Séminaire Lotharingien de Combinatoire **89B** (2023)
Article #84, 12 pp.

Proceedings of the 35th Conference on Formal Power Series and Algebraic Combinatorics (Davis)

Semistandard Parking Functions and a Finite Shuffle Theorem

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Abstract. We introduce the higher rank rational (q, t)-Catalan polynomials and prove these are equal to finite truncations of the Hikita polynomial. We also generalize results of Gorsky-Mazin-Vazirani and construct an explicit bijection between semistandard parking functions and affine compositions. Using these results we prove a finite analogue of the Rational Shuffle Theorem in the context of spherical double affine Hecke algebras.

Keywords: Catalan polynomial, Shuffle Theorem, elliptic Hall algebra, Hikita polynomial, DAHA, spherical DAHA.

1 Introduction

The *Catalan numbers* C_n are some of the most ubiquitous quantities throughout mathematics, naturally counting objects across a vast array of fields. Their many generalizations, ranging from the so-called *rational Catalan numbers* $C_{(m,n)}$ to their bivariate (q, t) counterpart $C_{(m,n)}(q, t)$, have played a central role in the deep connections between (q, t)-combinatorics and important problems arising in the K-theory of Hilbert schemes, the homology of torus knots, the geometry of Gieseker varieties, and other areas of algebraic geometry, topology, and representation theory.

A particularly important family of objects counted by Catalan numbers is the set of *Dyck paths*, lattice paths in a square that do not cross the diagonal. Particular labelings of these paths give rise to *parking functions*. Originally introduced by Konheim and Weiss [12] in their study of hashing problems, parking functions were afterwards generalized by Armstrong, Loehr and Warrington [2] to the rational setting. A new perspective on parking functions, inspired by Anderson [1] and studied extensively by Gorsky-Mazin-Vazirani [7], gives an explicit bijection between the set of parking functions $\mathsf{PF}(m, n)$ and a certain set of affine permutations. More recently, motivated by the

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representation theory of quantizations of the Gieseker moduli space [5] Simental, jointly with Etingof-Krylov-Losev, generalized these constructions further by introducing the *higher rank rational Catalan numbers* $C^{(r)}_{(m,n)}$ and *higher rank semistandard parking functions* $\mathsf{SSPF}^r(m,n)$. As in the classical and rational cases, $|\mathsf{SSPF}^r(m,n)| = C^{(r)}_{(m,n)}$.

We generalize this further by extending the bijection given in [7] to a bijection between $\mathsf{SSPF}^r(m,n)$ and affine compositions, which through an explicit standardization procedure, allows us to define a "`dinv`" statistic on these new sets of objects. We use this statistic to introduce the *higher rank rational* (q,t)*-Catalan polynomial* $C^{(r)}_{(m,n)}(X;q,t)$. In [6] we further show how the `dinv` statistic corresponds precisely to the dimension of an associated affine space in an affine paving of a parabolic Springer fiber.

Tying much of this theory together are the celebrated *Shuffle Theorems*. In the classical setting, the Shuffle Theorem gives a combinatorial formula for the bigraded Frobenius character of the space of diagonal harmonics. While the Frobenius character of this representation was computed by Haiman geometrically in the early 2000's [10] and its combinatorial expression, given as a sum over parking functions, was conjectured by Haglund-Haiman-Loehr-Remmel-Ulyanov [9] shortly thereafter, the conjecture was open for 15 years until Carlsson and Mellit [4] proved it. Unexpectedly arising in their studies of knot invariants via Cherednik algebras, Gorsky and Neguţ proposed a rational generalization of the Shuffle Conjecture. In particular, they conjectured that a certain *elliptic Hall algebra* element acting on 1 inside the ring of symmetric functions gave rise to the Frobenius character of a certain bigraded S_n representation. The combinatorial formula for this character, eponymously named the *Hikita polynomial*, had previously been computed by Hikita [11] as a certain sum over $\mathsf{PF}(m,n)$. Using similar methods as in the classical case, Mellit [13] successfully proved this Rational Shuffle Theorem.

We connect our results to the Shuffle Theorems by first proving that our higher rank rational (q,t)-Catalan polynomial corresponds precisely to the truncation of the Hikita polynomial to a finite number of variables. Then, using the fact that the elliptic Hall algebra arises as the inverse limit of the spherical *double affine Hecke algebra* (DAHA) $\mathbb{SH}(r)^{++}$, we show there is an action of $\mathbb{SH}(r)^{++}$ on the ring of symmetric polynomials whose action on 1 results in these higher rank rational (q,t)-Catalan polynomials. Since in the $r \to \infty$ limit this agrees with the rational generalization of Gorsky-Neguţ, our construction gives a finite analogue of the Rational Shuffle Theorem.

2 Background

Let $m,n \in \mathbb{Z}_{>0}$ satisfy $\gcd(m,n) = 1$. An (m,n)-***Dyck path*** is a lattice path consisting of only north and west steps from $(n,0)$ to $(0,m)$ bounded by the lines $x = 0$, $y = 0$, and $mx + ny = mn$. Denote by $\mathrm{D}(m,n)$ the set of all (m,n)-Dyck paths and let $\mathcal{N}(\mathrm{D})$ be the set of *north* steps for any $\mathsf{D} \in \mathrm{D}(m,n)$.

