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An SL_4-web basis from hourglass plabic graphs

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Abstract. We introduce *hourglass plabic graphs* and prove that certain of these graphs index a rotation-invariant SL_4-*web basis*, a structure that has been sought since Kuperberg's introduction of the SL_3-web basis in 1996. These graphs exhibit connections to the combinatorics of standard Young tableaux, crystals, alternating sign matrices, six-vertex configurations, and plane partitions.

Keywords: web, promotion, plabic graph, alternating sign matrix, six-vertex model

1 Introduction

The irreducible representations of the symmetric group S_n are the *Specht modules* S^λ indexed by integer partitions λ. For the case of 3-row rectangles, Kuperberg [17] famously introduced a diagrammatic "web" basis of the Specht module $S^{3\times b}$ (and more generally for other spaces of invariant tensors). Kuperberg's web basis has many important applications to quantum link invariants, cluster algebras, and algebraic geometry. From a combinatorial perspective, key properties of the web basis are that the long cycle $c = (12\ldots n)$ and the long element $w_0 = n(n-1)\ldots 1$ both act in this basis as scalar multiples of permutation matrices. Indeed, (up to scalars) c acts diagrammatically as a rotation [21] and w_0 as a reflection [19], yielding an easily accessible pictorial understanding of these actions. These facts are particularly powerful in dynamical algebraic

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combinatorics, where for example Petersen–Pylyavskyy–Rhoades [21] used them to obtain a much more direct proof of the 3-row case of Rhoades' [24] cyclic sieving result for rectangular Young tableaux under the action of tableau promotion.

The 2-row analogue of Kuperberg's web basis can be combinatorially identified with noncrossing matchings and is algebraically closely related to the Temperley–Lieb algebra (see [16]). Kuperberg also exhibits analogues in Lie types B_2 and G_2. Kuperberg [17] writes that "the main open problem [...] is how to generalize [these web bases] to higher rank." Despite progress on identifying generators and relations [4, 8, 14, 18], as well as constructions of non-rotating bases [5, 6, 9, 29], no rotation-invariant web basis for rectangular shapes of more than 3 rows has yet been obtained. However, see [20] for a putative web basis in a non-rectangular setting and [7] for the 2-column case.

Our main result is a rotation-invariant web basis for the 4-row rectangular Specht module $S^{4\times b}$ (and also for more general spaces of tensor invariants). Our diagrams are a generalization of Postnikov's plabic graphs (originally introduced for the study of totally positive parts of flag varieties), where our main combinatorial innovation is the introduction of *hourglass edges* that locally break planarity in a mild way. Our new hourglass webs have surprising connections to alternating sign matrices, the 6-vertex model of statistical physics, and plane partitions. A novel ingredient of our proof is the use of Kashiwara's crystal bases to identify and establish appropriate local growth rules for webs.

This paper is an extended abstract only; the full version will be available on the arXiv.

2 Background

2.1 Tableaux, promotion, and evacuation

A ***standard Young tableau*** of ***rectangular shape*** $a \times b$ is a bijective filling of an $a \times b$ grid of boxes with a rows and b columns using the values $1, 2, \ldots, ab$ such that each row is increasing left-to-right and each column is increasing top-to-bottom. Our running example will be

$$E = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 & 4 & 7 & 10 \\ \hline 8 & 9 & 11 & 14 \\ \hline 12 & 13 & 15 & 16 \\ \hline \end{array}, \tag{2.1}$$

which is a 4×4 standard Young tableau. We write $\mathrm{SYT}(a \times b)$ for the set of all standard Young tableaux of shape $a \times b$. In this paper, our focus is the case $a = 4$.

Promotion (Schützenberger [25]) is a bijection $\mathcal{P} : \mathrm{SYT}(a \times b) \to \mathrm{SYT}(a \times b)$ defined as follows. Given $T \in \mathrm{SYT}(a \times b)$, first delete the entry 1 from the box b. Fill the (now) empty box b by sliding into b the smaller of the two values appearing immediately to the right of b and immediately below b. This slide yields a new empty box b′. Repeat this sliding process until the empty box appears in the bottom right corner of the tableau. Finally, fill the bottom right corner with $ab + 1$ and then subtract 1 from all entries. The result is the promotion $\mathcal{P}(T) \in \mathrm{SYT}(a \times b)$. It is known [10] that for $\mathrm{SYT}(a \times b)$, we have $\mathcal{P}^{ab} = \mathrm{id}$. *Classical webs* (cf. Section 2.3) provide a pictorial proof of this fact for $a \leq 3$; our main result yields such a proof for $a = 4$. For the tableau E, this process is as follows:

