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From linear programming to colliding particles

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Abstract. Although simplices may appear trivial in the context of linear optimization, the simplex algorithm, guided by a pivot rule, can exhibit remarkably intricate dynamics on them when solving linear programs. In this paper we study the behavior of max-slope pivot rules on (products of) simplices and describe the associated pivot rule polytopes. For simplices, the pivot rule polytopes are combinatorially isomorphic to associahedra. To prove this correspondence, we interpret max-slope pivot rules in terms of the combinatorics of colliding particles on a line. For prisms over simplices, we recover Stasheff's multiplihedra. For products of two simplices we get new realizations of constrainahedra, that capture the combinatorics of certain particle systems in the plane.

Keywords: linear programming, geometry of pivot rules, particle collisions, associahedra, multiplihedra, constrainahedra

1 Introduction

A linear program (LP) is an optimization problem of the form

$$\begin{array}{lrcl} \text{maximize} & c_1x_1 + \cdots + c_nx_n & & \\ \text{subject to} & a_{i1}x_1 + \cdots + a_{in}x_n & \leq & b_i \quad \text{for } i = 1,\ldots,N\,. \end{array}$$

Geometrically, the set $P \subset \mathbb{R}^n$ of feasible solutions is a polyhedron and a generic objective function $c = (c_1, \ldots, c_n)$ induces an orientation on the vertex-edge graph $G(P)$ of P. The orientation is acyclic with a unique sink at the optimal vertex v_{opt}. The simplex algorithm starts at a given vertex v of P and proceeds along directed edges to v_{opt}. The choice of which edges to pursue is governed by a pivot rule.

A polytope $P \subset \mathbb{R}^n$ is a *simplex* if its vertices are affinely independent. Simplices are trivial from an optimization viewpoint as any two vertices of P are adjacent. However, sophisticated algorithms such as the simplex algorithm can exhibit complex and interesting dynamics on trivial instances. In this paper, we describe a beautiful and unexpected

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connection between the behavior of certain pivot rules on (products of) simplices and the combinatorics of colliding particles.

A pivot rule is memory-less if the decisions it makes on a linear program (P, c) are captured by an arborescence (or rooted tree) on $G(P)$; see, for example, beginning of Section 3. With De Loera, in [2] we studied families of memory-less pivot rules that are parametrized by weight vectors w. We showed that for any linear program (P, c) there is a polytope $\Pi(P, c)$, the *pivot rule polytope*, whose vertices correspond to the arborescences of (P, c) induced by the family of pivot rules. The facial structure of $\Pi(P, c)$ reflects the relation between the different rules on (P, c). For the *max-slope* pivot rule, a generalization of the shadow vertex algorithm of Gass and Saaty [9] that we introduced in [2], we were surprised to observe that the numbers of max-slope arborescences for simplices are given by the Catalan numbers. Much of the combinatorics surrounding the Catalan numbers is famously embodied by the **associahedron** Asso_{n-2}, a certain partially ordered set that is ubiquitous in geometric and algebraic combinatorics. There are many different realizations of Asso_{n-2} as the face lattice of a simple $(n-2)$-dimensional polytope (see [6]) and the present paper adds a new construction to the list.

Theorem 1.1. *Let P be an $(n-1)$-dimensional simplex and c a generic objective function. Then the max-slope pivot rule polytope $\Pi(P, c)$ is combinatorially isomorphic to the $(n-2)$-dimensional associahedron Asso_{n-2}.*

Associahedra describe the combinatorics of colliding particles. Consider n distinct and ordered particles on the real line. Particles move, collide, and merge until there is a single particle left. The various collisions can be time-independently recorded by a *bracketing*. For example (12)(345) states that at some point particles 1 and 2 collided and, before or after, 3, 4, and 5 simultaneously collided. Eventually, the two remaining particles collided. The associahedron Asso_{n-2} is the set of bracketings of $123\ldots n$ partially ordered by refinement. In order to prove Theorem 1.1, we describe a geometric correspondence between max-slope arborescences and bracketings. For that, the objective function c gives rise to velocities for the n particles and the weights w give each particle a location at time $t = 0$. For $t > 0$, the particles start to move from their locations at constant velocity. If two particles collide, the slower particle is absorbed by the faster one, which continues at its original velocity. For $t \gg 0$, only particle n is left. We record the particle $\mathcal{A}(i)$ that absorbs the particle i. Towards a proof of Theorem 1.1, we show that these maps $\mathcal{A}$, called *collision patterns*, are in bijection with bracketings and are precisely the max-slope arborescences of an $(n-1)$-simplex with objective function c.