Let $v_1, v_2, \ldots, v_m$ and $u_1, u_2, \ldots, u_n$ be the north and west steps of $\mathsf{D} \in \mathrm{D}(\mathrm{m}, \mathrm{n})$ read from $(n,0)$ to $(0,m)$. We denote by $\mathtt{dinv}(\mathsf{D})$ the number of ***diagonal inversions*** of D, which consist of pairs (u_r, v_s) over all $1 \le r \le n$ and $1 \le s \le m$, such that both u_r and v_s are intersected by some line ℓ of slope $\frac{-m}{n}$ and v_s lies southeast of u_r in D. The ***area*** of a Dyck path $\mathsf{D} \in \mathrm{D}(\mathrm{m}, \mathrm{n})$ is the number of lattice cells fully contained between the path D and the diagonal line $mx + ny = mn$. We set $\mathtt{co\text{-}dinv}(\mathsf{D}) = \frac{(m-1)(n-1)}{2} - \mathtt{area}(\mathsf{D})$.

Example 1. The Dyck path D in Figure 2 has $\mathtt{area}(\mathsf{D}) = 2$, $\mathtt{dinv}(\mathsf{D}) = 4$ and thus $\mathtt{co\text{-}area}(\mathsf{D}) = \frac{(m-1)(n-1)}{2} - 2 = 4$, $\mathtt{co\text{-}dinv}(\mathsf{D}) = \frac{(m-1)(n-1)}{2} - 4 = 2$.

It is a well known fact that the number of (m,n)-Dyck paths for coprime m, n is given by the *rational Catalan numbers*, $C_{(m,n)} := \frac{1}{n}\binom{n+m-1}{m}$. The (q,t)-*Catalan number* is the bivariate deformation given by the polynomial

$$C_{(m,n)}(q,t) := \sum_{\mathsf{D} \in \mathrm{D}(m,n)} q^{\mathtt{area}(\mathsf{D})} t^{\mathtt{dinv}(\mathsf{D})}. \tag{2.1}$$

2.1 Parking Functions and the Hikita Polynomial

An (m,n)***-parking function*** is a pair (D, φ), where D is an (m,n)-Dyck path and $\varphi : \mathcal{N}(\mathsf{D}) \to \{1, \ldots, m\}$ is a bijection that is strictly decreasing when reading upward on consecutive north steps. We will denote by $\mathsf{PF}(m,n)$ the set of (m,n)-parking functions.

For example, in Figure 1, the three rightmost diagrams correspond to all possible $(3,2)$-parking functions for the given Dyck path.

Parking functions are in bijection with a certain set of affine permutations. To explain this bijection we recall the ***affine symmetric group***,

$$\widetilde{S}_m := \left\{ \sigma : \mathbb{Z} \to \mathbb{Z} \mid \sigma \text{ is a bijection, } \sigma(x+m) = \sigma(x) + m, \text{ and } \sum_{i=1}^{m} \sigma(i) = \binom{m+1}{2} \right\}.$$

By the periodicity condition $\sigma \in \widetilde{S}_m$ is completely determined by $\sigma(1), \ldots, \sigma(m)$. We will use window notation and write $\sigma = [\sigma(1), \ldots, \sigma(m)]$. We say $\sigma \in \widetilde{S}_m$ is ***n-stable*** if $\sigma(x+n) > \sigma(x)$ for all $x \in \mathbb{Z}$ and denote by $\widetilde{S}_m^n$ the set of all n-stable affine permutations.

In [7], the third author jointly with Gorsky and Mazin study a bijection (see Section 3.1)

$$\mathcal{A} : \mathsf{PF}(m,n) \to \widetilde{S}_m^n, \tag{2.2}$$

Using this, for $\sigma = \mathcal{A}(\mathsf{D}, \varphi)$, we define

$$\mathtt{co\text{-}dinv}(\mathsf{D}, \varphi) := |\{(i,h) \in \{1, \ldots, m\} \times \{1, \ldots, n\} \mid \sigma(i+h) < \sigma(i)\}|. \tag{2.3}$$

and set $\mathtt{dinv}(\mathsf{D}, \varphi) := \frac{(m-1)(n-1)}{2} - \mathtt{co\text{-}dinv}(\mathsf{D}, \varphi)$.

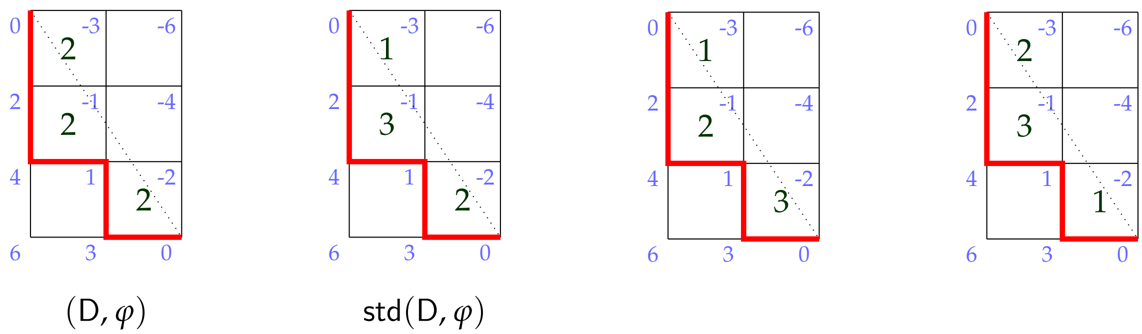


Figure 1: A $(3,2)$-semistandard parking function (D,φ) and all $(3,2)$-parking functions for D, of which the second from the left is the unique standardization $\mathsf{std}(\mathsf{D},\varphi)$ determined by γ. A box with upper right corner (x,y) is marked by $\gamma(x,y)$.