$$E = \begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4&7&10\\\hline 8&9&11&14\\\hline 12&13&15&16\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline \ &2&5&6\\\hline 3&4&7&10\\\hline 8&9&11&14\\\hline 12&13&15&16\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 2&\ &5&6\\\hline 3&4&7&10\\\hline 8&9&11&14\\\hline 12&13&15&16\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline 3&\ &7&10\\\hline 8&9&11&14\\\hline 12&13&15&16\\\hline\end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline 3&7&\ &10\\\hline 8&9&11&14\\\hline 12&13&15&16\\\hline\end{array}$$

$$\downarrow$$

$$\mathcal{P}(E) = \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6&9&13\\\hline 7&8&10&15\\\hline 11&12&14&16\\\hline\end{array} \leftarrow \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline 3&7&10&14\\\hline 8&9&11&16\\\hline 12&13&15&17\\\hline\end{array} \leftarrow \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline 3&7&10&14\\\hline 8&9&11&16\\\hline 12&13&15&\ \\\hline\end{array} \leftarrow \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline 3&7&10&14\\\hline 8&9&11&\ \\\hline 12&13&15&16\\\hline\end{array} \leftarrow \begin{array}{|c|c|c|c|}\hline 2&4&5&6\\\hline 3&7&10&\ \\\hline 8&9&11&14\\\hline 12&13&15&16\\\hline\end{array}$$

For $T \in \mathrm{SYT}(a \times b)$, the ***evacuation*** is the tableau $\mathcal{E}(T) \in \mathrm{SYT}(a \times b)$ obtained by rotating T by 180° and then replacing each entry i by $ab + 1 - i$. Note that evacuation is an involution. Together $\mathcal{P}$ and $\mathcal{E}$ generate a dihedral group action on $\mathrm{SYT}(a \times b)$, as $\mathcal{P}^{-1} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{P}$.

2.2 Plabic graphs

Definition 2.1 (Postnikov [22])**.** A ***plabic graph*** is a planar undirected graph, embedded in a disk, whose boundary vertices have degree 1 and are labelled $b_1, b_2, \ldots$ in clockwise order and whose vertices are each colored black or white.

Plabic graphs were introduced by Postnikov [22] in his study of the totally positive Grassmannian, and have since proven [23] to be fundamental objects in the theories of cluster algebras [26] and KP solitons [15] and in the physics of scattering amplitudes [1].

For each boundary vertex b_i of a plabic graph G, one may obtain another boundary vertex $b_{\pi(i)}$ by beginning a walk on the unique edge incident to b_i and turning left at each white vertex and right at each black vertex, until the boundary is reached. The function π is in fact a permutation, denoted $\mathrm{trip}(G)$. If this set of walks from the boundary vertices avoids certain *bad double crossings*, then G is called ***reduced***. Two reduced plabic graphs share the same trip permutation if and only if they are connected by a sequence of certain graphical transformations called *moves* (see [22]).

2.3 Classical webs

Consider an $a \times n$ matrix of distinct variables. Multiplication by the special linear group SL_a induces an SL_a-module structure on the space of polynomials in these variables. An important task in representation theory is to describe the submodule of invariant polynomials. This submodule is the homogeneous coordinate ring of the *Grassmannian* of a-planes in $\mathbb{C}^n$; moreover, various multihomogeneous components yield concrete constructions of the *Schur modules* (the irreducible polynomial representations of SL_n) and of the *Specht modules* (the irreducible representations of the symmetric group S_n). In this abstract, we restrict attention to Specht modules, although our results are more general.

Our Specht modules $S^{a\times b}$ are associated to rectangular shapes $a \times b$. To compute with Specht modules, it is useful to have explicit vector space bases of $S^{a\times b}$. Various useful choices of basis are known, but we are especially interested in *web bases* [17]. Web bases have a variety of useful features. For combinatorial purposes, the most important such features are that web bases are diagrammatic (meaning that the basis elements are indexed by planar diagrams) and that the actions of the long cycle c and the long element w_0 are by (scalar multiples of) permutation matrices that are pictorially apparent (up to straightforward signs, c acts by rotation of diagrams [21] and w_0 by reflection [19]). For example, in [21], these facts for web bases were used to give substantially more direct proofs of cyclic sieving phenomena for the action of promotion on 2- and 3-row rectangular tableaux (following the original Kazhdan–Lusztig theory proof of Rhoades [24] for the general rectangular case). Our main result may allow an analogous proof in the 4-row case.