Bottman and Poliakova [5] studied a more general setup for particle collisions. For $m, n \geq 1$, they consider $m \cdot n$ particles sitting at the intersections of m horizontal and n vertical lines in the plane. The particles are allowed to move horizontally or vertically but they must retain their colinearities. The collisions can be recorded by *rectangular brackets* or, equivalently, by a partially ordered set on the (spaces between) the horizontal and vertical lines. The resulting poset of rectangular bracketings is called the **constrainahedron** $C(m,n)$. For $m = 1$, this is the associahedron. For $m = 2$, $C(2,n)$ is isomorphic to the *multiplihedron*, a poset first described by Stasheff [16] and realized as a polytope by Forcey [8]. It is shown in [5], that $C(m,n)$ is the face poset of a generalized permutahedron [12]. Chapoton and Pilaud [7] introduced a remarkable operation on products of generalized permutahedra, called *shuffle products*, and gave a different realization of $C(m,n)$ as the shuffle product of Loday associahedra. We give new realizations of constrainahedra that, in particular, are not generalized permutahedra.

Theorem 1.2. *Let $P_{m-1,n-1}$ be the product of an $(m-1)$-simplex and an $(n-1)$-simplex and let c be a generic objective function. Then the max-slope pivot rule polytope $\Pi(P_{m-1,n-1}, c)$ is combinatorially isomorphic to the (m,n)-constrainahedron $C(m,n)$.*

The combinatorial construction of constrainahedra was motivated by questions in homotopical algebra [11] as well as Gromov compactifications of configuration spaces. More precisely, Bottman [4] constructed 2-*associahedra* as posets capturing the behavior of ordered particles on parallel lines in the plane without colinearities. Bottman asked in Section 1.3 of [4] whether 2-associahedra are face posets of convex polytopes and notes in particular that a connection to fiber polytopes would be especially interesting and could lead to an interpretation in terms of Fukaya categories. In [2] it is highlighted that max-slope pivot rule polytopes are generalizations of monotone path polytopes, which are fiber polytopes. We hope that our results provide a new point of view on the realizability of 2-associahedra.

Our techniques generalize to higher products of simplices.

Theorem 1.3. *Let $(\mathbb{A}, \cdot)$ be a non-associative monoid and let $f_1, \ldots, f_k : \mathbb{A} \to \mathbb{A}$ be morphisms. The vertices of the max-slope pivot polytope of the Cartesian product of an $(n-1)$-dimensional simplex and a k-dimensional cube are in bijection with the possible ways of evaluating*

$$(f_{\sigma(1)} \circ f_{\sigma(2)} \circ \cdots \circ f_{\sigma(k)})(a_1 \cdot a_2 \cdots a_n),$$

where $a_1, \ldots, a_n \in \mathbb{A}$ and $\sigma \in \mathfrak{S}_k$ is a permutation.

These are the vertices of the (m,n)-multiplihedra of [7]. Pilaud and Poullot [10] show that max-slope pivot rule polytopes of higher products of simplices are isomorphic to shuffle products of associahedra.

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2 Max-slope pivot rule polytopes

We recall max-slope pivot rules and pivot rule polytopes in Section 2 and refer to [2, 3] for proofs. Let $P \subset \mathbb{R}^n$ be a convex polytope with vertices $V(P)$ and edges $E(P)$. We call $c \in \mathbb{R}^n$ **edge-generic** if $\langle c, u \rangle \neq \langle c, v \rangle$ for all $uv \in E$. The vector c defines an objective function $x \mapsto \langle c, x \rangle$ and (P, c) is a **linear program** (LP).

Let $v_{\text{opt}} \in V(P)$ be the unique maximizer of c over P. A **c-arborescence** is a map $\mathcal{A} : V(P) \setminus v_{\text{opt}} \to V(P)$ such that $\mathcal{A}(v)$ is a c-improving neighbor of $v \neq v_{\text{opt}}$. If c is clear from the context, we simply call $\mathcal{A}$ an arborescence. Arborescences encode the behavior of memory-less pivot rules on the linear program (P, c): From a starting vertex $v \in V(P)$, the simplex algorithm constructs a monotonically increasing path $v = v_0 v_1 \ldots v_m = v_{\text{opt}}$ in the graph of P that satisfies $v_i = \mathcal{A}(v_{i-1})$ for all $i = 1, \ldots, m$.

