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Towards Butler's conjecture

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Abstract. For a partition ν, let $\lambda, \mu \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| = 1$. Butler conjectured that the divided difference $\mathrm{I}_{\lambda,\mu}[X;q,t] = (T_\lambda \widetilde{H}_\mu[X;q,t] - T_\mu \widetilde{H}_\lambda[X;q,t])/(T_\lambda - T_\mu)$ of modified Macdonald polynomials of two partitions λ and μ is Schur positive. By introducing a new LLT equivalence called column exchange rule, we give a combinatorial formula for $\mathrm{I}_{\lambda,\mu}[X;q,t]$, which is a positive monomial expansion. We also prove Butler's conjecture for some special cases.

Keywords: Butler's conjecture, Science Fiction conjecture, Modified Macdonald polynomials, Macdonald intersection polynomials, column exchange rule, Butler permutations, (q,t)-Kostka polynomial

1 Introduction

In his seminal paper [19], Macdonald introduced the *Macdonald P- polynomials* $P_\mu[X;q,t]$ which are (q,t)-extension of Schur functions indexed by partitions μ. The *modified Macdonald polynomials* $\widetilde{H}_\mu[X;q,t]$ are introduced as a combinatorial version of the Macdonald P-polynomials.

For a partition $\mu \vdash n$, the Garsia–Haiman module V_μ is defined by the subspace in the polynomial ring $\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ spanned by partial derivatives of the polynomial Δ_μ analogous to the Vandermonde determinant:

$$\Delta_\mu := \det\left[x_i^{p_j} y_i^{q_j}\right]_{i,j=1,\ldots,n},$$

where (p_j,q_j) runs over cells in μ and the symmetric group $\mathfrak{S}_n$ acts diagonally permuting x and y variables [8]. Later, Haiman [13] proved that the Garsia–Haiman module plays

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a role as a representation-theoretic model for the modified Macdonald polynomial, i.e., the (bigraded) Frobenius characteristic of V_μ coincides with $\widetilde{H}_\mu[X;q,t]$.

Bergeron and Garsia [4] studied relationships among Garsia–Haiman modules called *Science Fiction conjecture*. Among the variety of its implications, they studied the intersection of Garsia–Haiman modules. To elaborate, Science Fiction conjecture implies the following.

Conjecture 1.1. Let ν be a partition and $\mu^{(1)},\dots,\mu^{(k)} \subseteq \nu$ be k distinct partitions such that $|\nu/\mu^{(1)}| = \cdots = |\nu/\mu^{(k)}| = 1$. Then the bigraded $\mathfrak{S}_n$-module $\bigcap_{i=1}^k V_{\mu^{(i)}}$ is of dimension $\frac{n!}{k}$, and its Frobenius characteristic is

$$\operatorname{Frob}\left(\bigcap_{i=1}^k V_{\mu^{(i)}}, q, t\right) = \sum_{i=1}^k \prod_{j\neq i} \frac{T_{\mu^{(j)}}}{(T_{\mu^{(j)}} - T_{\mu^{(i)}})} \widetilde{H}_{\mu^{(i)}}[X;q,t], \tag{1.1}$$

where $T_\mu := \prod_{(i,j)\in\mu} t^{i-1}q^{j-1}$.

The first assertion is called $\frac{n!}{k}$*-conjecture*, and Armon recently proved the $\frac{n!}{2}$-conjecture for hook shapes [2]. The second implication of Science Fiction conjecture gives a formula for the Frobenius characteristic of the intersection of the Garsia–Haiman modules as a linear combination of the modified Macdonald polynomials. We will call the symmetric function in the right-hand side of (1.1) an *Macdonald Intersection polynomial* of $\mu^{(1)},\dots,\mu^{(k)}$ and denote it by $\mathrm{I}_{\mu^{(1)},\dots,\mu^{(k)}}[X;q,t]$. In the companion paper of the authors [15], we study a remarkable connection between $\mathrm{I}_{\mu^{(1)},\dots,\mu^{(k)}}[X;q,t]$, ∇e_{k-1} which is the Frobenius characteristic of the diagonal harmonics [14], and the Shuffle formula $D_{k-1}[X;q,t]$ in [10, 7]. In this paper, we focus more on the case when $k=2$, which is related to Butler's conjecture. In 1994, Butler observed a surprising behavior of the modified Macdonald polynomials.