Unfortunately, rotation-invariant web bases are only known in a small number of cases. For the Specht modules $S^{2\times b}$, the web diagrams are the noncrossing matchings of $2b$ points on the boundary of a disk [16]. For $S^{3\times b}$, Kuperberg [17] introduced SL_3-webs, which are the bipartite plane graphs embedded in a disk such that all boundary vertices have degree 1, all internal vertices have degree 3, all internal faces have at least 6 sides, and all boundary vertices share a color. It has long been desired to obtain web bases for other shapes, especially 4-row rectangles. For example, [4, 8, 14, 18] describe families of diagrams that are too large to be bases, together with the relations among the corresponding invariant polynomials, whereas [5, 6, 29] provide algorithms to construct non-rotating bases. A rotating basis for certain nonrectangular shapes was recently given in [20] and a web basis for 2-column rectangles in [7]. Our main result is an explicit description of a rotation-invariant web basis for $S^{4\times b}$, the first extension to higher-row rectangular shapes since Kuperberg's 1996 paper [17].

3 Main results

In this section, we present our main results and new constructs. Sections 3.1 and 3.2 contain new definitions and theorems needed for our main theorems in Sections 3.3 and 3.4. Theorem 3.6 gives our main bijection between standard tableaux and equivalence classes of hourglass plabic graphs, and the corresponding web basis is given in Theorem 3.7.

3.1 Hourglass plabic graphs

Definition 3.1. An ***hourglass plabic graph*** is a bipartite plabic graph with black boundary vertices, with hourglass edges allowed between internal vertices, whose internal vertices are all 4-valent.

Hourglass plabic graphs admit moves of two kinds: *square moves* and *benzene moves* (see Figure 1). Hourglass plabic graphs G and G' are ***equivalent*** if one may be obtained from the other by a sequence of square and benzene moves. We write $[G]$ for the equivalence class of G. Equivalence of classical plabic graphs and webs under square moves has appeared before, while benzene equivalence is a new and crucial feature of hourglass plabic graphs.

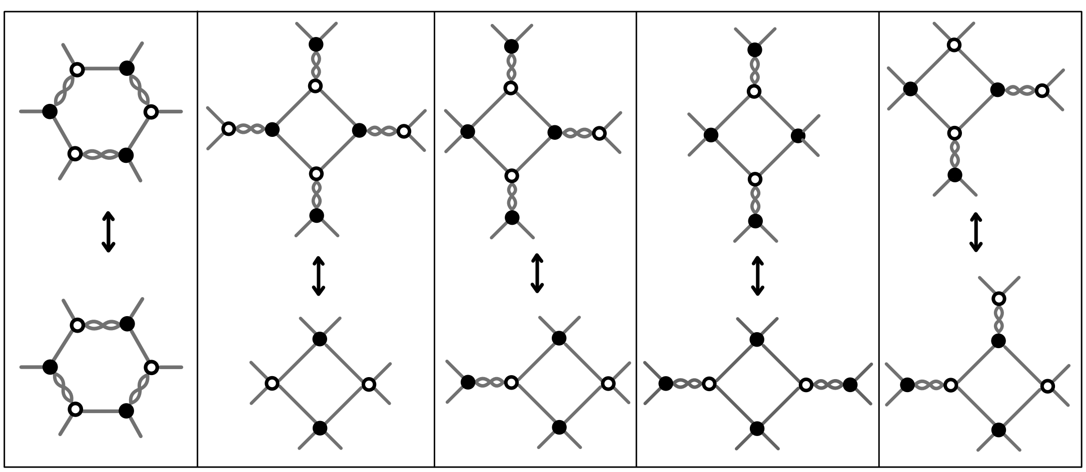


Figure 1: All moves that can be performed on an hourglass plabic graph, up to rotation and reversal of vertex colors. The leftmost move is a ***benzene move*** and the others are ***square moves***.