The max-slope pivot rule introduced in [2] generalizes the well-known shadow-vertex simplex algorithm. For generic $w \in \mathbb{R}^n$ we define the **max-slope** arborescence $\mathcal{A}^w$ of (P, c) by

$$\mathcal{A}^w(v) \ := \ \operatorname{argmax} \left\{ \frac{\langle w, u - v \rangle}{\langle c, u - v \rangle} : uv \in E(P), \langle c, u \rangle > \langle c, v \rangle \right\} . \tag{2.1}$$

As the notation suggests, argmax returns the improving neighbor of v that maximizes the given quantity. In particular *genericity* here means that the maximum is attained at a unique neighbor u for every vertex v. A geometric interpretation for a max-slope arborescence can be given as follows. Define the linear projection $\pi : \mathbb{R}^n \to \mathbb{R}^2$ by $\pi(x) = (\langle c, x \rangle, \langle w, x \rangle)$. The projection of an edge $uv \in E$ to the plane has a well-defined slope with respect to the x-axis and $\mathcal{A}^w(v)$ selects the edge with maximal slope.

For a c-arborescence $\mathcal{A}$ we define

$$\psi(\mathcal{A}) \ := \ \sum_{v \neq v_{\text{opt}}} \frac{\mathcal{A}(v) - v}{\langle c, \mathcal{A}(v) - v \rangle} . \tag{2.2}$$

and the **max-slope pivot rule polytope** of (P, c)

$$\Pi(P, c) \ := \ \operatorname{conv} \left\{ \psi(\mathcal{A}) : \mathcal{A} \ c\text{-arborescence of } (P, c) \right\} .$$

This is a $(\dim(P) - 1)$-dimensional polytope, that geometrically encodes the various max-slope arborescences of (P, c). For $w \in \mathbb{R}^n$ we write Q^w for the face of Q that maximizes $x \mapsto \langle w, x \rangle$.

Theorem 2.1 ([2, Theorem 1.4])**.** *Let P be a polytope and c an edge-generic objective function. Then $\Pi(P,c)^w = \{\psi(\mathcal{A}^w)\}$ for every generic w. In particular, the vertices of $\Pi(P,c)$ are in bijection with max-slope arborescences of (P,c).*

We record the following properties of max-slope pivot rule polytopes.

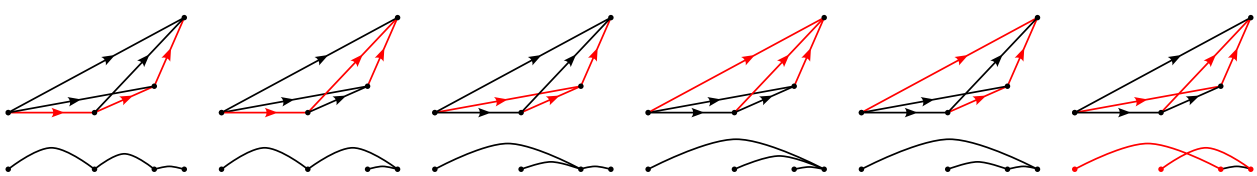
Corollary 2.2. *Let $P \subset \mathbb{R}^n$ be a polytope and c an edge-generic objective function.*

(i) *Let $w \in \mathbb{R}^n$ be generic and $\alpha \in \mathbb{R}$. Then $\mathcal{A}^{w+\alpha c} = \mathcal{A}^w$.*
(ii) *If $h \in \mathbb{R}^n$ such that $x \mapsto \langle h, x\rangle$ is constant on P, then $\Pi(P, c+h) = \Pi(P,c)$.*
(iii) *For any $b \in \mathbb{R}^n$, $\Pi(P+b,c) = \Pi(P,c)$.*
(iv) *If T is an invertible linear transformation, then $\Pi(TP,(T^{-1})^t c) = T\Pi(P,c)$.*

3 Max-slope pivot rules on simplices

Let P be an $(n-1)$-dimensional simplex and c an edge-generic objective function. By Corollary 2.2 we can assume that P is the standard simplex $\Delta_{n-1} := \mathrm{conv}(e_1,\dots,e_n)$ in $\mathbb{R}^n$ and that $c_1 < c_2 < \cdots < c_n$.

For $n \geq 1$, we write $[n] := \{1,2,\dots,n\}$. Since Δ_{n-1} has a complete graph, every vertex e_j with $j > i$ is a c-improving neighbor of e_i. Thus, arborescences of (Δ_{n-1}, c) bijectively correspond to maps $\mathcal{A} : [n-1] \to [n]$ such that $\mathcal{A}(i) > i$ for all i. Since every vertex can choose an improving neighbor independently, there are exactly $(n-1)!$ arborescences of (Δ_{n-1}, c). For the tetrahedron, it turns out that of the six arborescences only five are max-slope arborescences and the figure prompts the following definition:



Definition 3.1 (Noncrossing arborescence)**.** An arborescence $\mathcal{A} : [n-1] \to [n]$ is **noncrossing** if $\mathcal{A}(j) \leq \mathcal{A}(i)$ for all $1 \leq i < j < n$ with $j < \mathcal{A}(i)$.

