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Equivariant log-concavity of independence sequences of claw-free graphs

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Abstract. We show that the graded vector space spanned by independent vertex sets of any claw-free graph is strongly equivariantly log-concave, viewed as a graded permutation representation of the graph automorphism group. Our proof reduces the problem to the equivariant hard Lefschetz theorem on the cohomology of a product of projective lines, inspired by a combinatorial map of Krattenthaler. Both the result and the proof generalize our previous result on graph matchings. This also gives a strengthening and a new proof of results of Hamidoune, and Chudnovsky–Seymour.

Keywords: equivariant log-concavity, claw-free graphs, independence sequence

1 Introduction

A graph G is claw-free if no induced subgraph is the bipartite graph $K_{1,3}$. An independent set of a graph G is a set of nonadjacent vertices. The independence sequence of a claw-free graph is *log-concave*: for all $1 \leq k \leq \ell$, the numbers I_j of independent sets of size j satisfies that

$$I_{k-1} I_{\ell+1} \leq I_k I_\ell.$$

This was first proven by Hamidoune [5]. Then, Chudnovsky and Seymour [3] proved it by showing that the generating polynomial has only real roots, which is well-known to imply log-concavity with no internal zeros.

It is often interesting to ask if a certain behavior of a mathematical object respects the underlying symmetry. The notion of equivariant log-concavity was introduced by Gedeon, Proudfoot and Young [4] as a natural categorification of logarithmic concavity. Recently, it is used to study various log-concavity behaviors with respect to a natural group action in the contexts of topology, geometry and combinatorics.

Let Γ be a finite group, a Γ-representation $V_\bullet$ is **strongly equivariantly log-concave** if for all $1 \leq k \leq \ell$,

$$V_{k-1} \otimes V_{\ell+1} \subseteq V_k \otimes V_\ell$$

as a Γ-subrepresentation.

We highlight some known equivariantly log-concave graded representations that are of combinatorial, geometric, and topological interests in the literature:

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Theorem 1.1. *(A) The $V_\bullet^n$ given by the q-binomial coefficients for a fixed n as a $GL_n(\mathbb{F}_q)$-representation is strongly equivariantly log-concave [11, Proposition 6.7].*

(B) The rational cohomology $H^*(\mathrm{Conf}(n,\mathbb{C}),\mathbb{Q})$ of the configuration space of n points in $\mathbb{C}$ as an S_n-representation is strongly equivariantly log-concave for degrees ≤ 14 [10].

(C) The rational cohomology $H^*(\mathrm{Conf}(n,\mathbb{R}^3),\mathbb{Q})$ of the configuration space of n points in $\mathbb{C}$ as an S_n-representation is strongly equivariantly log-concave for degrees ≤ 14 [10].

(D) The $V_\bullet^n$ of even degrees of the intersection homology of the complex affine hypertoric variety of the root system of $\mathfrak{sl}_n$, viewed as an S_n-representation is strongly equivariantly log-concave for degrees ≤ 14 [10].

(E) The $V_\bullet^G$ given by matchings in a graph G as an $\mathrm{Aut}(G)$-representation is strongly equivariantly log-concave [9].

(F) The $V_\bullet^n$ given by k-subsets in $[n]$ as an S_n-representation is strongly equivariantly log-concave (as a special case of [9]).

The aim of this paper is to study the equivariant log-concavity of the following graded representation. Let G be a claw-free graph. Let $\mathbb{I}_k$ denote the set of independent vertex sets of size k. The automorphism group $\mathrm{Aut}(G)$ naturally acts on all independent vertex sets, and each $\mathbb{I}_k$ is invariant under this action. Define the graded representation of $\mathrm{Aut}(G)$

$$V_\bullet^G = \bigoplus_{k\geq 0, I\in\mathbb{I}_k} \mathbb{C}I,$$

and it admits a grading given by cardinalities. We have the following theorem.

