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A Catalanimal formula for Macdonald polynomials

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Abstract. Catalanimals are rational functions encoding virtual GL_l character series. When truncated to the polynomial characters, they have been shown to recover many important symmetric function quantities with coefficients in $\mathbb{Q}(q,t)$ that arise from the study of diagonal harmonics, including $\nabla^m e_k$ and, more generally, $\nabla^m s_\lambda$. Providing Catalanimal representatives of these quantities gave the necessary tools to show $\nabla^m s_\lambda$ is essentially a positive q,t-weighted sum of distinguished LLT polynomials, thereby resolving the Loehr-Warrington conjecture. Missing from this story was a Catalanimal description of the modified Macdonald polynomials $\tilde{H}_\mu$, which are intimately linked to the ∇ operator. In this abstract, we give a Catalanimal style expression for the modified Macdonald polynomials and provide a positivity conjecture on the entire GL_l character series.

Keywords: Symmetric functions, Macdonald polynomials, Catalanimals

1 Introduction

Macdonald polynomials are a basis of symmetric functions with coefficients in $\mathbb{Q}(q,t)$ exhibiting deep connections to representation theory and algebraic geometry. In particular, specific specializations of the q,t parameters recover various widely-studied bases of symmetric functions, such as Hall-Littlewood polynomials, Jack polynomials, q-Whittaker functions, and Schur functions, among others.

Since their introduction, the Macdonald Positivity Conjecture was central to the study of Macdonald polynomials. Precisely, the Macdonald Positivity Conjecture was that the Schur expansion coefficients of the modified Macdonald polynomials $\tilde{H}_\mu(X;q,t)$ lie in

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$\mathbb{Z}_{\geq 0}[q,t]$. In [8], it was shown that the Macdonald polynomials are doubly graded characters of certain symmetric group modules, known as Garsia-Haiman modules. In [16], the Macdonald Positivity Conjecture was proved by resolving the $n!$-Conjecture for the Garsia-Haiman modules.

Out of this study arose the space of diagonal harmonics, a bigraded S_k-module containing all the Garsia-Haiman modules associated to partitions of k as submodules. It was conjectured in [9] and proven in [17] that the doubly graded character of the diagonal harmonics can be succinctly described as ∇e_k for e_k the kth elementary symmetric function and ∇ an operator on symmetric functions with eigenfunctions given by the Macdonald polynomials. Precisely,

$$\nabla \tilde{H}_\mu = t^{n(\mu)} q^{n(\mu^*)} \tilde{H}_\mu \text{ where } n(\mu) = \sum_i (i-1)\mu_i \,. \tag{1.1}$$

Haglund et. al [13] conjectured a combinatorial identity expressing ∇e_k as a q,t-weighted sum of LLT polynomials, which are q-analogues of skew Schur functions related to Kazhdan-Lusztig polynomials and Fock space representations of $\mathcal{U}_q(\hat{\mathfrak{sl}}_n)$. This identity, now known as the Shuffle Theorem, was proven in [6]. After being conjectured in 2005, this formula was generalized in many directions, including the Rational Shuffle Theorem (conjectured in [1], proven in [21]), the Compositional Shuffle Theorem (conjectured in [14], proven in [6]), the Delta Theorem (conjectured in [15], proven in [7]), and the Loehr-Warrington Conjecture ([19]) for ∇s_μ, where s_μ is the Schur function indexed by any partition μ.

In several previous works, we provide a general class of functions called *Catalanimals*, special cases of which give quantities like $\nabla^m s_\mu$ and expressions in the Rational Shuffle Theorem. Catalanimals are defined with 3 "root ideals" and a weight, denoted $H(R_q, R_t, R_{qt}, \gamma)$. Catalanimals are properly thought of as infinite series of virtual GL_l characters, i.e. linear combinations of terms χ_λ for $\lambda \in \mathbb{Z}^l$ satisfying $\lambda_1 \geq \cdots \geq \lambda_l$ for χ_λ an irreducible GL_l-character. To recover symmetric functions, we define pol_X to be the truncation of this series to terms χ_λ satisfying $\lambda_l \geq 0$, called polynomial terms. This produces an element of Λ via the identification $\chi_\lambda \leftrightarrow s_\lambda$.