Conjecture 1.2. (Butler's conjecture [6]) Let ν be a partition and $\lambda,\mu \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| = 1$. Then the Macdonald intersection polynomial $\mathrm{I}_{\lambda,\mu}[X;q,t] = \frac{T_\lambda \widetilde{H}_\mu[X;q,t] - T_\mu \widetilde{H}_\lambda[X;q,t]}{T_\lambda - T_\mu}$ is Schur positive.

Note that Science Fiction conjecture implies Butler's conjecture, as mentioned in [4]. To be more precise, the Macdonald intersection polynomial $\mathrm{I}_{\lambda,\mu}[X;q,t]$ is given as a Frobenius characteristic of an $\mathfrak{S}_n$–module $V_\lambda \cap V_\mu$, thus Schur positive.

We divide the main results in this paper into three parts. The first part gives a combinatorial formula (F-expansion, monomial expansion) for $\mathrm{I}_{\lambda,\mu}[X;q,t]$ (Theorem 4.1, Corollary 4.2). The second part is to prove partial cases for Butler's conjecture (Theorem 5.1, Corollary 5.2). The third part provides combinatorial formulas for (q,t)-Kostka polynomials which are consistent with Butler's conjecture (Theorems 6.1 and 6.2).

2 Generalization of modified Macdonald polynomials

A (general) *diagram* D is a collection of points (cells) in $\mathbb{Z}_+ \times \mathbb{Z}_+$, and a *bottom cell* of D is a cell located in the lowest position on each column. A *filled diagram* (D, f) is a diagram together with a filling

$$f : D \setminus \{\text{bottom cells of } D\} \to \mathbb{F}$$

that assigns a scalar in $\mathbb{F}$ for each cell of D that is not a bottom cell. Throughout this paper, we let $\mathbb{F} = \mathbb{C}(q, t)$. We visualize (D, f) by writing the corresponding value of f for each cell. For example, see Figure 1.

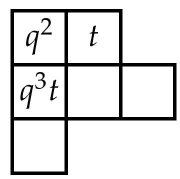


Figure 1: A filled diagram

We denote a diagram by its row numbers for each column: $D = [D^{(1)}, D^{(2)}, \dots,]$, where $D^{(j)}$ is the set of $i's$ such that $(i, j) \in D$. Throughout this paper, we assume that for every diagram D, each column $D^{(i)}$ is an interval $[a, b] = \{a, a+1, \dots, b\}$ for some $a \leq b$. For example, the diagram in Figure 1 can be represented as $D = [[1,3],[2,3],[2,2]]$, and the Young diagram for a partition $\mu = (\mu_1, \dots, \mu_\ell)$ can be written as $\mu = [[\mu'_1], \dots, [\mu'_{\mu_1}]]$, where μ' is the conjugate of μ.

Definition 2.1. We give a total order on cells in D row by row, top to bottom, and left to right within each row. We denote such total order with $N_D : D \to [|D|]$. For a filled diagram (D, f), we define functions $\mathrm{inv}_D : \mathfrak{S}_{|D|} \to \mathbb{F}$ and $\mathrm{maj}_{(D,f)} : \mathfrak{S}_{|D|} \to \mathbb{F}$ as follows: For a permutation $w \in \mathfrak{S}_{|D|}$, we say that a pair (u, v) of cells in D is an *inversion* with respect to w if $w_{N_D(u)} > w_{N_D(v)}$ and either

- $u = (i, j)$, $v = (i, j')$ and $j < j'$ or
- $u = (i, j)$, $v = (i-1, j')$ and $j > j'$.

