RST_\lambda$$

which restricts to a bijection

$$RSK_P : \mathfrak{S}_n \to \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times SYT_\lambda.$$

[3]This is the conjugate version of Gasharov's original definition [5]. We make this change to more naturally describe the algorithm as a row-insertion algorithm.

A *natural unit interval order* (or, NUIO) is a $(\mathbf{3}+\mathbf{1})$-free and $(\mathbf{2}+\mathbf{2})$-free poset with a certain natural labeling. Let $G = \mathrm{Inc}(P)$ for a natural unit interval order P. The Shareshian–Wachs [9, 10] *chromatic quasisymmetric function* for G is

$$X_G(\mathbf{x}, t) = \sum_{\kappa \in \mathcal{K}(G)} t^{\mathrm{des}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

where $\mathrm{des}(\kappa)$ is the number of edges $ij \in E$ such that $i < j$ and $\kappa(i) > \kappa(j)$. This quasisymmetric function specializes to Stanley's chromatic symmetric function by setting $t = 1$. Shareshian and Wachs prove that $X_G(\mathbf{x}, t)$ is in fact a symmetric function. For a P-tableaux $T \in \mathcal{T}_P$ define

$$\mathrm{inv}_G(T) = \#\{xy \in E \mid x < y \text{ and } x \text{ is to the right of } y\}.$$

Let $f^{\lambda}_{P,i}$ be the number of P-tableaux T with $\lambda(T) = \lambda$ and $\mathrm{inv}_G(T) = i$.

Theorem 2.3 ([10, Theorem 6.3]). *For a natural unit interval order P,*

$$X_{\mathrm{Inc}(P)}(\mathbf{x}, t) = \sum_{i \geq 0} t^i \sum_{\lambda \vdash n} f^{\lambda}_{P,i} s_{\lambda^*}. \tag{2.1}$$

Hence, for RSK_P to prove (2.1) bijectively, it must additionally satisfy:

$$\text{If } P \text{ is a NUIO and } RSK_P(\kappa) = (T, R) \text{ then } \mathrm{des}(\kappa) = \mathrm{inv}_G(T). \tag{2.2}$$

Remark 2.4. For $w \in \mathfrak{S}_n$, the Shareshian–Wachs *G-inversion statistic* [9, 10] is

$$\mathrm{inv}_G(w) = \#\{\{w(i), w(j)\} \in E \mid i < j, w(i) > w(j)\}.$$

The G-inversion statistic $\mathrm{inv}_G(w)$ is a natural extension of the classical inversion $\mathrm{inv}(w)$ statistic to the graphs we are considering. If $\kappa_w \in \mathcal{K}(G)$ is defined by $\kappa_w(i) = w^{-1}(i)$, then $\mathrm{des}(\kappa_w) = \mathrm{inv}_G(w)$. Hence, (2.2) for the restriction to $\mathfrak{S}_n$ is that if $RSK_P(w) = (T, R)$, then $\mathrm{inv}_G(w) = \mathrm{inv}_G(T)$.

In the next section, we will present an RSK_P solving Problem 2.2 which additionally satisfies (2.2) for **3**-free posets.

Remark 2.5. The class of **3**-free posets have been studied extensively in the literature of chromatic quasisymmetric functions under a few different, but equivalent, definitions.

- They are *length* 1 or 0 posets and their incomparability graphs are unit interval orders with bipartite complement as in Abreu–Nigro [1].
- They correspond to *abelian* Hessenberg varieties as in Harada–Precup [6].
- They correspond to Dyck paths with *bounce number* 1 or 0 as in Cho–Huh [3].

The commonality between each of the definitions and interpretations of **3**-free posets is that the resulting theory for chromatic quasisymmetric functions and Hessenberg varieties is quite nice and allows for one to prove e-positivity in a variety of ways. Therefore, one should expect an RSK for **3**-free posets to be just as nice.