Using Catalanimals, we gave an alternative proof of the Rational Shuffle Theorem and provided a generalization whose combinatorics are controlled by Dyck paths below a line of irrational slope in [5]. In [4], we provide a proof of the Extended Delta Conjecture by expressing it as a simple sum of Catalanimals. In [2], we used the Catalanimals introduced in [3] to resolve and generalize the Loehr-Warrington Conjecture for $\nabla^m s_\lambda$. Missing from these works was any treatment of Macdonald polynomials, the foundation upon which these shuffle theorems lie. In this extended abstract, we now fill in this missing piece by providing a Catalanimal style formula for a modified Macdonald polynomial. Our Catalanimal-style formula reads

$$\mathbf{H}_\mu(\mathbf{z};q,t) = \sigma\left(\frac{\prod_{\alpha_{ij}\in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(\mu[i])+1} t^{-\mathrm{leg}(\mu[i])} z_i/z_j\right) \prod_{\alpha_{ij}\in \widehat{R}_\mu} \left(1 - q\,t\,z_i/z_j\right)}{\prod_{\alpha_{ij}\in R_+} \left(1 - q\,z_i/z_j\right) \prod_{\alpha_{ij}\in R_\mu} \left(1 - t\,z_i/z_j\right)} \right), \quad (1.2)$$

where σ is the Weyl symmetrization operator defined in Section 2.2 and $R_\mu, \widehat{R}_\mu$ are subsets of the set of positive roots for GL_l explained in Section 3. Under polynomial truncation, our formula recovers $\omega \tilde{H}_\mu(X;q,t)$.

The proof begins with the Haglund-Haiman-Loehr formula [11] for $\tilde{H}_\mu(X;q,t)$ as a weighted sum of LLT polynomials. We then apply the operator ∇ and combine this with a formula for ∇ on an LLT polynomial established in our recent work [3].

This work serves as an extended abstract for a future paper in which we will elaborate on and further generalize this result.

2 Background

2.1 Partitions and symmetric functions

The (French style) diagram of a partition $\mu = (\mu_1 \geq \cdots \geq \mu_k > 0)$ is the set $\{(i,j) \in \mathbb{Z}_+^2 : 1 \leq j \leq k,\ 1 \leq i \leq \mu_j\}$. We identify (i,j) with the lattice square or *box* whose northeast corner has coordinates $(x,y) = (i,j)$, and refer to this box as being in *column* i and *row* j. We set $|\mu| = \mu_1 + \cdots + \mu_k$ and let $\ell(\mu) = k$ be the number of non-zero parts. We write μ^* for the transpose of a partition μ. The *arm* and *leg* of a box $b \in \mu$ are the number of boxes in μ strictly east of b and strictly north of b, respectively.

Example 2.1.1. Below we give the diagram of the partition $\mu = (2,2,1)$ and we fill box b with $(\mathrm{arm}(b), \mathrm{leg}(b))$:

0,0	
1,1	0,0
1,2	0,1

For this μ, we have $|\mu| = 5$ and $\ell(\mu) = 3$.

Let $\Lambda = \Lambda(X)$ be the algebra of symmetric functions in infinitely many variables $X = x_1, x_2, \ldots$, with coefficients in the field $\mathbf{k} = \mathbb{Q}(q,t)$. We follow Macdonald's notation [20] for the graded bases of Λ, and the automorphism $\omega\colon \Lambda \to \Lambda$ given on Schur functions by $\omega s_\lambda = s_{\lambda^*}$. We also work with series and symmetric functions in finitely many variables $\mathbf{z} = z_1, \ldots, z_l$. If $f(X) \in \Lambda$ is a formal symmetric function, then $f(\mathbf{z})$ or $f(z_1, \ldots, z_l)$ denotes its specialization with $X = z_1, \ldots, z_l, 0, 0, \ldots$.

Given a symmetric function $f \in \Lambda$ and any expression A involving indeterminates, the plethystic evaluation $f[A]$ is defined by writing f as a polynomial in the power-sums

p_k and evaluating with $p_k \mapsto p_k[A]$, where $p_k[A]$ is the result of substituting a^k for every indeterminate a occurring in A. The variables q, t from our ground field $\mathbf{k}$ count as indeterminates.

By convention, the name of an alphabet $X = x_1, x_2, \ldots$ stands for $x_1 + x_2 + \cdots$ inside a plethystic evaluation. Then $f[X] = f[x_1 + x_2 + \cdots] = f(x_1, x_2, \ldots) = f(X)$. For example, the evaluation $f[X/(1-t^{-1})]$ is the image of $f(X)$ under the $\mathbf{k}$-algebra automorphism of Λ that sends p_k to $p_k/(1-t^{-k})$.