Then we define

$$\mathrm{inv}_D(w) := \prod_{(u,v)} q,$$

where the product is over all pairs (u, v) of cells in D that are inversions with respect to w. For a permutation $w \in \mathfrak{S}_{|D|}$, we say that a cell $u = (i, j)$ in D is a *descent* with respect to w if $w_{N_D(u)} > w_{N_D(v)}$ where $v = (i-1, j)$ is the cell just below u. Then we define

$$\mathrm{maj}_{(D,f)}(w) := \prod_{u} f(u)$$

where the product is over all cells that are descents with respect to w. Finally we define a function $\mathrm{stat}_{(D,f)} : \mathfrak{S}_{|D|} \to \mathbb{F}$ as

$$\mathrm{stat}_{(D,f)}(w) := \mathrm{inv}_D(w)\,\mathrm{maj}_{(D,f)}(w).$$

We defined $\mathrm{stat}_{(D,f)}(w)$ for a permutation $w \in \mathfrak{S}_{|D|}$. However, we note that $\mathrm{stat}_{(D,f)}(w)$ can be defined in the same way for any length $|D|$ word of positive integers w. We define a generalization of the modified Macdonald polynomial $\widetilde{H}_{(D,f)}$ for a filled diagram (D, f) as

$$\widetilde{H}_{(D,f)}[X; q, t] = \sum_{w \in \mathfrak{S}_n} \mathrm{stat}_{(D,f)}(w) F_{\mathrm{iDes}(w)},$$

where $F_S[X]$ is the fundamental quasisymmetric functions.

Given a partition μ, we define the *standard filling* of μ,

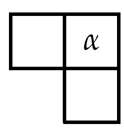
$$f_\mu^{\mathrm{st}} : \mu \setminus \{\text{bottom cells of } \mu\} \to \mathbb{F}$$

by $f_\mu^{\mathrm{st}}(u) = q^{-\mathrm{arm}_\mu(u)} t^{\mathrm{leg}_\mu(u)+1}$. Then the modified Macdonald polynomial for the filled diagram $(\mu, f_\mu^{\mathrm{st}})$ is the usual modified Macdonald polynomial

$$\widetilde{H}_{(\mu, f_\mu^{\mathrm{st}})}[X; q, t] = \widetilde{H}_\mu[X; q, t],$$

by Haglund–Haiman–Loehr formula [12].

Example 2.2. Let (D, f) be a filled diagram as depicted below,



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where $\alpha \in \mathbb{F}$. Then the statistics are given in the following table.

$w \in \mathfrak{S}_3$	123	132	213	231	312	321
$\mathrm{iDes}(w)$	$\emptyset$	$\{2\}$	$\{1\}$	$\{1\}$	$\{2\}$	$\{1,2\}$
$\mathrm{inv}_D(w)$	1	1	q	1	q	q
$\mathrm{maj}_{(D,f)}(w)$	1	α	1	α	1	α

Thus, the modified Macdonald polynomial for (D, f) is

$$\widetilde{H}_{(D,f)}[X; q, t] = F_\emptyset + (q + \alpha)F_{\{1\}} + (q + \alpha)F_{\{2\}} + q\alpha F_{\{1,2\}}.$$

Two distinct filled diagrams (D, f) and (D', f') can give the same modified Macdonald polynomials. For example, given a filled diagram (D, f) where

$$D = [D^{(1)}, D^{(2)}, \ldots, D^{(\ell)}].$$

Consider a diagram

$$D' = [D^{(2)}, \dots, D^{(\ell)}, D^{(1)} + 1],$$

where $I + 1 = \{a + 1 : a \in I\}$ for an interval I. In other words, D' is a diagram obtained by moving the leftmost column of D to the end of the right and placing it one cell up. Cells in D and D' are naturally in bijection, and we give a filling f' on D' inherited from the filling f on D. We define $\mathrm{cycling}(D, f) := (D', f')$. Then the following is well-known.

Lemma 2.3. *(Cycling rule) Let (D, f) be a filled diagram and $(D', f') = \mathrm{cycling}(D, f)$. Then we have*

$$\widetilde{H}_{(D,f)}[X; q, t] = \widetilde{H}_{(D',f')}[X; q, t].$$

In Section 3 and in the omitted proofs in Section 5, we exhibit a variety of relations between the modified Macdonald polynomials (or LLT polynomials).

3 Column exchange rule and Butler permutations

3.1 Column exchange rule

We introduce the *column exchange rule*, which concerns the equidistribution of two statistics over permutations. We use this rule to exchange the columns of filled diagrams.