Definition 3.2. An hourglass plabic graph G is ***fully reduced*** if no $G' \in [G]$ contains any of the following substructures:

1. An interior vertex incident to fewer than three other vertices,

2. A 2-cycle (treating an hourglass edge as a single edge), or
3. A 4-cycle containing an hourglass edge.

A key feature of hourglass plabic graphs is that they have *three* trip permutations $\mathrm{trip}_1(G), \mathrm{trip}_2(G)$, and $\mathrm{trip}_3(G)$. These are defined analogously to the trip permutation of a plabic graph, but we take the i-th left at white vertices and i-th right at black vertices when computing $\mathrm{trip}_i(G)$ (see Figure 2).

Theorem 3.3. *Two fully reduced hourglass plabic graphs G and G' are equivalent if and only if $\mathrm{trip}_i(G) = \mathrm{trip}_i(G')$ for $i = 1, 2, 3$.*

3.2 Promotion permutations

Motivated by trip permutations of hourglass plabic graphs, we want to associate tuples of permutations to rectangular standard Young tableaux. For $T \in \mathrm{SYT}(a \times b)$, $1 \leq i \leq a-1$ and $1 \leq j \leq ab$, denote with $p^{i,j}$ the unique entry of $\mathcal{P}^{j-1}(T)$ that moves from row $i+1$ to row i in the sliding process when applying promotion to $\mathcal{P}^{j-1}(T)$.

Definition-Theorem 3.4. *For $1 \leq i \leq a-1$, the map $\mathrm{prom}_i(T) : j \mapsto (p^{i,j} + j - 1) \pmod{ab}$ is a fixed-point-free permutation on the set $\{1, \ldots, ab\}$. We call $\mathrm{prom}_i(T)$ the i-th **promotion permutation** of T.*

The ***anti-exceedance set*** of a permutation π is $\mathrm{Aexc}(\pi) := \{i \mid \pi^{-1}(i) > i\}$. We denote with $\mathrm{rot}(\pi) = c^{-1} \circ \pi \circ c$ the ***rotation*** of π and we denote with $\mathrm{refl}(\pi) = w_0 \pi w_0$ its ***reverse-complement***. The promotion permutations of T satisfy the following properties.

Theorem 3.5. *Let $T \in \mathrm{SYT}(a \times b)$, then for $1 \leq i \leq a-1$*

1. $\mathrm{rot}(\mathrm{prom}_i(T)) = \mathrm{prom}_i(\mathcal{P}(T))$,
2. $\mathrm{refl}(\mathrm{prom}_i(T)) = \mathrm{prom}_i(\mathcal{E}(T))$,
3. $\mathrm{prom}_i(T) = \mathrm{prom}_{a-i}(T)^{-1}$, *and*
4. $\mathrm{Aexc}(\mathrm{prom}_i(T)) = \{e \mid e \text{ is in the first } i \text{ rows of } T\}$.

The promotion permutations in one-line notation of our running example (2.1) are

$$\begin{aligned}
\mathrm{prom}_1(E) &= 4\ 3\ 14\ 10\ 9\ 7\ 8\ 16\ 13\ 11\ 12\ \underline{6}\ \underline{5}\ 15\ \underline{2}\ \underline{1}, \\
\mathrm{prom}_2(E) &= 14\ 9\ 16\ 15\ 11\ 8\ 13\ \underline{6}\ \underline{2}\ 12\ \underline{5}\ \underline{10}\ \underline{7}\ \underline{1}\ \underline{4}\ \underline{3}, \text{and} \\
\mathrm{prom}_3(E) &= 16\ 15\ \underline{2}\ \underline{1}\ 13\ 12\ \underline{6}\ \underline{7}\ \underline{5}\ \underline{4}\ \underline{10}\ \underline{11}\ \underline{9}\ \underline{3}\ \underline{14}\ \underline{8}.
\end{aligned} \tag{3.1}$$

(The anti-exceedances of each permutation are underscored.) Note that $\mathrm{prom}_1(E)$ and $\mathrm{prom}_3(E)$ are inverses of each other and that $\mathrm{prom}_2(E)$ is an involution.

3.3 The main bijection

Given an hourglass plabic graph G, let $\mathrm{rot}(G)$ denote the hourglass plabic graph obtained from G by cyclically rotating the graph with respect to the boundary labels, and let $\mathrm{refl}(G)$ denote the hourglass plabic graph obtained by reflecting the graph (but not the boundary labels) with respect to a diameter of the disk intersecting the boundary between b_{4b} and b_1.