The *modified Macdonald polynomials* $\tilde{H}_\mu = \tilde{H}_\mu(X;q,t)$ of [10] are defined in terms of the Macdonald polynomials $Q_\mu(X;q,t)$ [20, VI (4.12)] or their integral forms $J_\mu(X;q,t)$ [20, VI (8.3)] by

$$\tilde{H}_\mu(X;q,t) = t^{\mathsf{n}(\mu)} J_\mu[\frac{X}{1-t^{-1}};q,t^{-1}] = t^{\mathsf{n}(\mu)} (\prod_{b\in\mu}(1-q^{\mathrm{arm}(b)+1}t^{-\mathrm{leg}(b)})) Q_\mu[\frac{X}{1-t^{-1}};q,t^{-1}], \tag{2.1}$$

where $\mathsf{n}(\mu) = \sum_i (i-1)\mu_i$. The $\tilde{H}_\mu(X;q,t)$ also have a direct combinatorial description [11], which we will recall in Theorem 4.1.1.

When $q = 0$, the modified Macdonald polynomials reduce to the *modified Hall-Littlewood polynomials*

$$\tilde{H}_\mu(X;0,t) = t^{\mathsf{n}(\mu)} Q_\mu[\frac{X}{1-t^{-1}};t^{-1}], \tag{2.2}$$

where the Hall-Littlewood polynomials $Q_\mu(X;t)$ are as defined in [20, III (2.11)]. At $t=1$ and $t=\infty$, the $\tilde{H}_\mu(X;0,t)$ specialize to the complete homogeneous symmetric functions $\tilde{H}_\mu(X;0,1) = h_\mu(X)$ and Schur functions $\tilde{H}_\mu(X;0,\infty) = s_\mu(X)$.

Example 2.1.2. The Macdonald polynomial $\tilde{H}_{22}(X;q,t)$ has the following expansion into the basis of Schur functions.

$$\tilde{H}_{22}(X;q,t) = s_4 + (q+t+qt)s_{31} + (q^2+t^2)s_{22} + (qt+q^2t+qt^2)s_{211} + q^2t^2 s_{1111} \tag{2.3}$$

2.2 Weyl symmetrization and related operators

The *Weyl symmetrization operator* $\boldsymbol{\sigma}$ for GL_l is defined by

$$\boldsymbol{\sigma}(f(z_1,\ldots,z_l)) = \sum_{w\in\mathcal{S}_l} w\left(\frac{f(z_1,\ldots,z_l)}{\prod_{i<j}(1-z_j/z_i)}\right) = \sum_{w\in\mathcal{S}_l} w\left(\frac{f(z_1,\ldots,z_l)}{\prod_{\alpha_{ij}\in R_+}(1-z_j/z_i)}\right), \tag{2.4}$$

where $f \in \mathbf{k}[z_1^{\pm 1},\ldots,z_l^{\pm 1}]$ is a Laurent polynomial, $\mathcal{S}_l$ acts by permuting the variables $z_1,\ldots,z_l$, and $R_+ = R_+(GL_l) = \{\alpha_{ij} : 1 \le i < j \le l\}$ denotes the set of positive roots for GL_l, with $\alpha_{ij} = \epsilon_i - \epsilon_j \in \mathbb{Z}^l$.

When $\mathbf{z}^\nu = z_1^{\nu_1}\cdots z_l^{\nu_l}$ for a dominant weight ν (a weight $\nu \in \mathbb{Z}^l$ is dominant if $\nu_1 \ge \cdots \ge \nu_l$), $\boldsymbol{\sigma}(\mathbf{z}^\nu) = \chi_\nu$ is an irreducible GL_l character. For an arbitrary weight

$\gamma \in \mathbb{Z}^l$, either $\boldsymbol{\sigma}(\mathbf{z}^\gamma) = \pm\chi_\nu$ for a suitable dominant weight ν, or $\boldsymbol{\sigma}(\mathbf{z}^\gamma) = 0$. We extend $\boldsymbol{\sigma}$ to an operator on formal $\mathbf{k}$-linear combinations $\sum_{\gamma\in\mathbb{Z}^l} c_\gamma \mathbf{z}^\gamma$ by applying it term by term, giving an infinite formal linear combination of irreducible GL_l characters $\sum_\nu a_\nu \chi_\nu = \sum_{\gamma\in\mathbb{Z}^l} c_\gamma \boldsymbol{\sigma}(\mathbf{z}^\gamma)$. This makes sense because for each dominant weight ν, the set of monomials $\mathbf{z}^\gamma$ such that $\boldsymbol{\sigma}(\mathbf{z}^\gamma) = \pm\chi_\nu$ is finite.