Gessel defined the *fundamental quasisymmetric functions* for any subset $S \in \{1,\ldots,m\}$ as the series

$$Q_S(X) := \sum_{\substack{i_1 \leq \cdots \leq i_m \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} \ldots x_{i_m}. \tag{2.4}$$

These functions interpolate between elementary and complete symmetric functions and form a basis for the space of quasisymmetric functions in an infinite number of variables.

Let $\mathtt{area}(\mathsf{D},\varphi) := \mathtt{area}(\mathsf{D})$ for any $(\mathsf{D},\varphi) \in \mathsf{PF}(m,n)$. Then, the ***Hikita polynomial*** is the q,t-symmetric function given by

$$\mathcal{H}_{(m,n)}(X;q,t) := \sum_{(\mathsf{D},\varphi)\in\mathsf{PF}(m,n)} q^{\mathtt{area}(\mathsf{D},\varphi)} t^{\mathtt{dinv}(\mathsf{D},\varphi)} Q_{\mathrm{Des}(\sigma^{-1})}(X), \tag{2.5}$$

where $\mathrm{Des}(\sigma^{-1}) := \{1 \leq j \leq m \mid \sigma^{-1}(j+1) < \sigma^{-1}(j)\}$ denotes the *descent set* of σ^{-1} for $\sigma = \mathcal{A}(\mathsf{D},\varphi) \in \widetilde{S}^n_m$. When paired with the complete symmetric function $h_m(X)$ under the Hall inner product, the Hikita polynomial returns the rational (q,t)-Catalan number, that is $\langle \mathcal{H}_{(m,n)}, h_m \rangle = C_{(m,n)}(q,t)$.

2.2 Higher Rank Catalans and Semistandard Parking Functions

Motivated by the representation theory of quantizations of the Gieseker moduli space, in [5] Etingof, Krylov, Losev and the second author introduced the following generalization of parking functions.

Definition 2. Let $r,m,n \in \mathbb{Z}_{>0}$. A ***rank*** r ***semistandard*** (m,n)***-parking function*** is a pair (D,φ) with $\mathsf{D} \in \mathsf{D}(m,n)$ and a function $\varphi : \mathcal{N}(\mathsf{D}) \to \{1,\ldots,r\}$ that is weakly decreasing when reading upward on consecutive north steps. Let $\mathsf{SSPF}^r(m,n)$ be the set of all rank r semistandard (m,n)-parking functions.

In the special case when $r = m$, and φ is a bijection, then the filling will automatically become standard, recovering the original set of (m,n)-*parking functions* $\mathsf{PF}(m,n)$ considered in [2].

Given any $(\mathsf{D},\varphi) \in \mathsf{SSPF}^r(m,n)$, we let $\texttt{area}(\mathsf{D},\varphi) := \texttt{area}(\mathsf{D})$, define the ***weight*** of (D,φ) as the composition of m given by $\texttt{wt}(\mathsf{D},\varphi) := (|\varphi^{-1}(1)|, |\varphi^{-1}(2)|, \ldots, |\varphi^{-1}(r)|)$. Let $\mathsf{SSPF}^r_{\mathbf{w}}(m,n)$ denote the set of all rank r semistandard (m,n)-parking functions with weight $\mathbf{w}$. Note that for any $(\mathsf{D},\varphi) \in \mathsf{PF}(m,n)$, we have $\texttt{wt}(\mathsf{D},\varphi) = (1^m)$.

Example 3. Consider (D,φ), the semistandard $(5,4)$-parking function in Figure 2 (left). Then (D,φ) has rank $r = 3$, $\texttt{area}(\mathsf{D},\varphi) = 2$, and $\texttt{wt}(\mathsf{D},\varphi) = (1,3,1)$. By comparison, the $(5,4)$-parking function in Figure 2 (right) has the same area but rank 5 and weight (1^5).

In [5, Theorem 2.28], Etingof, Krylov, Losev and the second author defined the ***rank*** r (m,n)***-Catalan number*** $C^{(r)}_{(m,n)}$ to be the cardinality of $\mathsf{SSPF}^r(m,n)$ and proved that when m and n are coprime,

$$C^{(r)}_{(m,n)} := |\,\mathsf{SSPF}^r(m,n)| = \frac{1}{n}\binom{nr+m-1}{m}. \tag{2.6}$$

3 Higher Rank Rational (q,t)-Catalan Polynomial

As mentioned above, there exists a map $\mathcal{A} : \mathsf{PF}(m,n) \to \widetilde{S}^n_m$. We will recall this bijection and construct a generalization for $\mathsf{SSPF}\,(m,n)$. Denote by $[r]$ the set $\{1,\ldots,r\} \subset \mathbb{Z}$.