Proposition 3.1. *(column exchange rule) Let (D, f) and (D', f') be filled diagrams that are the same except for the i-th and $i+1$-th columns. In addition, let i-th, and $i+1$-th column of (D, f) is of the form $\mu = [[n], [m]]$ and of (D', f') is of the form $\lambda = [[m], [n]]$ for some $n > m$, and $f_\mu : \mu \to \mathbb{F}$ and $f_\lambda : \lambda \to \mathbb{F}$ be fillings such that*

- *$f_\mu(i, 1) = f_\lambda(i, 2)$ for $i > m + 1$,*
- *$\alpha := q^{-1} f_\mu(m + 1, 1) = f_\lambda(m + 1, 2)$, and*
- *$f_\mu(i, 1) = \alpha f_\mu(i, 2) = \alpha f_\lambda(i, 1) = f_\lambda(i, 2)$ for $1 < i \leq m$,*

Then there is a stat, iDes, *and content-preserving bijection $\phi_i^{(D,f)}$ from $\mathfrak{S}_{|D|}$ to $\mathfrak{S}_{|D'|}$. In particular, we have*

$$\widetilde{H}_{(D,f)}[X; q, t] = \widetilde{H}_{(D',f')}[X; q, t].$$

For example, filled diagrams (μ, f_μ) and (λ, f_λ) depicted in Figure 2 satisfy the conditions in Proposition 3.1. In what follows, we abuse our notation to write $\phi_i^{(D,f)} = \phi_i$ for convenience.

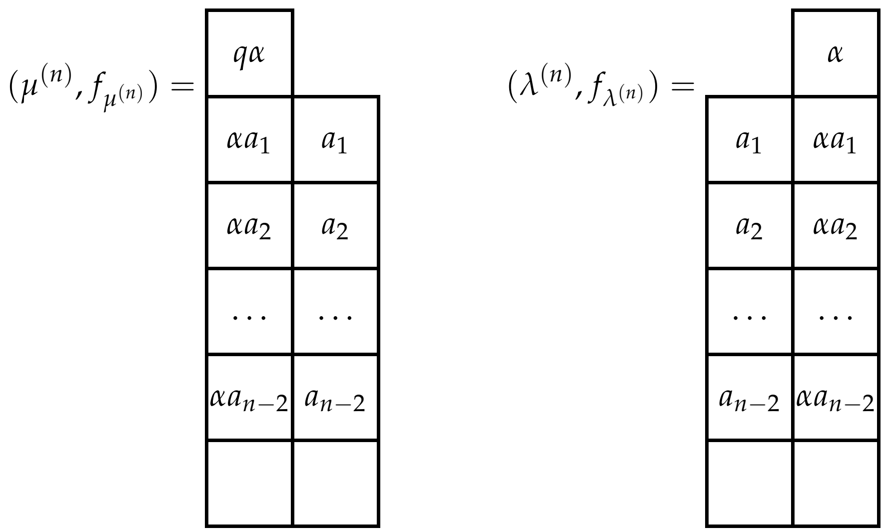

Figure 2: Filled diagrams (μ, f_μ) and (λ, f_λ)

Remark 3.2. When $(D, f) = (\mu, f_\mu^{\mathrm{st}})$ for a partition μ and standard filling f_μ^{st}, the implication

$$\widetilde{H}_{(D,f)}[X; q, t] = \widetilde{H}_{(D',f')}[X; q, t]$$

of the above Proposition follows from [11, Theorem 5.1.1]. Our proof gives a bijective proof of their theorem, answering a question of Haglund [1]. In addition, we would like to highlight a few differences and advantages of the column exchange rule (and its proof).

First of all, the bijection ϕ preserves not only stat, iDes, but also content. This fact will be a crucial point in Section 5, and in the upcoming paper [15]. Moreover, the bijections ϕ and ψ appear in the construction of the map ζ, which will be the main ingredient of Proposition 3.4. Finally, the conditions in Proposition 3.1 is more flexible than the conditions in [11, Theorem 5.1.1].

3.2 Butler permutations

For a directed perfect matching $M = \{(a_1, a_2), \ldots, (a_{2n-1}, a_{2n})\}$, we say an arc (a, b) is in the *forward direction* if $a < b$ and in the *reverse direction* otherwise. An arc (a, b) is *nested* by the other arc (c, d) if $\min\{c, d\} < a, b$ and $\max\{c, d\} > a, b$. We say that the pair of arcs (a, b) and (c, d) is *crossing* if $\min\{a, b\} < \min\{c, d\} < \max\{a, b\} < \max\{c, d\}$ or $\min\{c, d\} < \min\{a, b\} < \max\{c, d\} < \max\{a, b\}$. If not, we say the pair is *noncrossing*.

