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# Counting unicellular maps under cyclic symmetries

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**Abstract.** We count unicellular maps (matchings of the edges of a $2n$-gon) of arbitrary genus with respect to the $2n$-rotation symmetries of the polygon. An associated generating function that keeps track of the number of symmetric vertices of the resulting map generalizes the celebrated Harer-Zagier formula.

The answer to this enumerative question is not in the form of the usual cyclic sieving phenomenon (CSP), but does recover in the leading terms (genus-0 maps) a well known CSP for the Catalan numbers. The approach is representation theoretic, in that we relate symmetric unicellular maps with factorizations of the Coxeter element in a reflection group of type $G(m, 1, n)$.

**Keywords:** Harer-Zagier formula, unicellular maps, reflection groups, cyclic sieving

## 1 Introduction

Unicellular maps are the 3-constellations of the form $\sigma\alpha c = \mathbf{1}$ where $\sigma, \alpha, c \in S_{2n}$, $\sigma$ is a fixed point free involution, $\alpha$ an arbitrary permutation, and $c := (1, 2, \ldots, 2n)$ the long cycle. This corresponds to gluing the edges of a $2n$-gon (the gluing pattern is encoded in the involution $\sigma$).

The *genus* $g$ of a unicellular map is given as $2g = n + 1 - \text{cyc}(\alpha)$ (see also [6, p. 23]). The Harer-Zagier numbers $\varepsilon_g(n)$ count the unicellular maps with $n$ edges and genus $g$ and they have a very nice generating function formula:

$$\frac{1}{(2n-1)!!} \sum_g \varepsilon_g(n) \Phi_{n+1-2g}(X) = \frac{(1+X)^n}{(1-X)^{n+2}}, \tag{1.1}$$

where the polynomials $\Phi_n(X)$ are essentially the Eulerian polynomials; they are defined as follows:

$$\Phi_n(X) = \frac{\sum_{k=0}^{n-1} A(n,k) X^k}{(1-X)^{n+1}} \qquad \text{or equivalently} \qquad \Phi_n(X) = \sum_{k=0}^{\infty} (k+1)^n X^k, \tag{1.2}$$

where $A(n, k)$ is an Eulerian number (i.e., the number of permutations in $S_n$ with $k$ descents).

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**Definition 1.1** (Rotation of constellations)**.** There is a natural cyclic action $\Psi$ of order $2n$ on unicellular maps that corresponds to rotating the polygon. In terms of the constellation, the action is given as

$$\Psi\big[(\sigma,\alpha,c)\big] = (c^{-1}\sigma c, c^{-1}\alpha c, c).$$

To count symmetric 3-constellations, we essentially need to count the factorizations $\sigma\alpha c = \mathbf{1}$ that are fixed by *simultaneous* conjugation by some power $c^N$ of $c$. Equivalently this means counting factorizations $\sigma\alpha c = \mathbf{1}$ in $S_{2n}$ all of whose factors $\sigma,\alpha,c$ also belong to the centralizer $Z_{S_{2n}}(c^N)$. Now, the centralizer $Z_{S_{2n}}(c^N)$ is just the reflection group[1] $G(m,1,2n/m)$ where $m$ is the order of $c^N$ (i.e. $m = 2n/\gcd(2n,N)$). From now on, we will always assume that $N$ divides $2n$ and we will always have $mN = 2n$.

That is, the problem of counting 3-constellations fixed under $\Psi^r$ is equivalent to counting factorizations $\sigma\alpha c = \mathbf{1}$ in $G(m,1,N) = Z_{S_{2n}}(c^r)$ where $\sigma$ belongs to the conjugacy classes of $G(m,1,N) \leq S_{2n}$ into which the class $S_{2n}$ of fixed point free involutions has been decomposed. This problem turns out to be particularly easy because $c = (1,2,\dots,2n)$ is a Coxeter element *also* in $G(m,1,N)$.