Definition 4. Let $m,r \in \mathbb{Z}_{>0}$. An (m,r)***-affine composition*** is a function $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the following properties:

(1) $f(x+m) = f(x) + r$ for all $x \in \mathbb{Z}$.

(2) The set $f^{-1}[r]$ has exactly one element from each residue class mod m.

(3) $\sum_{x\in f^{-1}[r]} x = \binom{m+1}{2}$.

Denote by $\mathcal{AC}(m,r)$ the set of all (m,r)-affine compositions. See Examples 5 and 7.

Just as for affine permutations, by the periodicity of condition (1) above, we use window notation and write $f = [f(1),\ldots,f(m)]_r$. The ***weight*** of f is given by

$$\texttt{wt}(f) = (|f^{-1}(1)|, |f^{-1}(2)|, \ldots, |f^{-1}(r)|).$$

Note that an affine composition of weight (1^m) is simply an affine permutation. As before, $f \in \mathcal{AC}(m,r)$ is ***n-stable*** if $f(x+n) \geq f(x)$ for all $x \in \mathbb{Z}$. Then, let $\mathcal{AC}_{\mathbf{w}}(m,r)$ be the set of all (m,r)-affine compositions with weight $\mathbf{w}$ and denote by $\mathcal{AC}^n_{\mathbf{w}}(m,r)$ the subset of $\mathcal{AC}_{\mathbf{w}}(m,r)$ consisting of those affine compositions that are n-stable.

Example 5. Let $m, r \in \mathbb{Z}_{>0}$ and consider the (m,r)-affine composition $f \in \mathcal{AC}(m,r)$ that in window notation is $f = [1, 1, \ldots, 1]_r$. Note that $f^{-1}[r] = [m]$ so that f is indeed an affine composition of weight $(m, 0, \ldots, 0) \in \mathbb{Z}^r_{\geq 0}$. We leave it to the reader to verify that f is the function $f(x) = r\lfloor \frac{x-1}{m} \rfloor + 1$.

3.1 Semistandard Parking Functions are Affine Compositions

We now describe $\mathcal{A} : \mathsf{SSPF}^r(m,n) \to \mathcal{AC}^n(m,r)$ which is a generalization of the map $\mathcal{A} : \mathsf{PF}(m,n) \to \widetilde{S}^n_m$.

Given any $(\mathsf{D}, \varphi) \in \mathsf{SSPF}^r(m,n)$, index the north steps $v_1, \ldots, v_m \in \mathcal{N}(\mathsf{D})$ by their topmost coordinate, so that $v_j = (a_j, j)$ for each $1 \leq j \leq m$ and in particular $a_m = 0$. Define the function $f_\varphi : \mathbb{Z} \to \mathbb{Z}$ as follows:

Step 1: Define $\gamma : \mathbb{Z}^2 \to \mathbb{Z}$ by $\gamma(x,y) = mn - mx - ny$.

Step 2: Set $\tilde{f}(\gamma(v_j)) = \varphi(v_j)$ for $j \in [m]$, so that $\tilde{f}^{-1}(i) = \gamma(\varphi^{-1}(i))$ for $i \in [r]$.

Step 3: For any $x \notin \{\gamma(v_j)\}_{1 \leq j \leq m}$, write $x = \gamma(v_j) + pm$ for some $p \in \mathbb{Z}$ and $j \in [m]$ and set $\tilde{f}(x) := \tilde{f}(\gamma(v_j)) + pr$.

Step 4: Let $k = \frac{n(m-1)-(m+1)}{2} - \sum_{j=1}^m a_j$, and set $f_\varphi(x) := \tilde{f}(x+k)$ for all $x \in \mathbb{Z}$.

Then we set $\mathcal{A}(\mathsf{D}, \varphi) = f_\varphi$. Note that since m, n are coprime then the set $\{\gamma(v_j)\}_{1 \leq j \leq m}$ contains exactly one element per residue class mod m. Thus $\tilde{f}$ is well-defined.

Theorem 6. *Let $r, m, n \in \mathbb{Z}_{>0}$ with $\gcd(m,n) = 1$. The map $\mathcal{A} : \mathsf{SSPF}^r(m,n) \to \mathcal{AC}^n(m,r)$ sending $(\mathsf{D}, \varphi) \mapsto f_\varphi$ is a weight-preserving bijection. Hence, given any r-part weak composition $\mathbf{w}$ of m, the construction above gives an isomorphism of sets,*

$$\mathcal{A}_\mathbf{w} : \mathsf{SSPF}^r_\mathbf{w}(m,n) \to \mathcal{AC}^n_\mathbf{w}(m,r).$$

In the special case when $r = m$ and $\mathbf{w} = (1^m)$, so that $\mathcal{AC}^n_{(1^m)}(m,m) = \widetilde{S}^n_m$, Theorem 6 recovers the map $\mathcal{A} : \mathsf{PF}(m,n) \to \widetilde{S}^n_m$ constructed in [7]. For example, in Figure 4 we list the affine permutations for all the $(3,2)$-parking functions.