Recall that the *polynomial characters* of GL_l are the irreducible characters χ_ν for which ν is a partition, that is, $\nu_l \geq 0$. Given any formal $\mathbf{k}$-linear combination $\sum_\nu a_\nu \chi_\nu$ of irreducible GL_l characters, we define its *polynomial truncation* by

$$\mathrm{pol}_X\Big(\sum_\nu a_\nu \chi_\nu\Big) = \sum_{\nu_l \geq 0} a_\nu s_\nu(X). \tag{2.5}$$

In principle, the right hand side is an infinite formal sum of symmetric functions, but, for instance, if $\sum a_\nu \chi_\nu$ is homogeneous of degree d, then the right hand side is an ordinary symmetric function, homogeneous of degree d.

2.3 LLT polynomials

We recall the definition and basic properties of LLT polynomials [18], using the 'attacking inversions' formulation from [13].

A *skew diagram* is a difference $\nu = \lambda/\mu$ of partition diagrams $\mu \subseteq \lambda$. The *content* of a box $b = (i, j)$ in row j, column i of a (skew) diagram is $c(b) = i - j$.

Let $\boldsymbol{\nu} = (\nu_{(1)}, \ldots, \nu_{(k)})$ be a tuple of skew diagrams. We consider the set of boxes in $\boldsymbol{\nu}$ to be the disjoint union of the sets of boxes in the $\nu_{(i)}$, and define the *adjusted content* of a box $a \in \nu_{(i)}$ to be $\tilde{c}(a) = c(a) + i\epsilon$, where ϵ is a fixed positive number such that $k\epsilon < 1$.

A *diagonal* in $\boldsymbol{\nu}$ is the set of boxes of a fixed adjusted content, that is, a diagonal of fixed content in one of the $\nu_{(i)}$.

The *reading order* on $\boldsymbol{\nu}$ is the total ordering $<$ on the boxes of $\boldsymbol{\nu}$ such that $a < b \Rightarrow \tilde{c}(a) \leq \tilde{c}(b)$ and boxes on each diagonal increase from southwest to northeast. See Example 2.3.2. An *attacking pair* is an ordered pair of boxes (a, b) in $\boldsymbol{\nu}$ such that $a < b$ in reading order and $0 < \tilde{c}(b) - \tilde{c}(a) < 1$.

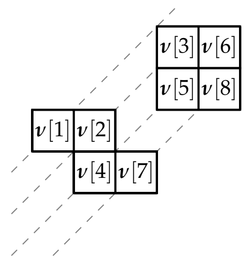
A *semistandard tableau* on the tuple $\boldsymbol{\nu}$ is a map $T \colon \boldsymbol{\nu} \to \mathbb{Z}_+$ which restricts to a semistandard Young tableau on each component $\nu_{(i)}$. The set of these is denoted $\mathrm{SSYT}(\boldsymbol{\nu})$. An *attacking inversion* in T is an attacking pair (a, b) such that $T(a) > T(b)$. The number of attacking inversions in T is denoted $\mathrm{inv}(T)$.

Definition 2.3.1. The *LLT polynomial* indexed by a tuple of skew diagrams $\boldsymbol{\nu}$ is the generating function

$$\mathcal{G}_{\boldsymbol{\nu}}(X; q) = \sum_{T\in \mathrm{SSYT}(\boldsymbol{\nu})} q^{\mathrm{inv}(T)} \mathbf{x}^T, \tag{2.6}$$

where $\mathbf{x}^T = \prod_{a\in\boldsymbol{\nu}} x_{T(a)}$. By [13, 18], $\mathcal{G}_{\boldsymbol{\nu}}(X; q)$ is known to be symmetric.

Example 2.3.2. Let $\boldsymbol{\nu} = ((32)/(10), (33)/(11))$ and let $\boldsymbol{\nu}[1], \dots, \boldsymbol{\nu}[8]$ denote the boxes of $\boldsymbol{\nu}$ in increasing reading order as illustrated in (2.7). We can visualize this by aligning the shapes of $\boldsymbol{\nu}$ along content diagonals and reading diagonals top to bottom and bottom left to top right along each diagonal.

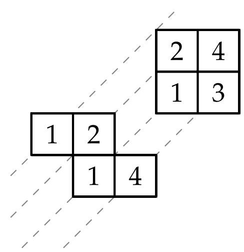


(2.7)

The tuple $\boldsymbol{\nu}$ has attacking pairs

$$\{(\boldsymbol{\nu}[2], \boldsymbol{\nu}[3]), (\boldsymbol{\nu}[3], \boldsymbol{\nu}[4]), (\boldsymbol{\nu}[4], \boldsymbol{\nu}[5]), (\boldsymbol{\nu}[4], \boldsymbol{\nu}[6]), (\boldsymbol{\nu}[5], \boldsymbol{\nu}[7]), (\boldsymbol{\nu}[6], \boldsymbol{\nu}[7]), (\boldsymbol{\nu}[7], \boldsymbol{\nu}[8])\}. \quad (2.8)$$