Theorem 3.6. *For an equivalence class* $[G]$ *of fully reduced hourglass plabic graphs with* $4b$ *boundary vertices, there is a unique* $4 \times b$ *standard Young tableau* $\mathcal{T}([G])$ *whose first* i *rows contain the entries* $\mathrm{Aexc}(\mathrm{trip}_i(G))$ *for* $i = 1, 2, 3$*. Furthermore, the map* $\mathcal{T}$ *is a bijection satisfying:*

1. $\mathrm{trip}_i(G) = \mathrm{prom}_i(\mathcal{T}([G]))$ *for* $i = 1, 2, 3$,
2. $\mathcal{T}([\mathrm{rot}(G)]) = \mathcal{P}(\mathcal{T}([G]))$*, and*
3. $\mathcal{T}([\mathrm{refl}(G)]) = \mathcal{E}(\mathcal{T}([G]))$.

See Figure 2 for an example.

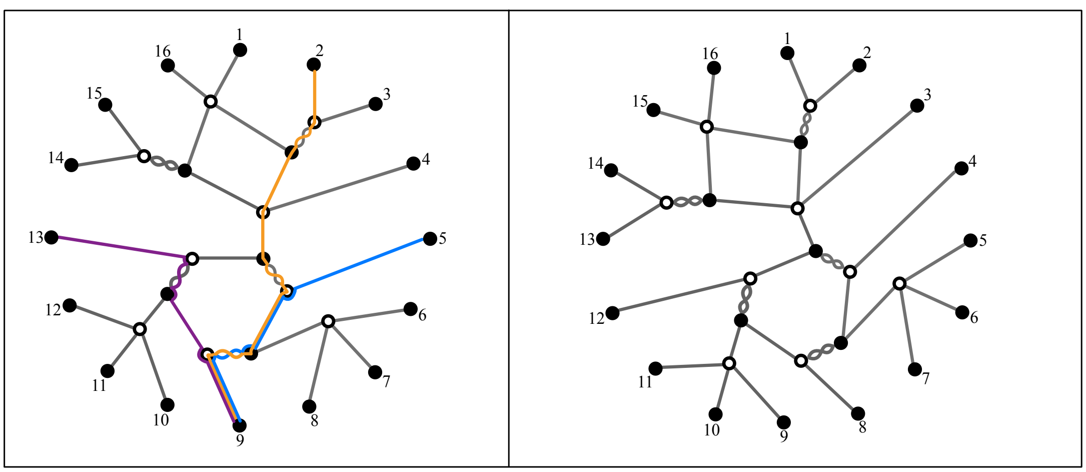


Figure 2: An hourglass plabic graph G representing $\mathcal{G}(E)$ for our running example tableau E (left) and its rotation $\mathrm{rot}(G)$ (right), corresponding to $\mathcal{P}(E)$. The paths computing $\mathrm{trip}_1(G)(9) = 13, \mathrm{trip}_2(G)(9) = 2$, and $\mathrm{trip}_3(G)(9) = 5$ are drawn in purple, orange, and blue respectively, matching the values of $\mathrm{prom}_i(E)(9)$ computed in (3.1).

The inverse $\mathcal{G}$ of the map $\mathcal{T}$ in Theorem 3.6 involves *growth rules*, which operate on the *lattice word* of a standard Young tableau T. In the lattice word $L = \ell_1 \ell_2 \ldots$ of T, ℓ_k is the row index of the box labeled k in T. The lattice condition requires that, for any prefix of L and for all $i \leq j$, the number of i's is not less than the number of j's. For example, the lattice word associated to (2.1) is $L = 1122112332344344$.

For $T \in \mathrm{SYT}(2 \times b)$, we may construct a non-crossing circular matching as follows. First place $2b$ vertices in a line, each with a dangling edge labeled ℓ_i from the lattice word L of T. Now iteratively "cap off" adjacent pairs of dangling strands labeled 1 on the left, 2 on the right. This is analogous to matching parentheses. After all dangling strands have been capped off, wrap the resulting figure around a circle.

For $T \in \mathrm{SYT}(3 \times b)$, Khovanov–Kuperberg [13] extended this construction to produce Kuperberg's trivalent non-elliptic webs. The algorithm starts with $3b$ black vertices and dangling strands again labeled by the lattice word of T. Strands are combined iteratively using a collection of 14 growth rules (see [21, Figure 4] under $1 \leftrightarrow 1, 0 \leftrightarrow 2, \bar{1} \leftrightarrow 3$). For instance, adjacent strands labeled $1, 2$ dangling from black vertices may be combined in a new trivalent white vertex with a new dangling strand labeled 3. Khovanov–Kuperberg showed that the algorithm is independent of all choices and is bijective.