Definition 3.3. For a permutation $w = w_1 \ldots w_{2n} \in \mathfrak{S}_{2n}$. Consider a directed perfect matching $M(w)$ on $\{0, 1, \ldots, 2n+1\}$ with directed arcs $\alpha_1(w), \alpha_2(w), \ldots$ as follows:

- $\alpha_1(w)$ is a directed arc from w_n to $n+1$,
- $\alpha_i(w)$ is a directed arc from $w_{n-2(i-1)}$ to $w_{n-2(i-1)+1}$ for $2 \leq i \leq \frac{n}{2}$, and
- $\alpha_{\frac{n}{2}+1}(w)$ is a directed arc from 0 to w_1.

Let $k(w)$ the smallest integer k such that the pair of arcs $\alpha_k(w)$ and $\alpha_{k+1}(w)$ is noncrossing if exists. We say that w is a *Butler permutation* if either

- $\alpha_{k+1}(w)$ is nested by $\alpha_k(w)$ and $\alpha_k(w)$ is in the reverse direction, or
- $\alpha_{k+1}(w)$ is not nested by $\alpha_k(w)$ and $\alpha_k(w)$ is in the forward direction.

If there is no such k, we let $k(w)$ to be $\frac{n}{2}$. We say that w is a *Butler permutation* if $\alpha_{k(w)}$ is in the reverse direction. We denote the set of Butler permutations of length $2n$ by $\mathfrak{B}_{2n}$.

Let w be a word of weight $\alpha = (\alpha_1, \alpha_2, \dots)$, i.e. number of i's in w is α_i. The *standardization* $\operatorname{std}(w)$ of w is given by replacing 1's with the numbers $[1, \alpha_1]$ in increasing order from left to right, replacing 2's with the numbers in $[\alpha_1+1, \alpha_1+\alpha_2]$ in increasing order from left to right, and so on. For example, we have $\operatorname{std}(3155) = 2134$.

3.3 Main Lemma

Proposition 3.4. *Suppose $n > m$. Let $\bar{\mu} = [[n], [m] \setminus \{1\}]$, $\bar{\nu} = [[m+1], [m] \setminus \{1\}] \subseteq \mu$, and $\bar{\lambda} = [[m], [n] \setminus \{1\}]$. Let $f_{\bar{\mu}} : \bar{\mu} \to \mathbb{F}$ and $f_{\bar{\lambda}} : \bar{\lambda} \to \mathbb{F}$ be fillings such that*

- $f_{\bar{\mu}}(i,1) = f_{\bar{\lambda}}(i,2)$ *for* $i > m+1$,
- $\alpha := q^{-1} f_{\bar{\mu}}(m+1,1) = f_{\bar{\lambda}}(m+1,2)$,
- $f_{\bar{\mu}}(i,1) = \alpha f_{\bar{\mu}}(i,2) = \alpha f_{\bar{\lambda}}(i,1) = f_{\bar{\lambda}}(i,2)$ *for* $2 < i \leq m$, *and*
- $f_{\bar{\mu}}(2,1) = \alpha f_{\bar{\lambda}}(2,1)$.

Then, for any filled diagram (D, f_D),

$$\frac{\widetilde{H}_{((D,\bar{\mu}),(f_D,f_{\bar{\mu}}))}[X;q,t] - \alpha \widetilde{H}_{((D,\bar{\lambda}),(f_D,f_{\bar{\lambda}}))}[X;q,t]}{1-\alpha} = \sum_{w} \operatorname{stat}_{(\bar{\mu},f_{\bar{\mu}})} F_{\operatorname{iDes}(w)},$$

where the sum is over all permutations $w \in \mathfrak{S}_{n+m-1+|D|}$ such that $\operatorname{std}(w\downarrow_{\bar{\mu}}^{(D,\bar{\mu})}) \in \mathfrak{B}_{2m}$.