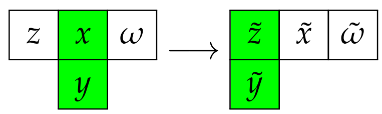
3 RSK for 3-free posets

In this section, we assume that P is a **3**-free poset with incomparability graph G. Hence, any P-tableau T has at most 2 rows.

3.1 RSK_P for permutations

We will define a row-insertion algorithm that inserts the letters of permutation $w = w_1 \cdots w_n$ one by one to make a series of pairs $(T_1, R_1), \ldots, (T_n, R_n)$ of P-tableau T_i and standard Young tableau R_i with same shape.

To do so, we must describe a series of *local moves* on P-arrays (which are defined to be P-tableaux in columns only). Given a $(1, 2, 1)$ subshape of a P-array T, we change it into a (2,1,1) shape with the same entries



Define the change according to the 4 cases as shown in Figure 2.

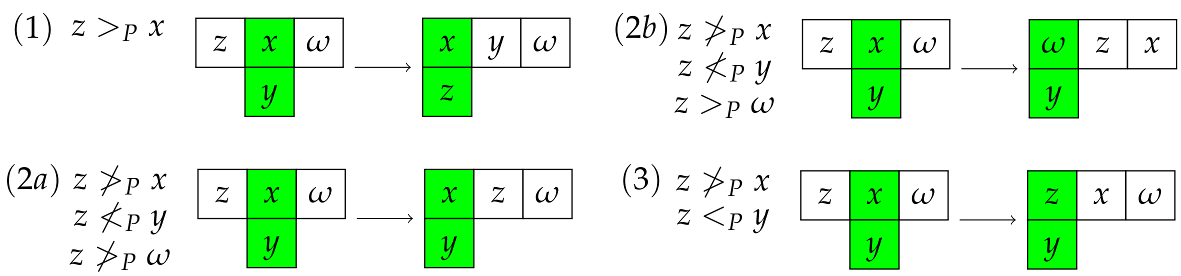


Figure 2: The 4 local moves

Definition 3.1 (RSK for **3**-free P)**.** Let P be a **3**-free poset and let $w \in \mathfrak{S}_n$.

1. For w_1, define
$$T_1 = \boxed{w_1} \qquad R_1 = \boxed{1}$$

2. Suppose $w_1, \ldots, w_{i-1}$ have been inserted forming a pair (T_{i-1}, R_{i-1}) of P-tableau and standard Young tableaux, respectively, of the same shape λ with $|\lambda| = i - 1$.

   (a) If w_i is not smaller in P than the last number of the first row of T_{i-1}, then place w_i at the end of the row creating T_i.

(b) Otherwise, suppose T_{i-1} looks like

$$\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
 & \cdots & & x_{k+1} & x_k & x_{k-1} & \cdots & x_1 \\
\hline
 & \cdots & \\
\cline{1-3}
\end{array}$$

and let x_k be the leftmost element of the first row such that $x_j >_P w_i$ for all $j \leq k$ (so, $x_{k+1} \not>_P w_i$). Then replace x_k by w_i and place x_k in the cell directly below its old cell (this is empty because P is **3**-free). Then, apply the *local moves* moving this new domino to the left until we have a partition shape. Call this new tableau T_i.

In either case, if c is the new cell of T_i, set $R_i(c) = i$.

We write $RSK_P(w) = (T, R)$ for the result after inserting the final letter of w.