Let $T \in \mathrm{SSYT}(\boldsymbol{\nu})$ be as follows:

$T =$ 

. (2.9)

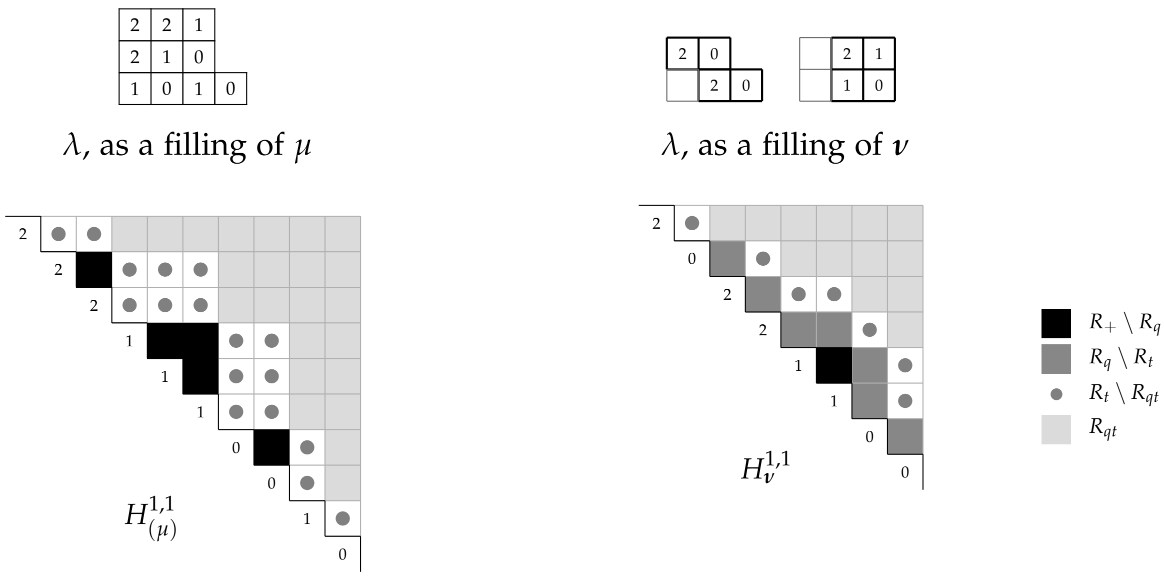
T has attacking inversions $\{(\boldsymbol{\nu}[3], \boldsymbol{\nu}[4]), (\boldsymbol{\nu}[7], \boldsymbol{\nu}[8])\}$, so $\mathrm{inv}(T) = 2$. Also, $\mathbf{x}^T = x_1^3 x_2^2 x_3 x_4^2$.

3 Catalanimals

In [3], we introduced *Catalanimals*—infinite series of irreducible GL_l characters in variables $\mathbf{z} = z_1, \dots, z_l$, defined by

$$H(R_q, R_t, R_{qt}, \lambda) = \boldsymbol{\sigma}\left(\frac{z_1^{\lambda_1} \cdots z_l^{\lambda_l} \prod_{\alpha_{ij} \in R_{qt}} (1 - q\, t\, z_i/z_j)}{\prod_{\alpha_{ij} \in R_q} (1 - q\, z_i/z_j) \prod_{\alpha_{ij} \in R_t} (1 - t\, z_i/z_j)} \right), \quad (3.1)$$

depending on a weight $\lambda \in \mathbb{Z}^l$ and subsets R_q, R_t, R_{qt} of the set of positive roots $R_+ = \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq l\}$ for GL_l. Our convention is always to expand the denominator factors $(1 - c\, z_i/z_j)$ for $c \in \mathbf{k}$ and $i < j$ as geometric series $(1 - cz_i/z_j)^{-1} = 1 + cz_i/z_j + \cdots$ before applying $\boldsymbol{\sigma}$.



$$\omega \nabla s_{433} = (q\,t)^9 (H^{1,1}_{(\mu)})_{\mathrm{pol}} \qquad \omega \nabla \mathcal{G}_{\boldsymbol{\nu}} = -(q\,t)^4 q^7 (H^{1,1}_{\boldsymbol{\nu}})_{\mathrm{pol}}$$

Figure 1: (i) The Catalanimal $H^{1,1}_{(\mu)}$ for $\mu = (433)$. (ii) The Catalanimal $H^{1,1}_{\boldsymbol{\nu}}$ for $\boldsymbol{\nu} = ((32)/(10), (33)/(11))$. These are illustrated by drawing the root sets in an $\ell \times \ell$ grid labeled by matrix-style coordinates, with the sets $R_+ \setminus R_q$, $R_q \setminus R_t$, $R_t \setminus R_{qt}$, R_{qt} specified according to the legend on the right; the weight λ is written on the diagonal with λ_1 in the upper left.