4 Combinatorial formulas for $\mathrm{I}_{\lambda,\mu}[X;q,t]$.

4.1 Proof of a combinatorial formula for the F-expansion for $\mathrm{I}_{\lambda,\mu}[X;q,t]$.

Theorem 4.1. *For a partition ν, let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\mu| = |\nu/\lambda| = 1$. Moreover, suppose the cells ν/μ, and ν/λ are in i-th and j-th column respectively for $i < j$. Then the fundamental quasisymmetric expansion of $\mathrm{I}_{\lambda,\mu}[X;q,t]$ is given by*

$$\mathrm{I}_{\lambda,\mu}[X;q,t] = \sum_{w \in \mathcal{B}_{\mu,\lambda}} \mathrm{stat}_{(\mu, f^{\mathrm{st}}_{\mu})}(w) F_{\mathrm{iDes}(w)}[X]. \tag{4.1}$$

Here, $\mathcal{B}_{\mu,\lambda}$ is the set of permutations w of $\mathfrak{S}_{|D|}$ such that restriction of

$$\mathrm{std}(\phi_{\nu_1 - 1} \cdots \circ \phi_j \circ \phi_1 \circ \cdots \circ \phi_{i-1}(w))$$

to the first $m+1$ rows of the last two columns is a Butler permutation.

We sketch a proof of Theorem 4.1 by giving an example. Let $\lambda = [[4],[3],[3],[2],[1]]$ and $\mu = [[4],[4],[3],[1],[1]]$. First of all, the filled diagram $(\lambda, f^{st}_{\lambda})$ is

t				
$q^{-2}t^2$	$q^{-1}t$	t		
$q^{-3}t^3$	$q^{-2}t^2$	$q^{-1}t^2$	t	

.

After applying the column exchange rule to move the second column to the left and the fourth column to the right, then cycling rule to move the first column to the last, we obtain the filled diagram $(D_\lambda, f_{D_\lambda})$

$q^{-1}t$				$q^{-1}t$
$q^{-2}t^2$	t			$q^{-2}t^2$
$q^{-3}t^3$	$q^{-1}t^2$		$q^{-1}t$	

.

On the other hand, the filled diagram (μ, f_μ^{st}) is

$$\begin{array}{|c|c|c|c|c|}
\cline{1-2}
q^{-1}t & t \\
\cline{1-3}
q^{-2}t^2 & q^{-1}t^2 & t \\
\cline{1-3}
q^{-2}t^3 & q^{-1}t^3 & t^2 \\
\hline
 & & & & \\
\hline
\end{array}\;.$$

After applying the column exchange rule to move the fourth column to the left and the second column to the right, then cycling rule to move the first column to the last, we obtain the filled diagram (D_μ, f_{D_μ})

$$\begin{array}{|c|c|c|c|c|}
\cline{1-1}\cline{4-4}
q^{-1}t & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & q^{-1}t & \multicolumn{1}{c}{} \\
\cline{1-2}\cline{4-4}
q^{-2}t^2 & t & \multicolumn{1}{c|}{} & q^{-1}t^2 & \multicolumn{1}{c}{} \\
\cline{1-2}\cline{4-5}
q^{-3}t^3 & q^{-1}t^2 & \multicolumn{1}{c|}{} & q^{-3}t^3 & \\
\hline
 & & & & \multicolumn{1}{c}{} \\
\cline{1-4}
\end{array}\;.$$

One can check that the filled diagrams $(D_\lambda, f_{D_\lambda})$ and (D_μ, f_{D_μ}) are the same except for the last two columns. In addition, the last two columns satisfy the condition of Proposition 3.4 where $\alpha = q^2t^2$.

We end this section with two implications of Theorem 4.1. The first one is a monomial expansion of $\mathrm{I}_{\lambda,\mu}[X;q,t]$, and the second one is about the specialization $\mathrm{I}_{\lambda,\mu}[X;1,1]$.

Corollary 4.2. *For a partition ν, let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| = 1$. Then the monomial expansion of $\mathrm{I}_{\lambda,\mu}[X;q,t]$ is given by*

$$\mathrm{I}_{\lambda,\mu}[X;q,t] = \sum_{\nu \vdash n} \sum_{w \in \mathcal{B}_{\mu,\lambda,\nu}} \mathrm{stat}_{(\mu, f_\mu^{\mathrm{st}})}(w) m_\nu[X]. \tag{4.2}$$

Here, $\mathcal{B}_{\mu,\lambda,\nu}$ is the set of words w of weight ν whose standardization $\mathrm{std}(w)$ is a Butler permutation of (μ, λ).