Example 7. Set $m = 5, n = 4$ and $r = 3$. Let (D, φ) be the rank 3 semistandard $(5,4)$-parking function in Figure 2 (left). Note that $\mathtt{wt}(D, \varphi) = (1,3,1)$. From the figure we can read that

$$\gamma(\varphi^{-1}(1)) = \{0\}, \qquad \gamma(\varphi^{-1}(2)) = \{3,4,7\}, \qquad \gamma(\varphi^{-1}(3)) = \{6\}.$$

Thus, setting $\tilde{f}(\gamma(v_j) + 5p) := \tilde{f}(\gamma(v_j)) + 3p$ we find that $\tilde{f}$ is the following map:

⋯	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10	⋯
⋯	−3	−4	−1	−1	**1**	0	−1	**2**	**2**	4	**3**	**2**	5	5	7	⋯

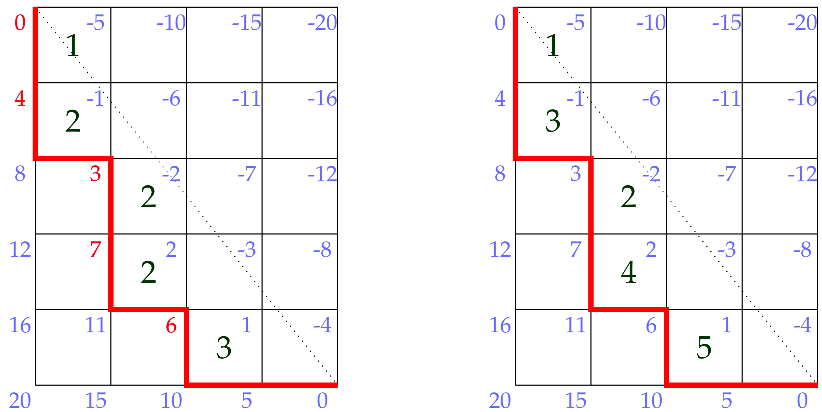


Figure 2: A semistandard $(5,4)$-parking function with rank $r=3$ (left), and its standardization (right). A box with upper right corner (x,y) is marked by $\gamma(x,y)$.

Now since the x-coordinates of the north steps $\{v_j=(a_j,j)\}_{1\le j\le 7}$ are given by $a_1=2$, $a_2=a_3=1$, and $a_4=a_5=0$, it follows that $k=1$. Thus, setting $f_\varphi(x)=\tilde{f}(x+1)$ we find that $f_\varphi=[-1,2,2,4,3]_3$, thus $\texttt{wt}(f_\varphi)=\texttt{wt}(\tilde{f})=(1,3,1)$, as expected. Notice that by construction $f_\varphi^{-1}[3]=\{\gamma(v_j)-1\}=\{-1,2,3,5,6\}$ which, modulo 5, is equivalent to the set $\{1,\dots,5\}$. Summing the entries in this set we get $\binom{6}{2}$, hence f_φ is indeed a $(5,3)$-affine composition, which by inspection can be seen to be 4-stable.

3.2 Standardization and Diagonal Inversions

The bijection in Theorem 6 allows us to compute $\texttt{co-dinv}$ via Equation (2.3), provided that $(\mathsf{D},\varphi)\in\mathsf{PF}(m,n)$. Thus, in order to define the diagonal inversions for any semistandard parking function we utilize a standardization map from $\mathsf{SSPF}(m,n)\to\mathsf{PF}(m,n)$.

For $\mathbf{w}=(w_1,\dots,w_r)$, an r-part weak composition of m, let $S_\mathbf{w}=S_{w_1}\times\cdots\times S_{w_r}$ be the parabolic subgroup of $\widetilde{S}_m$ indexed by $\mathbf{w}$. Denote by $(S_\mathbf{w}\backslash\widetilde{S}_m)^{\min}$ the set of minimal length right coset *representatives* and let $(S_\mathbf{w}\backslash\widetilde{S}^n_m)^{\min}=(S_\mathbf{w}\backslash\widetilde{S}_m)^{\min}\cap\widetilde{S}^n_m$.

Proposition 8. *Let $m,r\in\mathbb{Z}_{>0}$. Given $\mathbf{w}=(w_1,\dots,w_r)$, an r-part weak composition of m, let ${}_\mathbf{w}f$ be the affine composition of weight $\mathbf{w}$ given by ${}_\mathbf{w}f:=[\underbrace{1,\dots,1}_{w_1},\underbrace{2\dots,2}_{w_2},\dots,\underbrace{r,\dots,r}_{w_r}]_r$. Then the map*

$$\mathcal{S}_\mathbf{w}:(S_\mathbf{w}\backslash\widetilde{S}_m)^{\min}\to\mathcal{AC}_\mathbf{w}(m,r)$$

sending $\sigma\mapsto{}_\mathbf{w}f\circ\sigma$ is a bijection. Moreover, $\mathcal{S}_\mathbf{w}$ preserves n-stability and thus restricts to a bijection $\mathcal{S}^n_\mathbf{w}:(S_\mathbf{w}\backslash\widetilde{S}^n_m)^{\min}\to\mathcal{AC}^n_\mathbf{w}(m,r)$ for any $n\in\mathbb{Z}_{>0}$ coprime to m.