Theorem 3.2. *Let $\mathcal{A} : [n-1] \to [n]$ be an arborescence. Then $\mathcal{A}$ is a max-slope arborescence for (Δ_{n-1}, c) if and only if $\mathcal{A}$ is noncrossing.*

For the proof, we use a canonical decomposition of noncrossing arborescences; see Figure 1.

Lemma 3.3. *Let $\mathcal{A} : [n-1] \to [n]$ be a noncrossing arborescence on $n \geq 2$ nodes. Let r be minimal with $\mathcal{A}(r) = n$. Then $\mathcal{A}_1 : [r-1] \to [r]$ given by $\mathcal{A}_1(i) := \mathcal{A}(i)$ for $i < r$ and $\mathcal{A}_2 : [n-r-1] \to [n-r]$ given by $\mathcal{A}_2(i) := \mathcal{A}(r+i) - r$ are noncrossing arborescences. Moreover, $\mathcal{A}$ is uniquely determined by $(\mathcal{A}_1, \mathcal{A}_2)$.*

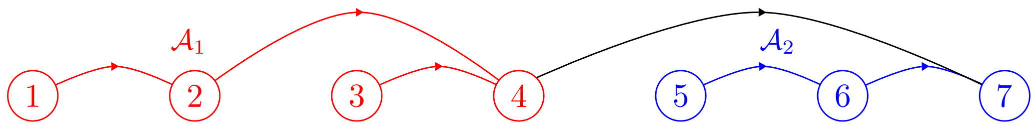


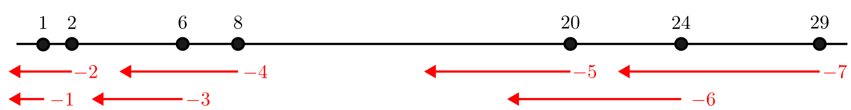
Figure 1: A noncrossing arborescence on 7 nodes. $r = 4$ is minimal with $\mathcal{A}(r) = 7$. The decomposition into $\mathcal{A}_1, \mathcal{A}_2$ is obtained by restricting to $1, \ldots, 4$ and $5, \ldots, 7$. The nodes $1, 3, 5$ are leaves and they are all immediate leaves.

It follows from Lemma 3.3 that the number of vertices is the Catalan number C_n: there is a bijection taking a noncrossing arborescence $\mathcal{A}$ on n nodes to a pair of noncrossing arborescences $(\mathcal{A}_1, \mathcal{A}_2)$ of sizes r and $n - r$, respectively, for $r = 1, \ldots, n - 1$. Since $\mathcal{A}_1, \mathcal{A}_2$ are arbitrary, the number of noncrossing arborescences satisfies the famous Catalan recurrence; see [15]. We call $k < n$ a **leaf** of $\mathcal{A}$ if there is no i with $\mathcal{A}(i) = k$. Induction on n yields

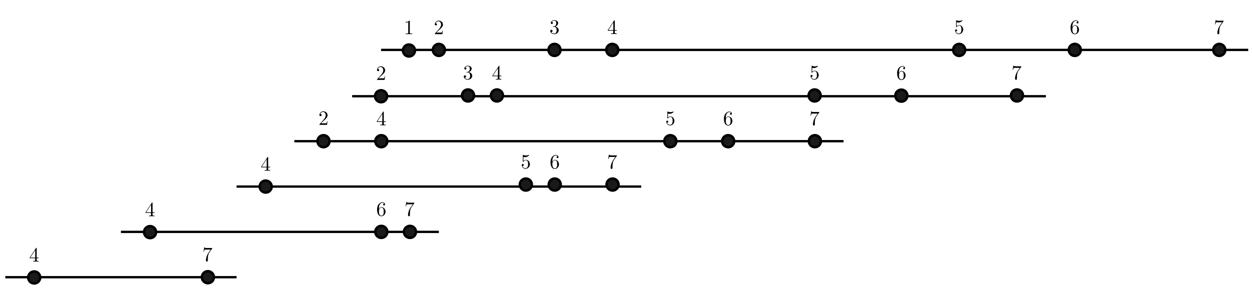
Lemma 3.4. *Let $\mathcal{A}$ be a noncrossing arborescence on $n \geq 2$ nodes. Then there is a leaf k with $\mathcal{A}(k) = k + 1$, called an **immediate leaf**.*

4 Particles with locations and velocities

Consider $n \geq 2$ labelled particles on the real line. Every particle $i = 1, 2, \ldots, n$ has a constant velocity $-c_i < 0$. We assume that the velocities satisfy $0 < c_1 < c_2 < \cdots < c_n$. At time $t = 0$, the particles are at locations $-w_1 \leq -w_2 \leq \cdots \leq -w_n$. For example