There is however a caveat: In the Harer-Zagier formula (1.1), the genus is directly related to the reflection length of $\alpha$ so we can keep track of it with representation theory. Here, the genus of a symmetric constellation is related to the length of $\alpha$ as an element in $S_{2n}$ but this is not the same as (or a multiple of) its length as an element in $G(m,1,N)$. There are two natural approaches here; track the length as an element in $G(m,1,N)$ and interpret it as a *combinatorial* statistic on the map (this succeeds with Theorem 3.8) or define a new length function to track the genus and attempt to express it representation-theoretically (a first attempt here fails; we discuss it in Section 4).

We present the first approach in Section 3, where we interpret the usual length function for $G(m,1,N)$ as a combinatorial (but sadly not topological) statistic on the maps. Then, Zagier's proof [14] of the Harer-Zagier formula (1.1) generalizes essentially out of the box; we have existing theorems that replace all the ingredients of the proof and we prove Theorem 3.8 which is a direct generalization of (1.1).

In Section 4 we define a new length function for $G(m,1,N)$ that corresponds to the topological genus; it is a class invariant and is even somewhat compatible with a factorization in the group algebra of $G(m,1,N)$ which gives us some control over the formulas coming from the Frobenius lemma. It is not clear though what the analog of the Eulerian polynomials $\Phi_n(X)$ of (1.2) should be in this case (nor whether such an analog should a priori exist!).

We first start with a mini review of Zagier's proof of the Harer-Zagier formula (1.1) to set up a pattern of how the proofs would go in these two approaches.

[1] Note that the reflections of $G(m,1,N)$ do not come from transpositions of $S_{2n}$; they come from some elements of type $(2^m, 1^{2n-2m})$ (the transposition-like ones) and some other ones –multiple cycle types– for the diagonal-like reflections; see Example 3.3 and Remark 3.4.

## 2 Main ingredients of Zagier's proof of the Harer-Zagier formula

We give in this section the main ingredients in Zagier's proof (or a re-imagining of Zagier's proof relying more on Jucys-Murphy elements). We will generalize each of them in the next section.

The first is a direct application of the Frobenius lemma from representation theory (recall: $n+1-2g = \mathrm{cyc}(\alpha) = 2n - \ell_R(\alpha)$).

$$\sum_g \varepsilon_g(n) X^{n+1-2g} = \frac{(2n-1)!!}{(2n)!} \cdot \sum_{\chi \in \widehat{S_{2n}}} \chi(\sigma)\chi(c) \cdot \widetilde{\chi}\left(\sum_{w \in S_{2n}} w X^{2n-\ell_R(w)}\right), \tag{A}$$

where $\sigma$ is any fixed point free involution in $S_{2n}$, $c$ any fixed long cycle, and $\widetilde{\chi}$ denotes the normalized character $\chi$ (i.e. $\widetilde{\chi}(a) := \chi(a)/\chi(1)$ for an element $a \in \mathbb{C}[S_{2n}]$).

The second ingredient is a well known factorization in the symmetric group algebra:

$$\sum_{w \in S_{2n}} w X^{2n-\ell_R(w)} = X(X+J_2)(X+J_3)\cdots(X+J_{2n}), \tag{B1}$$

where $J_i := (1i) + \cdots + (i-1i)$ is the $i$-th Jucys-Murphy element. As an application of this factorization we know for instance that the normalized traces appearing in (A) are just binomials:

$$\frac{1}{(2n)!} \cdot \widetilde{\chi}_k\left(\sum_{w \in S_{2n}} w X^{2n-\ell_R(w)}\right) = \binom{X+2n-1-k}{2n}, \tag{B2}$$

where $\chi_k$ is the $k$-th exterior power of the reflection representation of $S_{2n}$ (it is a direct application of the Murnaghan-Nakayama rule that only these irreducible characters are non-zero on the long cycle $c$).