Definition 9. For any $(\mathsf{D},\varphi)\in\mathsf{SSPF}^r_\mathbf{w}(m,n)$ the ***standardization*** of (D,φ) is the parking function $\mathsf{std}(\mathsf{D},\varphi) \in \mathsf{PF}(m,n)$ given by,

$$\mathsf{std}(\mathsf{D},\varphi) := \mathcal{A}^{-1}\mathcal{S}_{\mathbf{w}}^{-1}\mathcal{A}_{\mathbf{w}}(\mathsf{D},\varphi).$$

Theorem 10. *For any $(\mathsf{D},\varphi) \in \mathsf{SSPF}^r_{\mathbf{w}}(m,n)$, its standardization $\mathsf{std}(\mathsf{D},\varphi)$ is the (m,n)-parking function (D,α) with $\alpha : \mathcal{N}(\mathsf{D}) \to \{1,\dots,m\}$ the unique bijection satisfying:*

(1) $\alpha(\varphi^{-1}(a)) < \alpha(\varphi^{-1}(b))$ for all $1 \le a < b \le r$, and

(2) $\alpha(v_i) < \alpha(v_j)$ whenever $\varphi(v_i) = \varphi(v_j)$ and $\gamma(v_i) < \gamma(v_j)$.

As seen in Figure 1 conditions (1) and (2) ensure a unique standard representative of a semistandard parking function (D,φ).

Hence, for any rank r semistandard (m,n)-parking function (D,φ) we set

$$\mathtt{dinv}(\mathsf{D},\varphi) := \mathtt{dinv}(\mathsf{std}(\mathsf{D},\varphi)). \tag{3.1}$$

Example 11. Consider the rank $r = 3$ semistandard $(5,4)$-parking function (D,φ) in Figure 2 (left) with preimages $\varphi^{-1}(1) = \{v_5\}$, $\varphi^{-1}(2) = \{v_2,v_3,v_4\}$, and $\varphi^{-1}(3) = \{v_1\}$. Since $\gamma(v_2) = 7$, $\gamma(v_3) = 3$, and $\gamma(v_4) = 4$, then $\alpha(v_5) < \alpha(v_3) < \alpha(v_4) < \alpha(v_2) < \alpha(v_1)$. Hence α is the unique map from $\mathcal{N}(\mathsf{D}) \to \{1,\dots,5\}$ given in Figure 2 (right).

Example 12. Consider the rank 2 semistandard $(3,2)$-parking function in Figure 1 (left). We have that $\mathcal{A}(\mathsf{std}(\mathsf{D},\varphi)) = [1,2,3]$, the identity permutation which does not have inversions and thus has codinv 0. Then $\mathtt{dinv}(D,\varphi) = \frac{(2-1)(3-1)}{2} - 0 = 1$. By comparison, the affine permutations corresponding to the two rightmost parking functions in Figure 1 are $[1,3,2]$ and $[2,1,3]$ both of which have inversions.

While we do not expand upon this here, it is important to note that the affine spaces in an affine paving of a parabolic affine Springer fiber are indexed by the set $\mathsf{SSPF}^r(m,n)$ and the co-dinv statistic is the dimension of the corresponding affine space. Each parabolic affine Springer fiber admits a surjection from the affine Springer fiber in the full affine flag variety, and the latter variety has an affine paving indexed by parking functions, see [7]. The standardization procedure is related to properties of this surjection, see [6] for details.

Definition 13. Let $m,n,r \in \mathbb{Z}_{>0}$ with $\gcd(m,n) = 1$. The ***rank r rational (q,t)-Catalan polynomial*** is the polynomial in $\mathbb{Q}[q,t][x_1,\dots,x_r]$ given by

$$C^{(r)}_{(m,n)}(x_1,\dots,x_r;q,t) := \sum_{(\mathsf{D},\varphi)\in\mathsf{SSPF}^r(m,n)} q^{\mathtt{area}(\mathsf{D})} t^{\mathtt{dinv}(\mathsf{D},\varphi)} x^{\mathtt{wt}(\varphi)}.$$

Example 14. It can be shown that $C^{(1)}_{(m,n)}(x_1;q,t) = x_1^m C_{(m,n)}(q,t)$, where $C_{(m,n)}(q,t)$ is the usual (q,t)-Catalan polynomial. In this sense, Definition 13 generalizes that of (q,t)-Catalan polynomials. As seen in Figure 3, for $r = 2$, $m = 3$, $n = 2$ we have

$$C^{(2)}_{(3,2)} = (x_1^3 + x_2^3)(q+t) + (x_1^2x_2 + x_1x_2^2)(q+1+t).$$

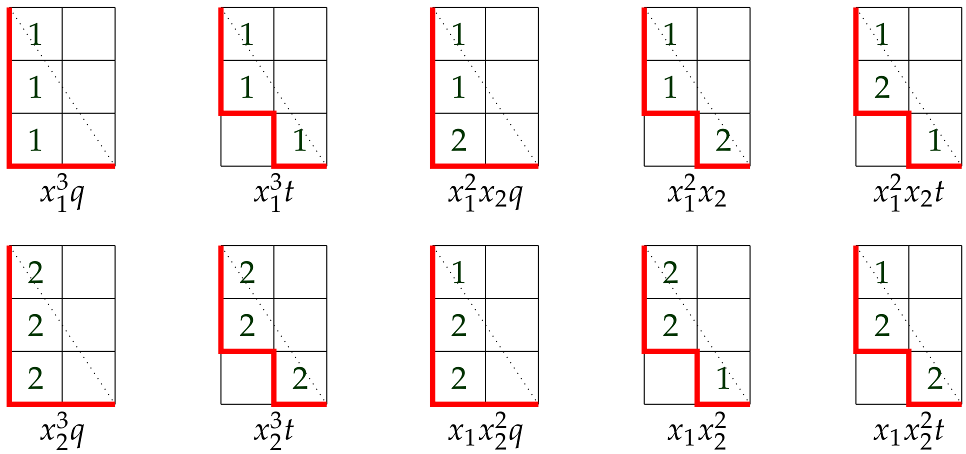


Figure 3: All the semistandard $(3,2)$-parking functions of rank 2 and their contribution to $\mathsf{C}^{(2)}_{(3,2)}(x_1,x_2;q,t)$.