Once the particles start moving from their initial locations $-w = (-w_1, \ldots, -w_n)$ with velocities $-c = (-c_1, \ldots, -c_n)$, they will eventually collide and merge. If particles $i < j$ collide, then particle i is absorbed by the faster particle j, which continues at velocity $-c_j$. For $t \gg 0$, the only remaining particle is n. For example



For now we assume that the locations $-w$ are chosen generically, so that at most two particles collide at any given point in time. We record the collisions by a map $\widetilde{\mathcal{A}}^{-w} : [n-1] \to [n]$ that we call a **collision pattern**: If particle i gets absorbed by particle j for the initial locations $-w$, then we set $\widetilde{\mathcal{A}}^{-w}(i) := j$. Note that any $w \in \mathbb{R}^n$ and $\alpha \gg 0$, $-(w - \alpha c)$ is strictly increasing and the associated collision pattern is independent of the choice of α. The connection to max-slope arborescences of simplices is as follows.

Theorem 4.1. *Let $n \geq 2$ and $c \in \mathbb{R}^n$ with $0 < c_1 < \cdots < c_n$. For $\mathcal{A} : [n-1] \to [n]$ and $w \in \mathbb{R}^n$, the following are equivalent.*

i) *$\mathcal{A} = \widetilde{\mathcal{A}}^{-(w-\alpha c)}$ is a collision pattern for n particles with given velocities $-c$ and locations $-(w - \alpha c)$ for some $\alpha \geq 0$.*

ii) *$\mathcal{A} = \mathcal{A}^w$ is a max-slope arborescence of (Δ_{n-1}, c) with respect to weight w.*

Proof. We can assume that $-w_1 < \cdots < -w_n$. If we fix $j > i$ and disregard all other particles for the moment, then the time t_{ij} of collision of i and j satisfies $-w_i - t_{ij}c_i = -w_j - t_{ij}c_j$, that is, $t_{ij} = \frac{-(w_j - w_i)}{c_j - c_i}$. By construction, i will be absorbed by particle j if t_{ij} is minimal among all t_{ik} with $k > i$. Thus, for $i < n$, we observe

$$\widetilde{\mathcal{A}}^{-w}(i) \;=\; \operatorname{argmin}\left\{\frac{-(w_j - w_i)}{c_j - c_i} : j > i\right\} \;=\; \operatorname{argmax}\left\{\frac{w_j - w_i}{c_j - c_i} : j > i\right\} \;=\; \mathcal{A}^w(i)\,,$$

where the last equation is the definition of $\mathcal{A}^w$ in (2.1). □

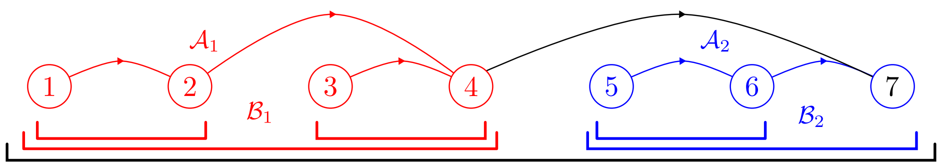
We use noncrossing arborescence and collision patterns interchangeably and write $\mathcal{A}^w$.

In the language of collision patterns, we can also interpret the decomposition of Lemma 3.3. If $\mathcal{A}^w$ is the collision pattern obtained from locations $-w$, then there is a time t_1 at which there are only two particles left. One of these two particles is clearly n, the other is the last particle that is absorbed by n, that is, the minimal $r \geq 1$ with $\mathcal{A}^w(r) = n$. In the time between $t = 0$ and $t = t_1$, the particles $1, \dots, r-1$ get absorbed by r and the particles $r+1, \dots, n-1$ get absorbed by n. The corresponding collision patterns are precisely $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively.

Collisions of particles can be encoded in terms of *bracketings*. A **bracket** is a subset of $[n]$ of the form $[i,j] := \{i, i+1, \dots, j\}$ for $i \leq j$. A **bracketing** $\mathcal{B}$ is a collection of distinct brackets $B_1, \dots, B_m$ such that for every $1 \leq r < s \leq m$, $B_r \subset B_s$ or $B_s \subset B_r$ or $B_s \cap B_r = \varnothing$. A bracket represents particles that have collided with each other at some point in time. If $B_r \subset B_s$ are contained in a bracketing $\mathcal{B}$, then this means at some point all the particles in B_r have collided and later all the particles in B_s will have collided. As there is ultimately only a single particle left, every bracketing has to contain $[1, n]$. The **associahedron** Asso_{n-2} is the set of bracketings of $[n]$ ordered by *reverse* inclusion. The unique *maximal* element is $\{[1, n]\}$. The minimal elements are the complete bracketings of $[n]$, that is, bracketings with $n-1$ brackets.