The third ingredient is that the eulerian polynomials of (1.2) give exactly the change-of-basis between the binomials in $X$ that appear above and the monomials $X^n$:

$$\sum_{k=1}^{n} \varepsilon_k X^k = \sum_{k=1}^{n} b_k \binom{X+n-k}{k} \quad \text{if and only if} \quad (1-X)^{n+1} \sum_{k=1}^{n} \varepsilon_k \Phi_k(X) = \sum_{k=1}^{n} b_k X^{k-1}. \tag{C}$$

This has many proofs but it is very conveniently stated in Theorems 2.5 and 2.10 in [8].

The final ingredient is the usual relation (as in [2] or [3]) between the characters $\chi$ such that $\chi(c) \neq 0$, the Coxeter numbers $c_\chi = k(2n)$, the exterior powers $\chi_k$, and hence the matrix of an element in the reflection representation of $S_{2n}$:

$$\sum_{\chi \in \widehat{S_{2n}}} \chi(\sigma)\chi(c) X^{\frac{c_\chi}{2n}} = \sum_{k=0}^{2n-1} \chi_k(\sigma)(-1)^k X^k = \frac{\mathfrak{p}(\sigma; X)}{1-X}, \tag{D}$$

where $\mathfrak{p}(\sigma; X)$ is the characteristic polynomial of $\sigma$ in the *standard* $(2n)$-dimensional representation of $S_{2n}$. Together (A),(B2),(C),(D) give us the Harer-Zagier formula (1.1) because $\mathfrak{p}(\sigma; X) = (1 - X^2)^n$.

# 3 Counting symmetric maps keeping track of $G(m, 1, N)$-length

In this section we generalize the Harer-Zagier formula (1.1) in a way that has all of the ingredients of Zagier's proof from the previous section working out of the box. To have a *meaningful interpretation* of the theorem however we will give first a combinatorial interpretation of the $G(m, 1, N)$-length.

Recall that the for the 3-constellation $\boldsymbol{\pi} = (\sigma, \alpha, c)$ the number $\text{cyc}(\alpha)$ of cycles of $\alpha$ equals the number of vertices $\boldsymbol{v}(\boldsymbol{\pi})$ of the combinatorial map $\boldsymbol{\pi}$ and also that

$$n + 1 - 2g = 2n - \ell_{S_{2n}}(\alpha) = \text{cyc}(\alpha) = \boldsymbol{v}(\boldsymbol{\pi}).$$

So, then the Harer-Zagier formula (1.1) can be rephrased as

$$\frac{1}{(2n-1)!!} \sum_{v} \mathcal{E}_v(n) \Phi_v(X) = \frac{(1+X)^n}{(1-X)^{n+2}}, \tag{3.1}$$

where $\mathcal{E}_v(n) = \varepsilon_{(n+1-2v)/2}(n)$ counts the number of unicellular maps $\boldsymbol{\pi}$ witn $n$ edges and $v$ vertices.

Now, we will give an explicit definition of unicellular maps with rotational symmetry at least $m$:

**Definition 3.1.** Let $n, m, N$ be positive integers such that $mN = 2n$. We denote by $C^m(N)$ the number of 3-constellations $\boldsymbol{\pi} = (\sigma, \alpha, c)$ with factors from $S_{2n}$ that are fixed by the operation $\Psi^N$ (i.e. have symmetry at least $m$):

$$C^m(N) = \Big\{ (\sigma, \alpha) \in S_{2n}^2 \mid \sigma\alpha c = \sigma^2 = \mathbf{1},\ \ell_{S_{2n}}(\sigma) = n,\ c^{-N}\sigma c^N = \sigma,\ c^{-N}\alpha c = \alpha \Big\}.$$

As we mentioned earlier, we can enumerate $C^m(N)$ by counting certain factorizations in $G(m, 1, N)$. The factors $\sigma, \alpha, c$ are still elements of $G(m, 1, N)$ and $c$ is its Coxeter element, but the class in $S_{2n}$ of fixed point free involutions $\sigma$ breaks into multiple conjugacy classes (see Remark 3.4) and the new length $\ell_{G(m,1,N)}(\alpha)$ is not a function of $g$ (or equivalently $\boldsymbol{v}(\boldsymbol{\pi})$). For this reason we define these two statistics:

**Definition 3.2.** Let $n, m, N$ be positive integers such that $mN = 2n$ and let $\sigma$ be a fixed point free involution of $S_{2n}$ such that $c^{-N}\sigma c^N = \sigma$. We write $\boldsymbol{d_m(\sigma)}$ for the number of $\Psi^N$-orbits of *centrally symmetric* 2-cycles of $\sigma$. (A *centrally symmetric* transposition is one of the form $(i, n+i)$.)

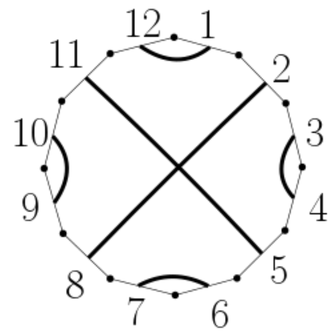


**Figure 1:** For the involution $\sigma$ of the figure, we have $d_4(\sigma) = 1$ but $d_2(\sigma) = 2$.

**Example 3.3.** Consider the involution $\sigma := (1,12)(2,8)(3,4)(5,11)(6,7)(9,10)$ of $S_{12}$. There are two *centrally symmetric* 2-cycles: $(2,8)$ and $(5,11)$. The involution is symmetric both under $\Psi^3$ (conjugation by $c^3$ or rotation of order $m = 4$) and under $\Psi^6$ (conjugation by $c^6$ or rotation of order $m = 2$). But the cycles $(2,8)$ and $(5,11)$ form two orbits under $\Psi^6$ but only one orbit under $\Psi^3$. See Figure 1.

**Remark 3.4** ($d_m$ detects conjugacy class in $G(m,1,N)$)**.** The point of this definition is that it detects the conjugacy class of the involution $\sigma$ as an element of $G(m,1,N)$. **The number $d_m(\sigma)$ counts on how many indices from $1$ to $N$ the involution $\sigma$ acts diagonally-like (maps $i$ to $-i$).** For Example 3.3 above, the centralizer $Z_{S_{2n}}(c^3)$ is isomorphic to the group $G(4,1,3)$ where the coordinates of the (3-dimensional ambient space) correspond to the three sets $\{1,4,7,10\}$, $\{2,5,8,11\}$, $\{3,6,9,12\}$. In this case $\sigma$ becomes $(1,3^{-i})(2,\bar{2})$: the first 2-cycle $(1,3^{-i})$ corresponds to the part $(1,12)(4,3)(7,6)(10,9)$ and the 2-cycle $(2,\bar{2})$ corresponds to the part $(2,8)(5,11)$. Then, the $d_4$ value here is $d_4(\sigma) = 1$ because the involution $\sigma$ has a single *diagonal* position in $G(4,1,3)$.

Similarly the centralizer $Z_{S_{2n}}(c^6)$ is isomorphic to the group $G(2,1,6)$ with coordinates corresponding to the three sets $\{1,7\}$, $\{2,8\}$, $\{3,9\}$, $\{4,10\}$, $\{5,11\}$, $\{6,12\}$. In this case $\sigma$ becomes $(1,\bar{6})(2,\bar{2})(3,4)(5,\bar{5})$ and thus $d_2(\sigma) = 2$ since $\sigma$ has two *diagonal* positions in $G(2,1,6)$.

We need to also replace the quantity $v(\boldsymbol{\pi})$ (the number of vertices of the map $\boldsymbol{\pi}$) with a new object that keeps track of the rotational symmetry of the vertices of the polygon that were identified into vertices of the map.