Example 3.2. Let P be as in Figure 1 and suppose we have

$$(T_5, R_5) = \left(\begin{array}{|c|c|c|c|c|}\hline 3 & 4 & 2 & 5 & 6 \\ \hline\end{array}\, , \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline\end{array} \right)$$

and we want to insert $w_6 = 1$. Then we need 3 local moves, which we indicate with an arrow.

$$\begin{array}{|c|c|c|c|c|}\hline 3 & 4 & 2 & 1 & 6 \\ \hline \multicolumn{3}{c|}{} & 5 & \multicolumn{1}{c}{} \\ \cline{4-4}\end{array} \xrightarrow{(3)} \begin{array}{|c|c|c|c|c|}\hline 3 & 4 & 2 & 1 & 6 \\ \hline \multicolumn{2}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{3-3}\end{array} \xrightarrow{(2b)} \begin{array}{|c|c|c|c|c|}\hline 3 & 1 & 4 & 2 & 6 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{3}{c}{} \\ \cline{2-2}\end{array} \xrightarrow{(2a)} \begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 2 & 6 \\ \hline 5 & \multicolumn{4}{c}{} \\ \cline{1-1}\end{array}$$

Example 3.3. Let P be as in Figure 1 and let $w = 361452$. We show the full RSK_P for w in Figure 3. If we apply a local move during the insertion step to compute T_i, we show them to the left of the corresponding step.

Theorem 3.4. *For **3**-free P, the above algorithm $w \mapsto (T, R)$ is a bijection*

$$RSK_P : \mathfrak{S}_n \to \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times SYT_\lambda$$

such that if P is additionally a natural unit interval order and $RSK_P(w) = (T, R)$, then $\mathrm{inv}_G(w) = \mathrm{inv}_G(T)$.

Proof (Idea). One needs to check that

(1) local moves create P-tableaux,

(2) each local move has a unique *inverse* local move, and

(3) each local move preserves G-inversions when P is a natural unit interval order.

Then (1) plus **3**-free gives a well-definition; (2) gives the bijective property; and (3) plus the definition of the insertion gives (2.2). □

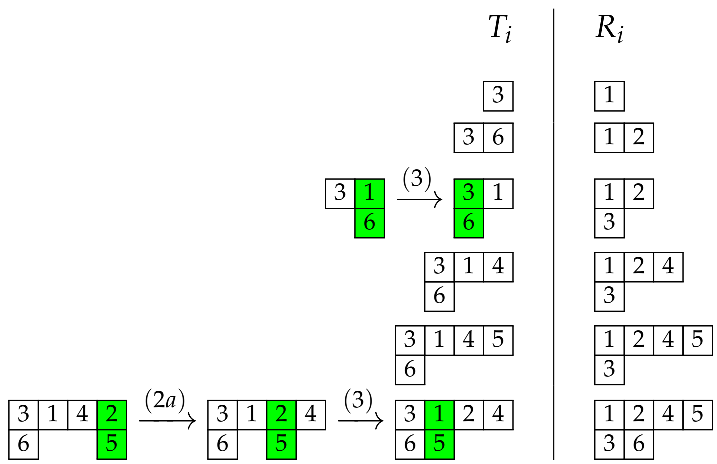


Figure 3: $RSK_P(361452)$

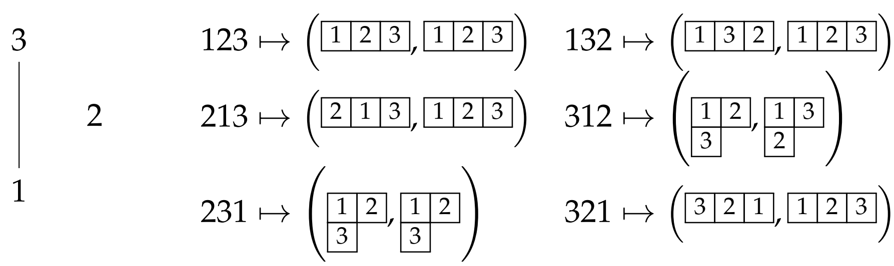


Figure 4: RSK_P for a poset P on $[3]$

Example 3.5. In Example 3.3, we have $\mathrm{inv}_G(w) = \mathrm{inv}_G(T) = 5$. See Figure 4 for a complete example for $n = 3$.