Using the standardization procedure above it can be shown that the restriction to r variables of Gessel's quasisymmetric function $Q_{\mathrm{Des}(\sigma^{-1})}(X)|_{x_1,\ldots,x_r}$ is naturally indexed by the rank r semistandard parking functions (D,φ) that standardize to (D,α), where $\mathcal{A}(\mathsf{D},\alpha)=\sigma$ and $\mathrm{Des}(\sigma^{-1})$ is the descent set of σ^{-1}. Namely,

$$Q_{\mathrm{Des}(\sigma^{-1})}(X)|_{x_1,\ldots,x_r} = \sum_{(\mathsf{D},\varphi)\in\mathsf{std}^{-1}(\mathsf{D},\alpha)} x^{\mathtt{wt}(\mathsf{D},\varphi)}. \tag{3.2}$$

Combining (2.5) and (3.2) we obtain the following theorem.

Theorem 15. *Given any $m,n,r\in\mathbb{Z}_{>0}$ with $\gcd(m,n)=1$, the truncation of the Hikita polynomial to r variables is given by the rank-r rational (q,t)-Catalan polynomial,*

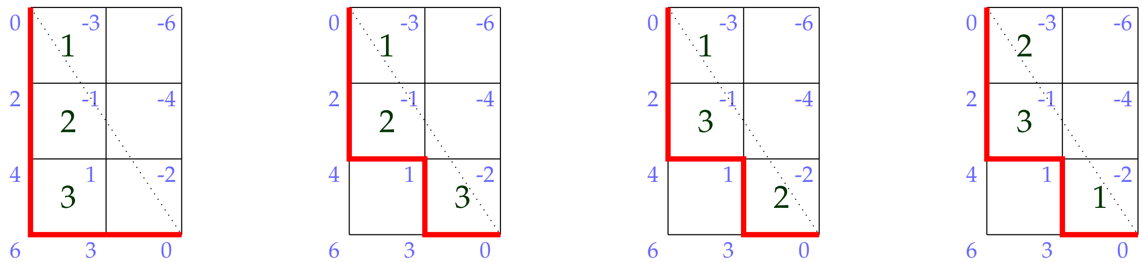
$$\mathcal{H}_{(m,n)}(X;q,t)|_{x_1,\ldots,x_r} = \mathsf{C}^{(r)}_{(m,n)}(x_1,\ldots,x_r;q,t).$$

Thus, $\mathsf{C}^{(r)}_{(m,n)}(x_1,\ldots,x_r;q,t)$ is a (q,t)-symmetric, Schur positive polynomial, and hence also symmetric in $x_1,\ldots,x_r$.

Example 16. Let us compute the Hikita polynomial $\mathcal{H}_{(3,2)}(X;q,t)$. There are four $(3,2)$-parking functions (D,φ), which are listed together with their associated affine permutations $\mathcal{A}(\mathsf{D},\varphi)$ in Figure 4. From here, we compute the Schur expansion:

$$\mathcal{H}_{(3,2)}(X;q,t) = qQ_{\{3\}} + Q_{\{2\}} + tQ_{\emptyset} + Q_{\{1\}} = (q+t)s_{(3)} + s_{(2,1)}.$$

It is straightforward to verify that the truncation to 2-variables of this polynomial coincides with the computation of $\mathsf{C}^{(2)}_{(3,2)}(x_1,x_2;q,t)$ in Example 14.



[0, 2, 4] [1, 3, 2] [1, 2, 3] [2, 1, 3]

Figure 4: The $(3,2)$-parking functions and their associated affine permutations under the bijection $\mathcal{A}$, which can easily be read from $\gamma(x,y)$ for each lattice point (x,y) on D.

4 Spherical DAHA and a Finite Shuffle Theorem

The ***elliptic Hall algebra*** $\mathcal{E}^{++}$ is the $\mathbb{C}(q,t)$-algebra consisting of an infinite family of generators $\{P_{(m,n)} \mid (m,n) \in \mathbb{Z}^2_{\geq 0} \setminus (0,0)\}$ modulo some relations, see e.g. [14]. Although $\mathcal{E}^{++}$ originally arose as the Hall algebra of the category of coherent sheaves on an elliptic curve, it has since had various topological and combinatorial incarnations and played a central role in connecting q,t-combinatorics and the Shuffle Theorems to many recent results in the theory of homological and geometric invariants for knots and links.

There is a well-known geometric action of $\mathcal{E}^{++}$ on the ring of symmetric functions $\Lambda_{q,t}$ whose fixed points correspond to the *modified Macdonald polynomials* $\widetilde{H}_\lambda(X;q,t)$ [14]. Denote this representation by V^{geom} and by $P^{\infty}_{(m,n)}(f)$ the image of $f \in \Lambda_{q,t}$ under this action. The *Rational Shuffle Theorem*, conjectured by Gorsky and Neguţ [8] and proved by Mellit [13], states the following.