Theorem 1.2. *For any claw-free graph G, the graded $\mathrm{Aut}(G)$-representation $V_\bullet^G$ is strongly equivariantly log-concave.*

Remark 1.3. Our proof uses combinatorics inspired by the work of Krattenthaler [8] to reduce the problem to the equivariant hard Lefschetz theorem on a Boolean algebra, or the cohomology of a product of projective lines, a generalization of the method in the author's previous work on graph matchings [9]. The result specializes to our previous result on graph matchings by taking the line graph $L(G)$ of a graph G: The line graph $L(G)$ of a graph G consists of vertices each for every edge in G and edges each for every common vertex shared by two edges in G. For example, every cycle graph C_n with n edges has its line graph isomorphic to itself, and the line graph of K_4 is the 1-skeleton of the hypersimplex $\Delta(2,4)$. A matching on G of size k yields an independent vertex set in $L(G)$ of size k. Line graphs are claw-free, by construction, but not all claw-free graphs are line graphs.

Remark 1.4. Taking dimensions immediately covers the previous results of Hamidoune, and Chudnovsky–Seymour, thus giving new proofs to these results.

Remark 1.5. Communicated by Eric Ramos and Nick Proudfoot, the group consisting of Melody Chan, Chris Eur, Dane Miyata, Nick Proudfoot, Eric Ramos, Lorenzo Vecchi, Claudia Yun, was studying if the graded $\mathrm{Aut}(T)$-representation of the independence sequence of a tree T is strongly equivariantly log-concave. They provided a counterexample, the star graph with 6 leaves, to disprove the statement. Note that this counterexample is "claw-ful", quite the opposite of "claw-free". Morally speaking, the enigmatic "claw" structure seems to be an obstruction to the equivariant log-concavity of independence sequence of a tree, but the lack thereof turns out to be crucial in our proof of Theorem 1.2.

Remark 1.6. During FPSAC 2023, an audience member and later the FPSAC committee chairs pointed out a third proof, prior to this paper, of log-concavity of the independence sequences of claw-free graphs by [15], of which we were not aware. The construction is similar to our construction in the reduction to the injectivity of the raising operator of an $\mathfrak{sl}_2(\mathbb{C})$-representation. One of the standard proofs of the hard Lefschetz theorem for a compact Kähler manifold is by exhibiting an $\mathfrak{sl}_2(\mathbb{C})$-action on the relevant vector space, where the hard Lefschetz operator coincides with the raising operator of the $\mathfrak{sl}_2(\mathbb{C})$-action. See, for example, Chern's proof [2]. Our setup features the presence of a group action on the graded vector space, which naturally arises from the underlying objects, and is respected by the hard Lefschetz operator. A priori, the raising operator of a $\mathfrak{sl}_2(\mathbb{C})$-representation or the hard Lefschetz operator needs not be equivariant with respect to the natural group action. For example, at the end of [15], Wagner recovers the well-known log-concavity of Stirling numbers of the second kind by constructing a claw-free graph for each n. However, we have shown, in an unpublished joint work with Siddarth Kannan, that the graded permutation representation of S_n spanned by partitions of the set $[n]$ is not S_n-equivariantly log-concave. The S_n-equivariant embeddings fail when $n = 7, 8$.[1] It would be fascinating to see if there exist other natural categorifications of the Stirling numbers of the second kind, perhaps inspired by Wagner's construction, that are S_n-equivariantly log-concave.

2 Proof of the main theorem

In this section, we prove the main theorem. The main idea is to construct a family of $\mathrm{Aut}(G)$-equivariant injections

$$V_{k-1} \otimes V_{\ell+1} \hookrightarrow V_k \otimes V_\ell$$

[1] Explicit computations are available at our GitHub repository: `https://github.com/shiyue-li/eqStir/blob/main/eqStir.sage`

for all $1 \leq k \leq \ell$ by reducing to the equivariant hard Lefschetz operator on a Boolean algebra, or the cohomology of a product of projective lines, via the combinatorics of the independent vertex sets. This method is inspired by Krattenthaler's combinatorial proof of the log-concavity of graph matching sequence [8].