Corollary 4.3. *For a partition ν, let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| = 1$. Then we have*

$$\mathrm{I}_{\lambda,\mu}[X;1,1] = h_{(2,1^{n-2})}[X],$$

which is independent of μ and λ. In particular, this is consistent with $\frac{n!}{2}$-conjecture.

5 Schur positivity of $\mathrm{I}_{\lambda,\mu}[X;q,t]$

The first Schur positivity result is the Schur positivity of $\mathrm{I}_{\lambda,\mu}[X;q,t]$ when we move a cell in the first or the second row of λ to obtain μ. The proof of the following theorem relies on the theory of LLT polynomials [16]. More precisely, we used Schur positivity of LLT polynomials [9] and LLT equivalences given in [17, 20, 18].

Theorem 5.1. *For a partition ν, let $\mu, \lambda \subseteq \nu$ be two distinct partitions such that $|\nu/\lambda| = |\nu/\mu| = 1$. Moreover, suppose ν/λ is a cell in the first or the second row. Then $\mathrm{I}_{\lambda,\mu}[X;q,t]$ positively expands in LLT polynomials. In particular, $\mathrm{I}_{\lambda,\mu}[X;q,t]$ is Schur positive.*

As a corollary, we prove Butler's conjecture for $q = 1$ or $t = 1$ (for general μ and λ).

Corollary 5.2. *Let λ be a partition and μ be a partition obtained from λ by moving a cell of a partition λ to the upper row. Then $\mathrm{I}_{\lambda,\mu}[X;q,1]$ (or $\mathrm{I}_{\lambda,\mu}[X;1,q]$) is Schur positive.*

6 Combinatorial formulas for (q,t)-Kostka polynomials

The modified (q,t)-Kostka polynomial $\widetilde{K}_{\mu,\nu}(q,t)$ is the Schur coefficient of the modified Macdonald polynomial:

$$\widetilde{H}_\mu[X;q,t] = \sum_\lambda \widetilde{K}_{\mu,\lambda}(q,t) s_\lambda[X].$$

This section provides combinatorial formulas for (q,t)-Kostka polynomials when μ is a two-column partition or a hook.

6.1 Two-column case

For $n \geq 1$, we define a directed weighted graph $G(n)$ on 2-bounded partitions p of size between n and $2n$, where there is a directed edge from p to q if and only if $p \Rightarrow q$ is a strong marked cover. The weight of each edge is the spin statistic for each corresponding strong marked cover. Then one may think of vertical strong marked tableaux of (2^n) as paths of the (hexagonal) graph, defined on 2-bounded partitions with edges corresponding to strong marked cover, from top to bottom. For a path P, we define the *end partition* $\mathrm{End}(P)$ of P as the 2-bounded partition corresponding to the end point of P and *weight* $\mathrm{wt}(P)$ of the path P as the sum of weights of the edges in P. We now give a combinatorial formula for 2$-$Schur expansion for the modified Macdonald polynomials of two-column partitions. The formula is motivated by the *vertical dual Pieri rule*, and the *shift invariance* of k-Schur functions studied by Blasiak, Morse, Pun, and Summers [5]. While working on this paper, we found that our formula is equivalent to Zabrocki's formula [21].

Theorem 6.1. *Let n and m be positive integers with $n \geq 2m$. Then we have*

$$\omega\left(\widetilde{H}_{(2^m,1^{n-2m})}[X;q,t]\right) = \sum_P \mathrm{wt}(P) \prod_{i \in L(P)} \frac{q}{t^i} s^{(2)}_{\mathrm{End}(P)}, \tag{6.1}$$

where ω is the conjugate map on the symmetric function ring given by $h_n \to e_n$ for $n \geq 1$ and $s^{(2)}_\mu[X]$ is the 2-Schur function. In particular, this formula is consistent with Butler's conjecture.

6.2 Hook case

In [3], Assaf defined the bijections $\mathcal{A}_k$ for $k \geq 1$. Using these bijections, she gave a combinatorial formula for the Schur expansion of the modified Macdonald polynomials indexed by hook partitions. This formula is indeed consistent with Butler's conjecture. By showing this, we provide the following formula, which only involves the 'ordinary' major statistic.