Hence, Theorem 3.4 provides a new combinatorial proof of (1.6) when P is **3**-free. For $w \in \mathfrak{S}_n$, let $\mathrm{DES}(w)$ be the usual descent set of a permutation. For a standard Young tableaux R define $\mathrm{DES}(R) = \{i \in [n-1] \mid i+1 \text{ is below } i\}$. Then classical RSK has the property that if $RSK(w) = (T, R)$, then $\mathrm{DES}(w) = \mathrm{DES}(R)$. For a poset P, define $\mathrm{DES}_P(w) = \{i \in [n-1] \mid w_i >_P w_{i+1}\}$. Then RSK_P also "preserves descents", meaning the following.

Proposition 3.6. *For* **3**-*free posets P, if $RSK_P(w) = (T, R)$, then* $\mathrm{DES}_P(w) = \mathrm{DES}(R)$.

Example 3.7. In Example 3.3, we have $\mathrm{DES}_P(w) = \mathrm{DES}(R) = \{2, 5\}$.

Remark 3.8. Chow [4] points out that the Sundquist–Wagner–West algorithm preserves descents only on a subclass of the posets considered in [13]. This smaller subclass does not contain **3**-free posets. The Kim–Pylyavskyy algorithm satisfies the similar property that $\mathrm{DES}(R) = \{n - d \mid d \in \mathrm{DES}_P(w)\}$.

3.2 Proper colorings

While section 3.1 was just for permutations, it serves as the foundation for an RSK_P of colorings. For each proper coloring $\kappa \in \mathcal{K}(G)$, form the generalized permutation $\{\binom{\kappa(i)}{i}\}_{i=1}^{n}$ of biletters. Then write this as a two-line array

$$\binom{\overline{\kappa}}{w} = \begin{pmatrix} \kappa(w_1) & \kappa(w_2) & \dots & \kappa(w_n) \\ w_1 & w_2 & \dots & w_n \end{pmatrix}$$

so that $\overline{\kappa} = (\kappa(w_1) \leq \kappa(w_2) \leq \cdots \leq \kappa(w_n))$ and if $\kappa(w_i) = \kappa(w_j)$ and $i < j$, then $w_i > w_j$.

Example 3.9. Let P be as in Figure 1 and let κ be the coloring of $\mathrm{Inc}(P)$ below

$$\begin{array}{ccccccc} \kappa & 3 & 2 & 1 & 3 & 2 & 1 \\ G & 1 \,—\, & 2 \,—\, & 3 \,—\, & 4 \,—\, & 5 \,—\, & 6 \end{array} \qquad \binom{\overline{\kappa}}{w} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 6 & 3 & 5 & 2 & 4 & 1 \end{pmatrix}$$

Then, the general RSK_P inserts the bottom word w (which is a permutation) to get a P-tableau T as in Definition 3.1 and records the procedure in R with the top word $\overline{\kappa}$. Our choice of 2-line array ensures that if $\kappa(w_i) = \kappa(w_{i+1})$, then $w_i >_P w_{i+1}$, so $\kappa(w_i)$ and $\kappa(w_{i+1})$ will be in separate rows of R, and hence R will be *row-strict*.

Theorem 3.10. *For* **3***-free P, the above algorithm $\kappa \mapsto (T, R)$ is bijection*

$$RSK_P : \mathcal{K}(G) \to \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,\lambda} \times RST_\lambda$$

such that if $RSK_P(w) = (T, R)$ and P is additionally a natural unit interval order, then $\mathrm{des}(\kappa) = \mathrm{inv}_G(T)$. *Moreover, restricting this map to $\mathfrak{S}_n$ gives the bijection in Theorem 3.4.*

Theorem 3.10 hence provides new combinatorial proofs of the Schur bases expansions given in Theorems 2.1 and 2.3 when P is **3**-free.