Fix a graph G and $1 \leq k \leq \ell$, for each pair I, J in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$, consider the induced subgraph on the symmetric difference of I and J, i.e., $(I \setminus J) \cup (J \setminus I)$, denoted by $G_{I,J}$. The components in $G_{I,J}$ can only be either a path or a cycle, because G is claw-free and I, J are independent vertex sets. Consider all the components in $G_{I,J}$ that are paths of even lengths, i.e., paths that contain odd number of vertices in $I \cup J$, denoted as $C_{I,J}$. ("C" for "chains".) Color vertices from I with blue, and J with pink. Note that each path in $C_{I,J}$ has both endpoints colored blue or pink. Now we do some counting: Let $P_{I,J}$ resp. $B_{I,J}$ be the number of paths with pink resp. blue endpoints in $C_{I,J}$. We have that

$$P_{I,J} + B_{I,J} = |C_{I,J}|, \text{ and } P_{I,J} - B_{I,J} = (\ell + 1) - (k - 1) \geq 2.$$

From these, we know

$$2B_{I,J} \leq B_{I,J} + P_{I,J} - 2 = |C_{I,J}| - 2, \text{ and therefore, } B_{I,J} \leq \frac{|C_{I,J}|}{2} - 1. \tag{2.1}$$

Our next step is to decompose each of $V_{k-1} \otimes V_{\ell+1}$ and $V_k \otimes V_\ell$ into a direct sum of Boolean algebras on certain partitions in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$ and $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$.

Definition 2.1. Two pairs $(I, J), (I', J')$ of independent vertex sets are equivalent if $I \cup J = I' \cup J'$ and I resp. J agrees with I' resp. J' outside of $C_{I,J}$ or $C_{I',J'}$.

One verifies using arguments in [9] that this indeed gives partitions $\Pi_{k-1,\ell+1}$ and $\Pi_{k,\ell}$ on $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$ and $\mathbb{I}_k \times \mathbb{I}_\ell$ respectively.

For any part $P \in \Pi_{k-1,\ell+1}$ and each pair (I, J) in P, we associate a set of pairs of independent vertex sets in $\mathbb{I}_{k-1} \times \mathbb{I}_{\ell+1}$ as follows. For each path in $C_{I,J}$ with endpoints colored blue, we swap the colors on all the vertices in this path from pink to blue and from blue to pink. This swapping produces a path in $C_{I,J}$ with endpoints colored pink. Now collect all the blue resp. pink vertices in $G_{I,J}$ and record that as I' resp. J'. Since I' resp. J' now has k resp. ℓ vertices, the pair (I', J') is in $\mathbb{I}_k \times \mathbb{I}_\ell$. Repeat for every path in $C_{I,J}$, we obtain a subset $N_{I,J}$ in $\mathbb{I}_k \times \mathbb{I}_\ell$. See Figure 1 for an example on the 6-cycle graph for $k = \ell = 2$. Using a similar argument as in [9, Section 2.2], we verify that $N_{I,J}$ is a part of $\Pi_{k,\ell}$, denoted as P'. Now define a map

$$\Phi_{k,\ell} \colon V_{k-1} \otimes V_{\ell+1} \to V_k \otimes V_\ell, \quad I \otimes J \mapsto \frac{1}{|N_{I,J}|} \sum_{(I',J') \in N_{I,J}} I' \otimes J'.$$

Using similar argument as in [9, Section 2.2], one verifies that $\Phi_{k,\ell}$ is Aut(G)-equivariant.