We extend these constructions to $T \in \mathrm{SYT}(4 \times b)$. More precisely, we give a bijection $\mathcal{G} : SYT(4 \times b) \to \{[G] \mid G$ a fully reduced hourglass plabic graph$\}$ inverse to the bijection $\mathcal{T}$ from Theorem 3.6. The algorithm involves approximately 100 growth rules; see Figure 3 for some examples. An intriguing feature of the 4-row growth rules is the presence of "witnesses," which are dangling strands which are not themselves combined. Moreover, the result is *not* independent of choices, though the result is well-defined as an equivalence class of fully reduced hourglass plabic graphs. Our key technique for generalizing the Khovanov–Kuperberg growth rules is to observe that each rule corresponds to a pair of corresponding vertices in isomorphic Kashiwara crystal graphs.

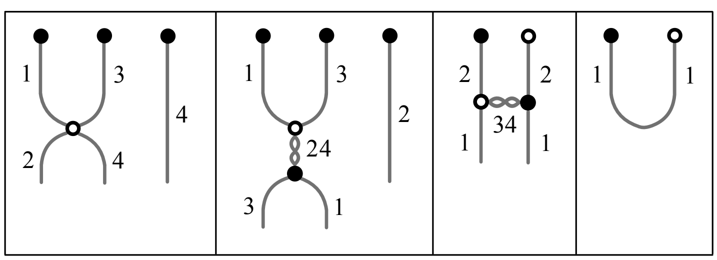


Figure 3: Some examples of 4-row growth rules.

3.4 The SL_4-web basis

Kuperberg [17] introduced SL_3-webs as a combinatorial model for the category of $\mathrm{SL}_3(\mathbb{C})$-modules. In particular, bipartite trivalent webs with $3b$ black boundary vertices correspond to ***invariant polynomials***. These are elements of

$$\mathrm{Inv}((\mathbb{C}^3)^{\otimes b}) = \mathrm{Hom}_{\mathrm{SL}_3(\mathbb{C})}((\mathbb{C}^3)^{\otimes b}, \mathbb{C}) \subset \mathbb{C}[x_{i1}, x_{i2}, x_{i3} : 1 \leq i \leq b],$$

where $\mathbb{C}^3$ is the defining representation. For example, 3 black boundary vertices connected to a single white vertex corresponds to the 3×3 determinant polynomial in

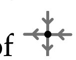
9 variables x_{ij} for $1 \leq i, j \leq 3$. Kuperberg showed that the invariants associated to SL_3-webs form a basis for $\mathrm{Inv}((\mathbb{C}^3)^{\otimes b})$. A group based at LACIM [2] showed more directly that Kuperberg–Khovanov's growth rules applied to $T \in \mathrm{SYT}(3 \times b)$ produce webs whose invariants have grevlex-leading terms that correspond naturally to T itself. See [2, Thm. 5.2] for details.

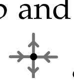
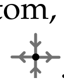
Using Fraser–Lam–Le's monomial expansion [8, Lemma 5.4], we extend the result of [2] to hourglass plabic graphs.

Each equivalence class $[G]$ of fully reduced reduced hourglass plabic graphs contains a ***top*** hourglass plabic graph G^{top}, whose corresponding *symmetrized six-vertex configuration* (see Section 4) has all triangles oriented counterclockwise. For example the graphs in Figure 2 are top fully reduced hourglass plabic graphs. The top element G^{top} is unique up to square moves, and thus the web invariant polynomial $[\![G^{\mathrm{top}}]\!]$ is well-defined.

Theorem 3.7. *The web invariant polynomials* $[\![G^{\mathrm{top}}]\!]$ *of top fully reduced hourglass plabic graphs with* $4b$ *boundary vertices are a rotation-invariant basis for the invariant space* $\mathrm{Inv}((\mathbb{C}^4)^{\otimes b})$.