**Definition 3.5.** For any 3-constellation $\boldsymbol{\pi} = (\sigma,\alpha,c)$ in $S_{2n}$, and any numbers $m,N$ such that $mN = 2n$, we define $v^m_{\text{free}}(\boldsymbol{\pi})$ to be the number of vertices of $\boldsymbol{\pi}$ (equivalently cycles of $\alpha$) that are not fixed by *any* power of $\Psi^N$ (apart from of course $\Psi^{Nm} = \text{Id}$).

**Proposition 3.6.** *If a 3-constellation $\boldsymbol{\pi} = (\sigma,\alpha,c)$ in $S_{2n}$ is fixed under some power $\Psi^N$, then if $m$ is such that $mN = 2n$,*

$$\ell_{G(m,1,N)}(\alpha) = \frac{2n - v^m_{\text{free}}(\boldsymbol{\pi})}{m}.$$

Before finally stating the main theorem of this section, we need to define the generalizations of the polynomials $\Phi_n(X)$ of (1.2). We will be using a well known generalization of Eulerian polynomials for $G(m,1,N)$ that encodes the notion of descent due to Steingrímsson [12].

**Definition 3.7.** For any two positive integers $m, N$ we define the polynomials

$$\Phi_{m,N}(X) = \frac{\sum_{k=0}^{N} A(m,N,k)X^k}{(1-X)^{N+1}} \quad \text{or equivalently} \quad \Phi_{m,N}(X) = \sum_{k=0}^{\infty}(mk+1)^N X^k,$$

where $A(m,N,k)$ is the number of elements in $G(m,1,N)$ with $k$ descents, see [12, Thm. 17].

With these interpretations, we are ready to state and give a (sketch of the) proof of the following generalization of the Harer-Zagier theorem (1.1) that counts maps that remain invariant under a given rotation of the initial polygon.

**Theorem 3.8.** *For any $n, m, N, k \in \mathbb{Z}_{>0}$ such that $2n = mN$, the numbers $\mathcal{E}_{k,v}(m,N)$ of 3-constellations $\boldsymbol{\pi} = (\sigma, \alpha, c)$ in $S_{2n}$ with $d_m(\sigma) = k$ and $\boldsymbol{v}^m_{\text{free}}(\boldsymbol{\pi}) = mv$ (see Defn. 3.2 and Defn. 3.5) such that $\Psi^N(\boldsymbol{\pi}) = \boldsymbol{\pi}$ (see Defn. 1.1) can be calculated via:*

$$\frac{1}{\binom{N}{k}\cdot(N-k-1)!!\cdot m^{\frac{N-k}{2}}}\sum_{v}\mathcal{E}_{k,v}(m,N)\cdot\Phi_{m,v}(X) = \frac{1}{1-X}\cdot\left(\frac{1+X}{1-X}\right)^{\frac{N-k}{2}},$$

*where the polynomials $\Phi_{m,v}(X)$ are as in Defn. 3.7.*

*Sketch.* All the ingredients (A),(B2),(C),(D) are readily available. (A) is just the Frobenius lemma. For (B2) see [8, Prop. 3.2] but it can also be shown using the following version of (B1):

$$\sum_{w\in G(m,1,N)} wX^{N-\ell_{G(m,1,N)}(w)} = (X+J_1)(X+J_2)\cdots(X+J_n),$$

where $J_i = (1,i) + \cdots + (i-1, i^{\bar{\xi}}) + (i, i^{\xi}) + \ldots + (i, i^{\bar{\xi}})$ are a version of the JM elements. The approach of [10, Prop. 4.8] expresses the character values on these generalized Jucys-Murphy elements as certain content calculations, see also [9, Section 4.2] or [15].