Theorem 4.2. *Collision patterns on n particles are in bijection with complete bracketings.*

The proof is by induction on n as illustrated here



The proof of Theorem 1.1 proceeds by showing that Asso_{n-2} and $\Pi(\Delta_{n-1}, c)$ have the same vertices-in-facets incidences. To that end, we need to determine the facets of $\Pi(\Delta_{n-1}, c)$, which correspond to the coarsest non-trivial bracketings.

Proposition 4.3. *Let $1 \leq r < s \leq n$ with $(r,s) \neq (1,n)$. The bracketing $\mathcal{B} = \{[r,s],[1,n]\}$ is obtained from locations $-w$ if and only if*

$$-\mu w + \gamma \mathbf{1} - \alpha c \;=\; -w^{r,s} \;:=\; (c_1, \ldots, c_{r-1}, c_s, \ldots, c_s, c_{s+1}, \ldots, c_n)$$

for some $\gamma, \alpha, \mu \in \mathbb{R}$ with $\alpha \geq 0$ and $\mu > 0$.

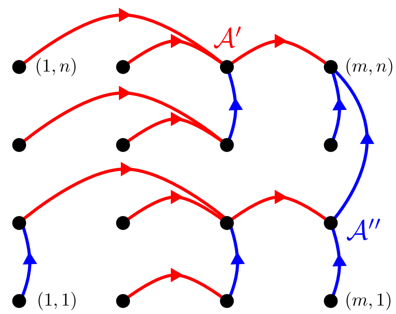
With the locations $-w^{r,s}$, the particles $r, r+1, \ldots, s$ have already collided at time $t = 0$. The remaining particles collide at time $t = 1$.

5 Products of simplices and constrainahedra

For $m, n \geq 1$ consider the Cartesian product $P_{m-1,n-1} = \Delta_{m-1} \times \Delta_{n-1} \subset \mathbb{R}^m \times \mathbb{R}^n$. This is a simple $(m+n-2)$-dimensional polytope whose graph is on nodes $V_{m,n} = [m] \times [n]$ visually arranged in an $m \times n$-grid. An edge-generic objective function $c = (c', c'') \in \mathbb{R}^m \times \mathbb{R}^n$ with $0 < c'_1 < c'_2 < \cdots < c'_m$ and $0 < c''_1 < c''_2 < \cdots < c''_n$ induces an orientation on the graph with oriented vertical edges $ri \to rj$ for $1 \leq i < j \neq n$ and horizontal edges $ri \to si$ for $1 \leq r < s \leq m$. The unique sink is the node mn and we write $V^\circ_{m,n} := V_{m,n} \setminus \{mn\}$.

Arborescences of $(P_{m-1,n-1}, c)$ can be identified with maps $\mathcal{A} : V^\circ_{m,n} \to V_{m,n}$ with the property that if $\mathcal{A}(ri) = sj$ then $r < s$ and $i = j$ or $r = s$ and $i < j$. $\mathcal{A}$ determines an arborescence $\mathcal{A}'$ of (Δ_{m-1}, c') by $\mathcal{A}'(r) := s$ if $\mathcal{A}(rn) = sn$ and, analogously, an arborescence $\mathcal{A}''$ for (Δ_{n-1}, c'').

Every generic $w = (w', w'') \in \mathbb{R}^m \times \mathbb{R}^n$ determines a max-slope arborescence of $P_{m-1,n-1}$ such as the following for $(m,n) = (4,4)$:



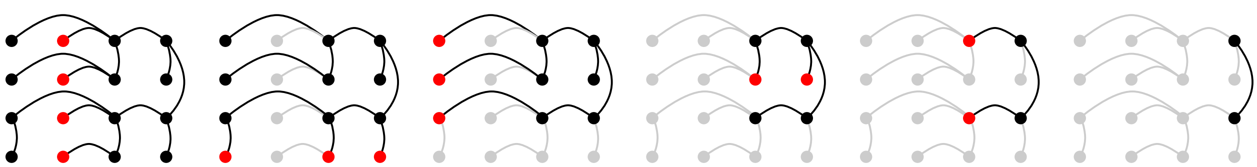
Inspired from this way of illustrating arborescences of $P_{m-1,n-1}$, we call $\mathcal{A}$ **consistent** if $\mathcal{A}(ri) = si$ implies $\mathcal{A}'(r) = s$ and $\mathcal{A}(ri) = rj$ implies $\mathcal{A}''(i) = j$ for all $ri \in V^\circ_{m,n}$. We call $\mathcal{A}$ **grid-noncrossing** if $\mathcal{A}'$ and $\mathcal{A}''$ are noncrossing and there are no $1 \le r \le s < t \le m$ and $1 \le i \le j < k \le n$ with $(r,i) \ne (s,j)$ and with $\mathcal{A}(si) = sk$ and $\mathcal{A}(rj) = tj$.