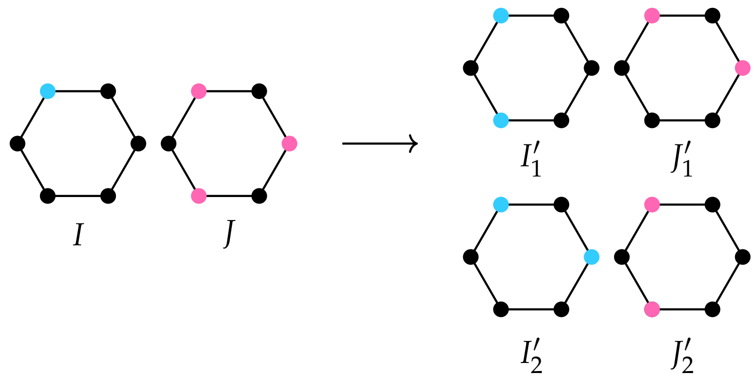

Figure 1

To show injectivity, we consider the following vector space for any part P in $\Pi_{k-1,\ell+1}$

$$V_{k-1,\ell+1}(P) := \mathrm{Span}_{\mathbb{F}}\{I \otimes J \mid (I,J) \in P\}.$$

We now realize $V_{k-1,\ell+1}(P)$ as a categorification of the B_Pth level of the Boolean lattice on C_P. Consider the map

$$\beta_P \colon P \to \binom{C_P}{B_P}, \quad (I,J) \mapsto \text{the set of paths with blue endpoints in } C_{I,J}.$$

It is well-defined by the construction of paths of blue endpoints in C_P and bijective using a similar argument in [9]. Next, we consider the vector space

$$V_{C_P,B_P} := \mathrm{Span}_{\mathbb{F}}\left\{B \mid B \in \binom{C_P}{B_P}\right\},$$

and define

$$\underline{\beta_P} \colon V_{k-1,\ell+1}(P) \to V_{C_P,B_P}, \quad I \otimes J \mapsto \text{the set of paths with blue endpoints in } C_{I,J}.$$

It is an isomorphism of vector spaces, because β_P is a bijection on the bases.

Then, we do the same procedure for $\mathbb{I}_k \times \mathbb{I}_\ell$. We define vector spaces $V_{k,\ell}(P')$, $V_{C_P,B_{P'}}$ and the maps $\beta_{P'}$ and $\underline{\beta_{P'}}$ similar to those for P. Note that, by construction,

$$B_{P'} = B_P + 1 \quad \text{and} \quad C_{P'} = C_P.$$

Finally, for each P in $\Pi_{k-1,\ell+1}$, define the linear map

$$L_P \colon V_{C_P,B_P} \to V_{C_{P'},B_P+1}, \quad B \mapsto \frac{1}{|C_P| - B_P} \sum_{B \subseteq B' \in \binom{C_P}{B_P+1}} B'.$$

Crucially, L_P is a hard Lefschetz operator on the Boolean algebra spanned by all subsets of C_P, where the grading is given by cardinality. It is injective for degrees

$$B_P \leq |C_P|/2 - 1.$$

This operator and its injectivity on the lower half graded pieces have been studied in various contexts. We invite the reader to see proofs of various flavors: [12], [13, The hard Lefschetz theorem], [6, Proposition 7], [7], [14, Theorem 4.7] and [1, Theorem 1.1(3)].

By construction, the following diagram commutes:

$$\begin{array}{ccc} V_{k-1,\ell+1}(P) & \xrightarrow[\cong]{\beta_P} & V_{C_P,B_P} \\ \Big\downarrow \Phi_{k,\ell} & & \Big\downarrow L_P \\ V_{k,\ell}(P') & \xrightarrow[\cong]{\beta_{P'}} & V_{C_{P'},B_P+1}. \end{array}$$

Therefore, $\Phi_{k,\ell}$ is injective from $V_{k-1,\ell+1}(P)$ to $V_{k,\ell}(P')$.

Note that by construction,

$$V_{k-1} \otimes V_{\ell+1} = \bigoplus_{P \in \Pi_{k-1,\ell+1}} V_{k-1,\ell+1}(P) \cong \bigoplus_{P \in \Pi_{k-1,\ell+1}} V_{C_P,B_P}.$$

Then the last sentence of the previous paragraph implies that $\Phi_{k,\ell}$ is injective on the tensor product $V_{k-1} \otimes V_{\ell+1}$.

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