4 The six-vertex model and combinatorial connections

Fully reduced hourglass plabic graphs may be transformed to directed graphs in which all boundary vertices are degree 1 and oriented inward and all interior vertices are degree 4 and are sources, sinks, or any rotation of  (***transmitting vertices***). The bijection proceeds by orienting all non-hourglass edges from black to white vertex, removing the vertex coloring, and collapsing each hourglass edge and incident vertices to a single vertex, producing a transmitting vertex configuration. This is invertible, as each source corresponds to a black vertex, each sink corresponds to a white vertex, and each transmitting vertex uniquely expands to an hourglass edge with one black and one white vertex. Condition 3 of Definition 3.2 corresponds to each three-cycle being oriented (in the entire equivalence class). See Figure 4. A similar construction appears in [9].

These graphs can be seen as configurations of a symmetrized version of the *six-vertex model with domain wall boundary conditions*. Configurations of this model on a square grid involve boundary edges oriented inward along the left and right boundary and outward along the top and bottom, while interior edge configurations include transmitting vertices as well as  and . Such a configuration may be transformed to our convention by reversing the direction of all vertical edges, permuting transmitting vertices and producing a source or sink at each non-transmitting vertex.

On a square grid, these graphs are well-known to correspond to ***alternating sign matrices (ASM)***, matrices whose entries are in $\{0, \pm 1\}$, whose rows and columns sum to 1, and whose nonzero entries alternate in sign along each row and column. In our convention, sinks correspond to 1, sources to -1, and transmitting vertices to 0. In fact,

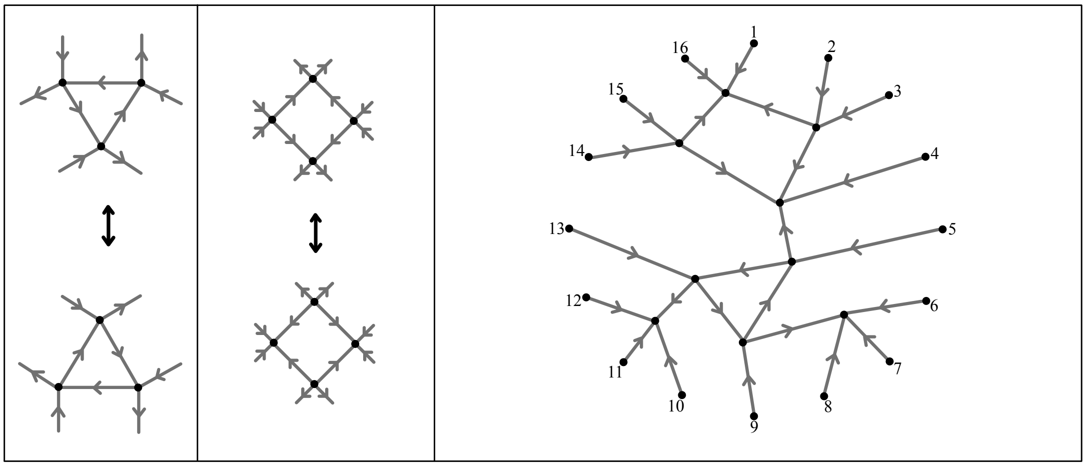


Figure 4: The six-vertex configuration of our running example (2.1) (right), together with six-vertex versions of the first two moves in Figure 1 (left).

this connection yields an intriguing enumeration; the $4 \times n$ standard Young tableau in which the numbers appear sequentially, left-to-right then top-to-bottom has $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ elements in its equivalence class, corresponding to all $n \times n$ alternating sign matrices.

The moves of Figure 1 transform to studied moves on six-vertex configurations and alternating sign matrices: the benzene move corresponds to the *star-triangle relation* associated to the *Yang-Baxter equation*, while the square move may be seen as a *fiber toggle* in the alternating sign matrix tetrahedral poset [28] or as travelling along an edge of the *ASM polytope* [27]. Composing these toggles at even squares followed by odd squares produces the well-studied *gyration* action of [30] on *fully-packed loops*, objects in bijection with alternating sign matrices. Fully-packed loops on generalized domains (such as ours) were considered by Cantini–Sportiello in their proof of the *Razumov–Stroganov conjecture* [3] and relate to *chained alternating sign matrices* [11].

The benzene move (in the plabic formulation of Figure 1) can be seen as a transformation on a *dimer cover* (perfect matching) of the hexagonal lattice, where the hourglass edges correspond to the included edges of the dimer cover. The transformation inverts a cycle of alternate hourglass and non-hourglass edges. This is well-known to correspond to adding and removing boxes from an associated *plane partition*.

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