A node $ri \in V_{m,n}$ is an **immediate leaf** of $\mathcal{A}$ if it has no incoming edges and $\mathcal{A}(ri) = (r+1)i$ or $\mathcal{A}(ri) = r(i+1)$.

Definition 5.1. An arborescence $\mathcal{A} : V^\circ_{m,n} \to V_{m,n}$ is **reducible** if $m = n = 1$ or

- there exists $r \in [m]$ such that ri is a immediate leaf for every i and the restriction of $\mathcal{A}$ to $V^\circ_{m,n} \setminus (r \times [n])$ is reducible, or
- there exists $i \in [n]$ such that ri is a immediate leaf for every r and the restriction of $\mathcal{A}$ to $V^\circ_{m,n} \setminus ([m] \times i)$ is reducible.

The above example is reducible as is illustrated here



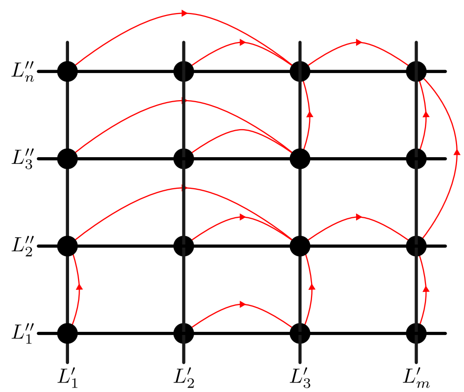
Reducibility implies that $\mathcal{A}$ is consistent and grid-noncrossing but the converse does not hold. For $m = 1$ or $n = 1$, reducible arborescences are precisely the noncrossing arborescences.

Theorem 5.2. *An arborescence $\mathcal{A} : V^\circ_{m,n} \to V_{m,n}$ is a max-slope arborescence of $(P_{m-1,n-1}, c)$ if and only if $\mathcal{A}$ is reducible.*

To view max-slope arborescences of $P_{m-1,n-1}$ as particle collisions, we use the following generalization of particles on a line due to Bottman and Poliakova [5]. We consider m vertical lines $L'_1, \ldots, L'_m$ and n horizontal lines $L''_1, \ldots, L''_n$. The lines are labelled left-to-right and bottom-to-top. We place a particle ri at the point of intersection of L'_r and L''_i. The particle movements are induced by parallel displacements of the $m+n$ lines. In this scenario parallel lines collide and particles contained in the colliding lines merge.

We equip every line L'_r and L''_i with a location $-w'_r$ and $-w''_i$ at time $t = 0$ and we assume that $-w'_1 \le -w'_2 \le \cdots \le -w'_m$ and $-w''_1 \le -w''_2 \le \cdots \le -w''_n$. For

$t > 0$, the lines L'_r move to the left with constant velocity $-c'_r$, the lines L''_i move down with constant velocity $-c''_i$. As before we assume that $0 < c'_1 < c'_2 < \cdots < c'_m$ and $0 < c''_1 < c''_2 < \cdots < c''_n$. If two or more lines collide, they are absorbed by the line with the largest index and this line continues at its original speed. With this setup, we can interpret $\mathcal{A}^{-w}$ as a collision pattern akin to Theorem 4.1. For example



The combinatorics of collisions of $m \cdot n$ particles sitting on m vertical and n horizontal lines was modelled in [5] by certain preorders on $\mathcal{L} = \{L'_1, \ldots, L'_{m-1}, L''_1, \ldots, L''_{n-1}\}$. Recall that a **preorder** on $\mathcal{L}$ is a reflexive and transitive relation $\preceq$. For $x \in \mathcal{L}$, the equivalence class $[x]$ is the collection of elements $y \in \mathcal{L}$ with $x \preceq y$ and $y \preceq x$. On the collection of equivalence classes $\preceq$ yields a partial order. Bottman and Poliakova define a preorder $\preceq$ on $\mathcal{L}$ to be a **good rectangular** preorder [5, Definition 2.1] if

- L'_r and L''_i are comparable for all r, i (orthogonal comparability)

and $L'_o \preceq L'_q$ (respectively $L''_o \preceq L''_q$) if and only if

- $L'_o \preceq L''_p \preceq L'_q$ (respectively $L''_o \preceq L'_p \preceq L''_q$) for some p, or (orthogonal link)
- there is no $\min(o, q) < p < \max(o, q)$ with $L'_o \prec L'_p \succ L'_q$ (respectively $L''_o \prec L''_p \succ L''_q$). (no gaps)

A good rectangular preorder $\preceq_2$ is **refined** by $\preceq_1$ if $x \preceq_1 y$ implies $x \preceq_2 y$ for all $x, y \in \mathcal{L}$.