Remark 3.11. By applying (P, ω)-partition reciprocity, the monomials of $\omega X_G(\mathbf{x}, t)$ are seen to be indexed by pairs $(\overline{\mathbf{o}}, \kappa)$ of acyclic orientations $\overline{\mathbf{o}}$ and *weak colorings* κ of $\overline{\mathbf{o}}$, meaning $x \to y$ in $\overline{\mathbf{o}}$ implies that $\kappa(x) \leq \kappa(y)$. In the full paper [2], we extend RSK_P to this collection.

3.3 Proper multicolorings

We now come to the fullest generality of RSK_P, mirroring the dual of the RSK bijection (1.2). A *proper multicoloring* of a graph G is a map κ that assigns each vertex i a finite set $\kappa(i) \subset \mathbb{P}$ such that if $ij \in E$ then $\kappa(i) \cap \kappa(j) = \emptyset$. Let $\mathcal{MK}_n(G)$ be the set of proper colorings so that $\sum_{v \in G} |\kappa(v)| = n$. We can turn a multicoloring $\kappa \in \mathcal{MK}_n(G)$ into a generalized permutation $\binom{\overline{\kappa}}{w}$ of length n analogously to how we did colorings. See Example 3.12.

Example 3.12. Let P be as in Figure 1 and let κ be the multicoloring of $\mathrm{Inc}(P)$ below

$$\begin{array}{ccccccccccccc} & & & 7 & & & & & & & & & \\ & 4 & & 5 & & 6 & & & & 4 & & & \\ \kappa & 3 & & 2 & & 1 & & 3 & & 2 & & 1 \\ G & 1 & — & 2 & — & 3 & — & 4 & — & 5 & — & 6 \end{array} \qquad \begin{pmatrix} \overline{\kappa} \\ w \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 6 & 3 & 5 & 2 & 4 & 1 & 1 & 5 & 2 & 3 & 2 \end{pmatrix}$$

Hence, the analogous version of (1.2)[4] is a bijection

$$RSK_P : \mathcal{MK}_n(G) \to \bigsqcup_{\lambda \vdash n} \mathcal{SST}_{P,\lambda} \times RST_\lambda$$

where $\mathcal{SST}_{P,\lambda}$ is the set of *semistandard* P-tableaux (meaning you can repeat elements of P) of shape λ. For a multicoloring κ of G, let the *type* μ of a multicoloring κ be the composition $\mu = (|\kappa(1)|, |\kappa(2)|, \dots)$ and let $\mathrm{des}(\kappa)$ be the number of pairs $\{(a,i),(b,j)\}$ such that if $a \in \kappa(i)$, $b \in \kappa(j)$, $ij \in E(G)$ and $i < j$, then $a > b$. The definition of $\mathrm{inv}_G(T)$ is easily extended to $T \in \mathcal{SST}_{P,\lambda}$.

Theorem 3.13. *If P is $\mathbf{3}$-free, then for each n, RSK_P is a bijection*

$$RSK_P : \mathcal{MK}_n(G) \to \bigsqcup_{\lambda \vdash n} \mathcal{SST}_{P,\lambda} \times RST_\lambda$$

such that if $RSK_P(\kappa) = (T, R)$, then the type of κ is the content of T. If P is additionally a natural unit interval order, then $\mathrm{des}(\kappa) = \mathrm{inv}_G(T)$.

When P is $\mathbf{3}$-free, Theorem 3.13 provides new bijective proofs of Gasharov's Schur expansion [5, Theorem 4] of the multichromatic symmetric function $X_G(\mathbf{x}; \mu)$ and when P is a natural unit interval order, Hwang's expansion [7, Theorem 4.19] of the multichromatic quasisymmetric function $X_G(\mathbf{x}, t; \mu)$ of type μ. These are both defined analogously to their chromatic versions,