Distinguished among Catalanimals are the *LLT Catalanimals*, denoted $H^{1,1}_{\boldsymbol{\nu}}$ in [3], whose polynomial part is equal to $\omega\nabla$ applied to the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}$, up to a q,t-monomial scalar factor. The Catalanimal $H^{1,1}_{\boldsymbol{\nu}}$ has root sets $R_+ \supseteq R_q \supseteq R_t \supseteq R_{qt}$, determined as follows using the same attacking inversion combinatorics as in the definition of the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}$:

- $R_+ \setminus R_q \leftrightarrow$ pairs of boxes in the same diagonal in $\boldsymbol{\nu}$,
- $R_q \setminus R_t \leftrightarrow$ the attacking pairs in $\boldsymbol{\nu}$,
- $R_t \setminus R_{qt} \leftrightarrow$ pairs going between adjacent diagonals $\boldsymbol{\nu}$,

where the boxes of $\boldsymbol{\nu}$ are numbered $1, \ldots, l$ in reading order (see Example 2.3.2). The weight λ is obtained by filling each diagonal D of $\boldsymbol{\nu}$ with the value $1 + \chi(D \text{ contains a row start}) - \chi(D \text{ contains a row end})$, where $\chi(P) = 1$ if P is true or 0 if P is false, and then reading this filling in the reading order—see Figure 1.

Example 3.1.1. Consider $\boldsymbol{\nu} = (\nu^{(1)})$ for $\nu^{(1)} = (111)$. To construct $H_{\boldsymbol{\nu}}^{(1,1)}$, we note that every box is on a different diagonal in $\boldsymbol{\nu}$ and no boxes attack each other, so $R_q = R_t = R_+$. Furthermore, in reading order, boxes 1 and 2 are adjacent and boxes 2 and 3 are adjacent, so $R_{qt} = \{\alpha_{13}\}$. Finally, every box is both a row start and a row end, so $\lambda = (111)$. Therefore, $H_{\boldsymbol{\nu}}^{(1,1)}$ is given by

$$H(R_+, R_+, \{\alpha_{13}\}, (111)) = \boldsymbol{\sigma}\left(\frac{\mathbf{z}^{111}(1 - q\,t\,z_1/z_3)}{\prod_{1\le i<j\le 3}(1 - q\,z_i/z_j)(1 - t\,z_i/z_j)}\right) \tag{3.2}$$

Taking the polynomial part, this becomes

$$\begin{aligned} &\mathrm{pol}_X\, H(R_+, R_+, \{\alpha_{13}\}, (111)) \\ &= s_{111} + (q + t + q^2 + qt + t^2)s_{21} + (qt + q^3 + q^2t + qt^2 + t^3)s_3 \\ &= \omega\nabla e_3 \end{aligned} \tag{3.3}$$

Note that the LLT polynomial with $\boldsymbol{\nu} = (\nu^{(1)})$ satisfying $\nu^{(1)} = (111)$ is equal to e_3.

For the purposes of this work, we will need only LLT Catalanimals for $\boldsymbol{\nu}$ a tuple of ribbon shapes. A *ribbon* is a connected skew shape containing no 2×2 block of boxes. If we only consider ribbons with m boxes such that the bottom-right box has content -1, then the contents of all the boxes are consecutive integers $-1, -2, \ldots, -m$. Define the *descent set* of a ribbon ν to be the set of contents $c(u)$ of the boxes $u = (i,j) \in \nu$ such that the box $v = (i, j-1)$ directly below u also belongs to ν.

Then, to a partition μ and a subset $D \subset \{(i,j) \in \mu \mid j > 1\}$, we associate a tuple of ribbons $\boldsymbol{\nu}(\mu, D) = (\nu^{(1)}, \ldots, \nu^{(k)})$ where $k = \mu_1$ is the number of columns of μ, and $\nu^{(i)}$ has size μ_i^*, box contents $\{-1, -2, \ldots, -\mu_i^*\}$, and descent set $Des(\nu^{(i)}) = \{-j \mid (i,j) \in D\}$. See Example 4.1.2 for examples of $\boldsymbol{\nu}(\mu, D)$ for fixed μ as D varies. Let $\boldsymbol{\mu}[1], \ldots, \boldsymbol{\mu}[l]$ denote the boxes of $\boldsymbol{\nu}(\mu, D)$ in increasing reading order. Also, let A_μ denote the number of attacking pairs in $\boldsymbol{\nu}(\mu, D)$. Note that both the reading order of the boxes of $\boldsymbol{\nu}(\mu, D)$ and the number of attacking pairs depend only on μ and not on D. Then, for any box $b = (i,j) \in \mu$, let $\mathrm{south}(b) = (i, j-1)$ and let $V_\mu = \{(\boldsymbol{\mu}[i], \boldsymbol{\mu}[j]) \mid \mathrm{south}(\boldsymbol{\mu}[i]) = \boldsymbol{\mu}[j]\}$ be the set of vertical dominoes in μ.