The change-of-basis (C) is in Theorems 3.17 and 3.18 of [8]. The final ingredient (D) comes from our previous work, joint with Chapuy, in [2, Section 9.5.2] where we prove an *equality* in $G(m,1,N)$ between $\sum \chi(c)\chi$ and a virtual character that involves the exterior powers of certain $N$-dimensional representations that are analogues of the standard representation of $S_N$. □

**Remark 3.9.** The genus 0 case, or equivalently $\text{cyc}(\alpha) = n+1$, appears only if $\boldsymbol{v}_{\text{free}}(\boldsymbol{\pi}) = n+1$ (no symmetry) or $\boldsymbol{v}_{\text{free}}(\boldsymbol{\pi}) = n$ ($\boldsymbol{\pi}$ has some symmetry). In this way, Theorem 3.8 recovers the known symmetry count in the form of a CSP [11, §7] in the genus-0 case (there the matchings must be non-crossing and determine a (different) noncrossing partition of the odd vertices $1, 3, \ldots 2n-1$; it is this object that is studied in [11]).

**Remark 3.10.** The approach described above can give a complete version of Zagier's main theorem from [14] (i.e. for any conjugacy class of $G(m,1,N)$ not just the fixed point free involutions).

**Remark 3.11.** The approach of this section can be generalized to other factorization counting questions, where conjugation by the long cycle is a natural symmetry. For instance, in works of Goupil-Schaeffer [4] and Bernardi-Morales [1], one could try to count symmetric factorizations by transfering the question to some $G(m,1,n)$ group. Factorizations of the Coxeter element $c \in G(m,1,n)$ have been extensively studied by Lewis-Morales, where the authors also observe [7, §8.2] in their setting that the $G(m,1,n)$-factorizations cannot keep track of the topological genus of a corresponding map.

# 4 Counting symmetric maps keeping track of genus

The main disadvantage of Theorem 3.8 is that the enumeration cannot keep track of the topological genus of the map $\boldsymbol{\pi}$. We discuss in this section a partial attempt to resolve this. We define a new length function in $G(m,1,N)$ given as

$$\ell_{sp}(w) := \ell_{S_{mN}}(w),$$

that is the *symmetric length* of $w \in G(m,1,N)$ is its length as an element of $S_{mN}$. Notice that this is a class function since if two elements are conjugate in $G(m,1,N)$ then they are also conjugate in $S_{mN}$ hence have the same length.

Then, a generalization of (1.1) in the spirit of Theorem 3.8 but using $\ell_{sp}(w)$ instead of $\ell_{G(m,1,N)}(w)$ would rely on understanding

$$\sum_{\chi \in \widehat{G(m,1,N)}} \chi(\sigma_k)\chi(c) \cdot \sum_{w \in G(m,1,N)} \frac{\chi(w)}{\chi(1)} X^{\ell_{sp}(w)},$$

where $\sigma_k$ is any involution with $d_m(\sigma_k) = k$.

It is not difficult to see that there is a factorization

$$\sum_{w \in G(m,1,N)} wX^{\ell_{sp}(w)} = \left[\mathbf{1} + X^{m-1}(11^{\xi}) + \ldots + X^{m-1}(11^{\bar{\xi}})\right] \times$$
$$\times \left[\mathbf{1} + X^{m}(12) + \ldots + X^{m}(12^{\xi}) + \ldots + X^{m-1}(22^{\bar{\xi}})\right] \cdots$$

where each reflection $\tau$ contributes the term $X^{\ell_{sp}}(\tau)$.

This factorization might be seen as an analogue of $(B1)$ and we can certainly calculate the corresponding traces for irreducible characters (either manually or by the techniques of [13, Lemma 3.7], or even by following Jucys original argument [5, Section 4] and relying on existing determinations of the eigenvalues of these generalized Jucys-Murphy elements on eigenvectors indexed by tuples of Young tableau as for instance [10]).

However, we have no analogue of $(B2)$: Even though Sage experiments suggest that we always have nice formulas for $\sum_{w\in G(m,1,N)} \widetilde{\chi}(w)X^{\ell_{sp}}(w)$, it is not clear that there exists a change of basis analogous to (C) (or even that one *might exist*: we need to transform more than $n$ polynomials; the corresponding polynomials with $X^{\ell_{G(m,1,N)}}$ depend only on the Coxeter number of $\chi$ when $\chi(c)\neq 0$ but with $\ell_{sp}$ this is no longer true.


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