Theorem 6.2. *For $\mu = (n-k, 1^k)$ a hook partition, we have*

$$\widetilde{H}_\mu[X;q,t] = \sum_{\lambda \vdash n} \sum_{w \in \mathrm{SS}(\lambda)} \left(\prod_{\substack{1 \leq i \leq k-1, \\ i \in \mathrm{Des}(w)}} t^{\mathrm{maj}(w|_{[k]})} \prod_{\substack{k \leq i \leq n-1, \\ i \in \mathrm{Des}(w)}} \frac{q^{n-i}}{t^i} \right) s_\lambda[X],$$

where $\mathrm{SS}(\lambda)$ is the set of superstandard tableaux of shape λ. In particular, this formula is consistent with Butler's conjecture.

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References

[1] P. Alexandersson and M. Sawhney. "A major-index preserving map on fillings". *The Electronic Journal of Combinatorics* **24**.4 (2017), P4–3.

[2] S. Armon. "A proof of the $\frac{n!}{2}$ conjecture for hook shapes". *Annals of Combinatorics* (2022), pp. 1–14.

[3] S. Assaf. "Toward the Schur expansion of Macdonald polynomials". *The Electronic Journal of Combinatorics* **25**.2 (2018), pp. 2–44.

[4] F. Bergeron and A. M. Garsia. "Science Fiction and Macdonald's polynomials". *CRM Proceedings & Lecture Notes, American Mathematical Society* **22** (1999), pp. 1–52.

[5] J. Blasiak, J. Morse, A. Pun, and D. Summers. "Catalan functions and k-Schur positivity". *Journal of the American Mathematical Society* **32**.4 (2019), pp. 921–963.

[6] L. Butler. "Subgroup lattices and symmetric functions". *Memoirs of the American Mathematical Society* (1994).

[7] E. Carlsson and A. Mellit. "A proof of the shuffle conjecture". *Journal of the American Mathematical Society* **31**.3 (2018), pp. 661–697.

[8] A. M. Garsia and M. Haiman. "A graded representation model for Macdonald's polynomials." *Proceedings of the National Academy of Sciences* **90**.8 (1993), pp. 3607–3610.

[9] I. Grojnowski and M. Haiman. "Affine Hecke algebras and positivity of LLT and Macdonald polynomials". *preprint* (2007).

[10] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. "A combinatorial formula for the character of the diagonal coinvariants". *Duke Mathematical Journal* **126**.2 (2005), pp. 195–232.

[11] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for nonsymmetric Macdonald polynomials". *American Journal of Mathematics* **130**.2 (2008), pp. 359–383.

[12] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for Macdonald polynomials". *Journal of the American Mathematical Society* **18**.3 (2005), pp. 735–761.

[13] M. Haiman. "Hilbert schemes, polygraphs and the Macdonald positivity conjecture". *Journal of the American Mathematical Society* **14**.4 (2001), pp. 941–1006.

[14] M. Haiman. "Vanishing theorems and character formulas for the Hilbert scheme of points in the plane". *Inventiones mathematicae* **149** (2002), pp. 371–407.

[15] D. Kim, S. J. Lee, and J. Oh. "Shuffle formula in Science Fiction for Macdonald polynomials". in preparation.

[16] A. Lascoux, B. Leclerc, and J.-Y. Thibon. "Ribbon tableaux, Hall–Littlewood functions, quantum affine algebras, and unipotent varieties". *Journal of Mathematical Physics* **38**.2 (1997), pp. 1041–1068.

[17] S. J. Lee. "Linear relations on LLT polynomials and their k-Schur positivity for $k = 2$". *Journal of Algebraic Combinatorics* **53**.4 (2021), pp. 973–990.

[18] S. J. Lee, J. Oh, and B. Rhoades. "Haglund's conjecture for multi-t Macdonald polynomials". *Discrete Mathematics* **346**.6 (2023), p. 113360.

[19] I. G. Macdonald. "A new class of symmetric functions." *Séminaire Lotharingien de Combinatoire* **20** (1988), B20a–41.

[20] C. R. Miller. "On the k-Schur Positivity of k-Bandwidth LLT Polynomials". PhD thesis. UC Berkeley, 2019.

[21] M. Zabrocki. "A Macdonald vertex operator and standard tableaux statistics". *The Electronic Journal of Combinatorics* (1998), R45.