Theorem 17 ([13])**.** *For coprime $m,n \in \mathbb{Z}_{>0}$, $P^{\infty}_{(m,n)}(1) = \mathcal{H}_{(m,n)}(X;q,t)$.*

The ***spherical DAHA*** $\mathbb{SH}(r)$ is the spherical subalgebra of the double affine Hecke algebra of type $\mathfrak{gl}(r)$. We consider the subalgebra $\mathbb{SH}(r)^{++}$ of $\mathbb{SH}(r)$ generated by elements $P^{(r)}_{(m,n)}$ with $(m,n) \in \mathbb{Z}^2_{\geq 0} \setminus (0,0)$. Schiffmann-Vasserot show that under the natural map $\mathbb{SH}(r)^{++} \to \mathbb{SH}(r-1)^{++}$ sending $P^{(r)}_{(m,n)} \mapsto P^{(r-1)}_{(m,n)}$, the algebra $\mathcal{E}^{++}$ arises as an inverse limit of the spherical DAHAs. Namely,

$$\mathcal{E}^{++} \cong \varprojlim_r \mathbb{SH}(r)^{++} \tag{4.1}$$

under which

$$P_{(m,n)} = \varprojlim_r P^{(r)}_{(m,n)}.$$

The algebra $\mathbb{SH}(r)^{++}$ comes equipped with a faithful polynomial representation on $\mathbb{Q}(q,t)[x_1 \ldots, x_r]^{S_r}$, denoted $Pol(r)$, that is compatible with (4.1) above and the inverse limit $\Lambda_{q,t} \cong \varprojlim_r \mathbb{Q}(q,t)[x_1 \ldots, x_r]^{S_r}$. Thus, there is an induced polynomial representation V^{alg} of $\mathcal{E}^{++}$ on $\Lambda_{q,t}$. This representation is nontrivially isomorphic to the geometric one of Schiffmann-Vasserot.

Proposition 18 ([14])**.** *There is an isomorphism of $\mathcal{E}^{++}$ representations $\Phi : V^{\text{alg}} \to V^{\text{geom}}$ given by the plethystic substitution that sends $F(X;q,t) \mapsto F\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right]$.*

By combining this proposition with Theorems 15 and 17 we obtain the following finite analogue of the Rational Shuffle Theorem.

Theorem 19. *Given any $m,n,r \in \mathbb{Z}_{>0}$ with $\gcd(m,n) = 1$, there exists an action of $\mathbb{SH}(r)^{++}$ on $\mathbb{Q}(q,t)[x_1 \ldots, x_r]^{S_r}$ satisfying*

$$P^{(r)}_{(m,n)}(1) = C^{(r)}_{(m,n)}(x_1, \ldots, x_r; q, t).$$

In particular, this action is nontrivially isomorphic to the polynomial representation $Pol(r)$.

This action is compatible with the geometric action of $\mathcal{E}^{++}$ on $\Lambda_{q,t}$ under all the inverse limits above, with $P^{\infty}_{(m,n)} = \varprojlim_r P^{(r)}_{(m,n)}$ as operators, so that as $r \to \infty$ it recovers the Rational Shuffle Theorem.

Given the complicated nature of plethystic substitution, finding the explicit action of $\mathbb{SH}(r)^{++}$ on $\mathbb{Q}(q,t)[x_1 \ldots, x_r]^{S_r}$ in Theorem 19 is quite difficult. In particular, even describing the action for $r = 1$ explicitly is unclear since although $\mathbb{SH}(1)^{++}$ is straightforward, the rational Catalan numbers $C_{(m,n)}(q,t)$ are not. Nonetheless, it would be very interesting to find an explicit description of this action.

5 The Non-Coprime Case and a Bizley-Type Formula

Naturally, one asks how much of these constructions can be extended to the non-coprime case. Unfortunately it is neither easy nor obvious how to proceed. A starting point, however, is the computation of $C^{(r)}_{(m,n)} = |\,\mathsf{SSPF}^r(m,n)|$ in the case m,n are not coprime. Below we give a Bizley-type formula for these numbers.

Theorem 20. *Let $m,n,r \in \mathbb{Z}_{>0}$ with $\gcd(m,n) = 1$. There is an equality of formal power series:*

$$1 + \sum_{k=1}^{\infty} C^{(r)}_{(mk,nk)} z^k = \exp\left(\sum_{k=1}^{\infty} \frac{1}{nk}\binom{nkr + mk - 1}{mk} z^k\right).$$

One can apply similar techniques to obtain a Bizley-type formula for the number of parking functions. This is, however, much more elegantly expressed by Aval and Bergeron in [3], who adapt Bizley's proof incorporating symmetric function arguments to find the Frobenius characteristic of the S_{mk}-representation $\mathbb{C}\mathsf{PF}(mk, nk)$. One can adapt the arguments in this paper to show that

$$C^{(r)}_{(mk,nk)} = \mathrm{Frob}(\mathbb{C}\mathsf{PF}(mk, nk))|_{x_1=\cdots=x_{mk}=1, x_{mk+1}=\cdots=0}.$$

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