$$X_G(\mathbf{x}, t; \mu) = \sum_{\lambda \vdash n} |\mathcal{SST}_{P,\lambda,\mu}| s_\lambda \qquad X_G(\mathbf{x}, t; \mu) = \sum_{i \geq 0} t^i \sum_{\lambda \vdash n} |\mathcal{SST}_{P,\lambda,\mu,i}| s_\lambda$$

where $\mathcal{SST}_{P,\lambda,\mu,i}$ is the set of $T \in \mathcal{SST}_{P,\lambda}$ with content μ and $\mathrm{inv}_G(T) = i$ and $\mathcal{SST}_{P,\lambda,\mu} = \bigsqcup_{i \geq 0} \mathcal{SST}_{P,\lambda,\mu,i}$. In particular, we get a direct combinatorial proof that for each composition $\mu = (\mu_1, \mu_2, \dots)$ of n, we have

$$\binom{n}{\mu} = \sum_{\lambda \vdash n} |\mathcal{SST}_{P,\lambda,\mu}| f^\lambda.$$

[4]More precisely, a *dual* form of *RSK*.

3.3.1 Equivalence Classes of Acyclic Orientations

Hwang [7] gives a further refinement of $X_G(\mathbf{x}, t; \mu)$, but we will just discuss the case of $\mu = (1^n)$ for clarity here. A *local flip* of an acyclic orientation $\overline{\mathbf{o}}$ changes $a \to b \leftarrow c$ to $a \leftarrow b \to c$ provided a and c are covered by b in the poset induced by $\overline{\mathbf{o}}$. Say $\overline{\mathbf{o}}'$ is *equivalent* to $\overline{\mathbf{o}}$ if it is related by a sequence of local flips, and let $[\overline{\mathbf{o}}]$ be its equivalence class. Note that if $\overline{\mathbf{o}}' \in [\overline{\mathbf{o}}]$ then $\operatorname{des}(\overline{\mathbf{o}}') = \operatorname{des}(\overline{\mathbf{o}})$. Let $\mathcal{K}(G, [\overline{\mathbf{o}}])$ be the set of colorings that are *compatible* (in a sense) with an $\overline{\mathbf{o}}' \in [\overline{\mathbf{o}}]$ and define a quasisymmetric function

$$X_{G,[\overline{\mathbf{o}}]}(\mathbf{x}) = \sum_{\kappa \in \mathcal{K}(G,[\overline{\mathbf{o}}])} x_\kappa$$

Note that $X_G(\mathbf{x}, t) = \sum_{[\overline{\mathbf{o}}]} t^{\operatorname{des}(\overline{\mathbf{o}})} X_{G,[\overline{\mathbf{o}}]}(\mathbf{x}, t)$. When $G = \operatorname{Inc}(P)$ for a natural unit interval order P, Hwang uses noncommutative symmetric function theory to prove that this is a symmetric function with Schur expansion

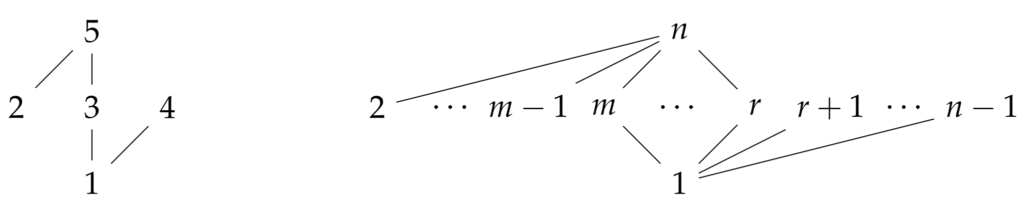
$$X_{G,[\overline{\mathbf{o}}]}(\mathbf{x}) = \sum_{T \in \mathcal{T}_{P,[\overline{\mathbf{o}}]}} s_{\lambda(T)}.$$

where $\mathcal{T}_{P,[\overline{\mathbf{o}}]}$ is some set of P-tableaux that are *compatible* (in a sense) with an $\overline{\mathbf{o}}' \in [\overline{\mathbf{o}}]$. All $T \in \mathcal{T}_{P,[\overline{\mathbf{o}}]}$ have $\operatorname{inv}_G(T) = \operatorname{des}(\overline{\mathbf{o}})$. We prove this expansion bijectively using RSK_P. Let $\mathcal{T}_{P,[\overline{\mathbf{o}}],\lambda}$ be the set of $T \in \mathcal{T}_{P,[\overline{\mathbf{o}}]}$ with shape λ.