In this case, the root sets for the LLT Catalanimal indexed by $\boldsymbol{\nu}(\mu, D)$ become $R_q = R_+$, and

$$R_t = R_\mu \overset{def}{=} \{\alpha_{ij} \in R_+ \mid \tilde{c}(\boldsymbol{\mu}[i]) + 1 \le \tilde{c}(\boldsymbol{\mu}[j])\} \tag{3.4}$$

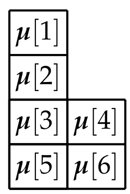
$$R_{qt} = \widehat{R}_\mu \overset{def}{=} \{\alpha_{ij} \in R_+ \mid \tilde{c}(\boldsymbol{\mu}[i]) + 1 < \tilde{c}(\boldsymbol{\mu}[j])\} \tag{3.5}$$

Then, we have the following special case of [3, Corollary 8.4.1].

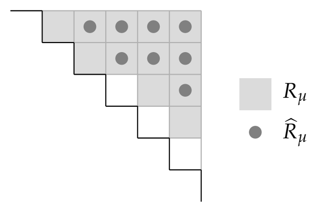
Theorem 3.1.2. *For a partition μ and a subset $D \subset \{(i,j) \in \mu \mid i > 1\}$, let $\boldsymbol{\nu} = \boldsymbol{\nu}(\mu, D)$ and $S = \{(\boldsymbol{\mu}[i], \boldsymbol{\mu}[j]) \in V_\mu \mid \boldsymbol{\mu}[i] \in D\}$. Then, we have the following formula for the operator ∇ applied to the LLT polynomial $\mathcal{G}_{\boldsymbol{\nu}}(X;q)$:*

$$\nabla \mathcal{G}_{\boldsymbol{\nu}}(X;q) = \omega \operatorname{pol}_X \boldsymbol{\sigma} \Bigg((-qt)^{|\mu|-\mu_1-|D|} q^{A_\mu} \times \frac{z_1 \cdots z_l \prod_{(\boldsymbol{\mu}[i],\boldsymbol{\mu}[j]) \in V_\mu \setminus D} z_i/z_j \prod_{\alpha_{ij} \in \widehat{R}_\mu} (1 - qtz_i/z_j)}{\prod_{\alpha_{ij} \in R_+} (1 - qz_i/z_j) \prod_{\alpha_{ij} \in R_\mu} (1 - tz_i/z_j)} \Bigg) \tag{3.6}$$

Example 3.1.3. For $\mu = (2,2,1,1)$, $l = 6$, the root sets R_μ and $\widehat{R}_\mu$ are visualized below drawn in an $l \times l$ grid, labeled by matrix-style coordinates and specified by the legend.



reading order



4 Macdonald Catalanimals

4.1 The Haglund-Haiman-Loehr formula

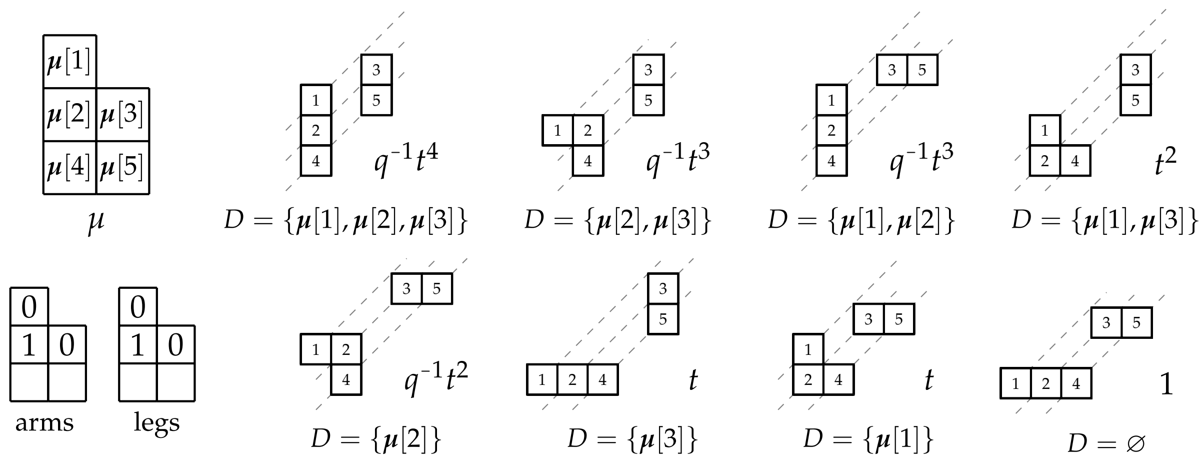
Haglund-Haiman-Loehr [11] gave a formula for the modified Macdonald polynomials $H_\mu(X;q,t)$ as a positive sum of LLT polynomials indexed by a tuple of ribbons. We now recall this formula.