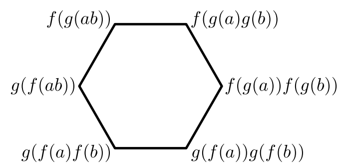
Theorem 5.3 ([5, Theorem 4.1]). *The good rectangular preorders on the set $\mathcal{L}$ of $m + n - 2$ lines partially ordered by refinement is the face poset of an $(m + n - 3)$-dimensional polytope, called the (m, n)-**constrainahedron** $\mathsf{C}(m, n)$.*

In contrast to the case of a single line for which the constrainahedron is the associahedron, constrainahedra are in general not simple polytopes. Facets correspond to **elementary collisions**, which are simultaneous collision of horizontal and vertical lines that are not the result of at least two simultaneous and disjoint collisions of sets of lines. There are three distinct types of elementary collisions. Locations for elementary collisions can be given similar to Proposition 4.3. The proof of Theorem 1.2 follows a similar strategy as that of Theorem 1.1 but is significantly more complicated.

6 Higher products and multiplihedra

In this last section, we focus on the max-slope pivot rule polytopes of the product $Q_{n,k} = \Delta_{n-1} \times [0,1]^k \subset \mathbb{R}^n \times \mathbb{R}^k$ together with an objective function (c, r) with $c = (c_1 < c_2 < \cdots < c_n)$ and $r = (r_1, \ldots, r_k) \in \mathbb{R}^k$ with $r_i > 0$ for all i.

For $k = 1$ the max-slope pivot polytope $\Pi(Q_{n,k}, (c, r))$ is the constrainahedron $C(2, n)$, which is Stasheff's **multiplihedron**; see [16]. For $n = k = 2$ the polytope $\Pi(Q_{n,k}, (c, r))$ is as follows with a vertex labelling that illustrates Theorem 1.3



Consider an evaluation of $(f_{\sigma(1)} \circ f_{\sigma(2)} \circ \cdots \circ f_{\sigma(k)})(a_1 \cdot a_2 \cdots a_n)$ as in Theorem 1.3 with σ fixed. Disregarding the morphisms for a moment, the order in which the $n-1$ multiplications have to be carried out is encoded by a partial order $\preceq_T$ on $[n-1]$ whose Hasse diagram is well-known to be a tree T rooted at the maximum (i.e., the last multiplication to be carried out). Every node in the tree has at most two children. With the labelling given by $[n-1]$, the possible posets are precisely the binary search trees on $n-1$ nodes. Now, for the j-th multiplication, we record the number $\phi(j)$ of morphisms that have to be applied before the multiplication can take place. The poset $([n-1], \preceq_T)$ together with the map $\phi : [n-1] \to \{0, 1, \ldots, k\}$ completely determines the evaluation. Note that if $i \prec j$, then $\phi(i) \leq \phi(j)$ and hence ϕ is an **order-preserving map**. A max-slope arborescence of $(Q_{n,k}, c)$ determines a noncrossing arborescence $\mathcal{A} : [n-1] \to [n]$ and thus a bracketing, which is another representation of $\preceq_T$. The map ϕ comes from the additional information provided by the arborescence.

The **order polynomial** $\Omega_P(l)$ of a partially ordered set $(P, \preceq)$ is a polynomial of degree $|P|$ such that $\Omega_P(l)$ is the number of order-preserving maps $P \to [l]$; see [1, 14]. For fixed $n \geq 2$, we define $V_n(k) := \sum_T \Omega_T(k+1)$, where the sum is over all binary search trees T on $[n-1]$. Our encoding of an evaluation of $f_{\sigma(1)} \circ f_{\sigma(2)} \circ \cdots \circ f_{\sigma(k)}$ on $a_1 \cdot a_2 \cdots a_n$ by a binary search tree and an order-preserving map into $\{0, 1, \ldots, k\}$ yields the following.

Corollary 6.1. *The number of evaluations of* $(f_{\sigma(1)} \circ f_{\sigma(2)} \circ \cdots \circ f_{\sigma(k)})(a_1 \cdot a_2 \cdots a_n)$ *is* $V_n(k)$*. In particular, the number of vertices of* $\Pi(Q_{n,k}, (c, r))$ *is* $k! V_n(k)$*.*

The number of vertices of the (k, n)-multiplihedron is given by Proposition 126 of [7] in terms of generating functions. It is remarkable that the number of vertices for varying k is essentially given by a polynomial. It can be shown that the leading coefficient of $V_n(k)$ is 1 and the constant coefficient is the n-th Catalan number C_n. Experiments show that all coefficients are non-negative and, of course, we tested if the polynomials are real-rooted. This seems to be true for $n \leq 10$ but unknown beyond.

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