Theorem 3.14. *Let P be a **3**-free natural unit interval order. For each acyclic orientation $\overline{\mathbf{o}}$ of $G = \operatorname{Inc}(P)$, RSK_P restricts to a bijection*

$$RSK_P : \mathcal{K}(G, [\overline{\mathbf{o}}]) \to \bigsqcup_{\lambda \vdash n} \mathcal{T}_{P,[\overline{\mathbf{o}}],\lambda} \times RST_\lambda$$

Remark 3.15. We show that RSK_P can be extended to the *beast* poset (Figure 5a) or more generally the *beastly* poset[5] (Figure 5b). The beast poset is explicitly avoided in the previous algorithms [13, 8].



(a) The beast poset (b) The beastly poset

Figure 5

[5]This is the second e-positive class of posets considered by Cho and Huh [3] with **3**-free being the first.

Acknowledgements

I would like to thank my advisor Michelle Wachs for her great help in the preparation of this extended abstract. I would also like to thank the reviewers for their helpful comments.

References

[1] A. Abreu and A. Nigro. "Chromatic symmetric functions from the modular law". *Journal of Combinatorial Theory, Series A* **180** (2021), p. 105407. DOI.

[2] K. Celano. "Chromatic symmetric functions and RSK for 3-free and beastly posets". 2023. In preparation.

[3] S. Cho and J. Huh. "On e-Positivity and e-Unimodality of Chromatic Quasi-symmetric Functions". *SIAM Journal on Discrete Mathematics* **33** (2019), pp. 2286–2315.

[4] T. Y. Chow. "Descents, quasi-symmetric functions, Robison-Schensted for posets, and the chromatic symmetric function". *Journal Algebraic Combainatorics* **10**.3 (1999), 227–240.

[5] V. Gasharov. "Incomparability graphs of (3 + 1)-free posets are S-positive". *Discrete Mathematics* **157**.1-3 (1996), 107–125. DOI.

[6] M. Harada and M. E. Precup. "The cohomology of abelian Hessenberg varieties and the stanley–stembridge conjecture". *Algebraic Combinatorics* **2**.6 (2019), 1059–1108. DOI.

[7] B. Hwang. "Chromatic quasisymmetric functions and noncommutative P-symmetric functions". 2022. arXiv:2208.09857.

[8] D. Kim and P. Pylyavskyy. "Robinson-Schensted correspondence for unit interval orders". 2020. arXiv:2003.12123.

[9] J. Shareshian and M. L. Wachs. "Chromatic quasisymmetric functions and Hessenberg varieties". *Configuration spaces*. Vol. 14. CRM Series. Ed. Norm., Pisa, 2012, pp. 433–460.

[10] J. Shareshian and M. L. Wachs. "Chromatic quasisymmetric functions". *Advances in Mathematics* **295** (2016), 497–551. DOI.

[11] R. P. Stanley. "A Symmetric Function Generalization of the Chromatic Polynomial of a Graph". *Advances in Mathematics* **111**.1 (1995), 166–194. DOI.

[12] R. P. Stanley. *Enumerative combinatorics. volume 2*. Cambridge University Press, 1999.

[13] T. S. Sundquist, D. G. Wagner, and J. West. "A Robinson–Schensted Algorithm for a Class of Partial Orders". *Journal of Combinatorial Theory, Series A* **79**.1 (1997), pp. 36–52. DOI.