Theorem 4.1.1. *For partition μ,*

$$\tilde{H}_\mu(X;q,t) = \sum_D \left(\prod_{u \in D} q^{-\operatorname{arm}(u)} t^{\operatorname{leg}(u)+1} \right) \mathcal{G}_{\boldsymbol{\nu}(\mu,D)}(X;q)\,, \tag{4.1}$$

where the sum runs over all subsets $D \subset \{(i,j) \in \mu \mid j > 1\}$.

Example 4.1.2. We illustrate the theorem for $\mu = (2,2,1)$. Below we draw $\boldsymbol{\nu}(\mu, D)$ for the 8 possible values of D with boxes labeled in reading order and with the corresponding coefficient $\prod_{u \in D} q^{-\mathrm{arm}(u)} t^{\mathrm{leg}(u)+1}$.



4.2 Proving the Macdonald Catalanimal formula

Applying ∇ to (4.1) and substituting our Catalanimal formula (3.6) for $\nabla \mathcal{G}_\nu$ for each summand, the resulting sum simplifies and yields the following formula for the modified Macdonald polynomials $\tilde{H}_\mu(X; q, t)$ (previewed in (1.2)):

Theorem 4.2.1. *For every partition μ,*

$$\tilde{H}_\mu(X; q, t) = \omega \, \mathrm{pol}_X \, \mathbf{H}_\mu(\mathbf{z}; q, t) \tag{4.2}$$

for $\mathbf{H}_\mu$ given by

$$\mathbf{H}_\mu(\mathbf{z}; q, t) = \boldsymbol{\sigma}\left(\frac{\prod_{\alpha_{ij} \in R_\mu \setminus \widehat{R}_\mu} \left(1 - q^{\mathrm{arm}(\mu[i])+1} t^{-\mathrm{leg}(\mu[i])} z_i/z_j\right) \prod_{\alpha_{ij} \in \widehat{R}_\mu} \left(1 - q\,t\,z_i/z_j\right)}{\prod_{\alpha_{ij} \in R_+} \left(1 - q\,z_i/z_j\right) \prod_{\alpha_{ij} \in R_\mu} \left(1 - t\,z_i/z_j\right)} \right). \tag{4.3}$$

The work of [11, 12] actually gives many formulae for $\tilde{H}_\mu$, one for each rearrangement of the columns of μ. In turn, following the same proof technique, we obtain a Catalanimal style formula for $\tilde{H}_\mu$ for each rearrangement of the columns of μ. More details will be given in the full version of this paper.

4.3 Positivity of the Macdonald Catalanimal

Finally, we provide a conjecture on the series $\mathbf{H}_\mu$ in (4.3). It is a theorem of [16] that the expansion $\tilde{H}_\mu(X; q, t) = \sum_\lambda \tilde{K}_{\lambda,\mu}(q, t) s_\lambda$ has positive coefficients $\tilde{K}_{\lambda,\mu}(q, t) \in \mathbb{Z}_{\geq 0}[q, t]$. However, computer experimentation suggests that $\mathbf{H}_\mu(\mathbf{z}; q, t)$ exhibits a stronger positivity as a series of GL_l-characters.

Conjecture 4.3.1. *For every partition μ of l, the series $\mathbf{H}_\mu(\mathbf{z}; q, t)$ is a positive sum of irreducible GL_l characters; that is, the coefficients in*

$$\mathbf{H}_\mu(\mathbf{z}; q, t) = \sum_\nu \mathbf{K}_{\nu,\mu}(q, t)\, \chi_\nu \tag{4.4}$$

are polynomials $\mathbf{K}_{\nu,\mu}(q, t) \in \mathbb{Z}_{\geq 0}[q, t]$ with non-negative integer coefficients.

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