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A combinatorial proof of an identity involving Eulerian numbers

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Abstract. We give a combinatorial proof of an identity that involves Eulerian numbers and was obtained algebraically by Brenti and Welker (2009). To do so, we study alcoved triangulations of dilated hypersimplices. As a byproduct, we describe the dual graph of these triangulations for the dilated standard simplex and the hypersimplex in terms of combinatorial objects, and conjecture their structure for dilated hypersimplices.

Keywords: Alcoved polytope, Eulerian numbers, Hypersimplex, Dual graphs.

1 Introduction

Brenti and Welker [2] studied how the numerator of a rational formal power series transforms after taking a subsequence of the coefficients and computing its generating series. In order to do so, they study a linear transformation with fixed bases related to the following combinatorial objects. First, we denote by $[r]^d$ the set of *words* of length d in the elements $[r] := \{1, 2, \ldots, r\}$.

Definition 1.1. For $d, r, i \in \mathbb{N}$ and $d \geq 1$, let

$$\mathfrak{C}(r, d, i) := \left\{ (c_1, c_2, \ldots, c_d) \in \mathbb{N}^d \mid c_1 + c_2 + \cdots + c_d = i \ , \ c_j \leq r \text{ for } 1 \leq j \leq d \right\}.$$

Denote by $C(r, d, i)$ the size of the set $\mathfrak{C}(r, d, i)$.

The other set of objects comes from the power series $\sum_{k\geq 0}(k+1)^d z^k = \frac{A_d(z)}{(1-z)^{d+1}}$ that was considered by Euler while studying the Riemann ζ-function [3]. The numerators $A_d(z)$ are the *Eulerian polynomials* and it is well-known that $a_{d,k}$ counts the permutations of d elements with k descents. These numbers are commonly called *Eulerian numbers*. For our purposes, and to make notation line-up in Section 3, we adopt the following convention.

Definition 1.2. Let $d \geq 1$ and $1 \leq j \leq d$. Define

$$\mathfrak{A}(d, j) := \left\{ \sigma \in \mathfrak{S}_d \;\middle|\; \operatorname{des}(\sigma) = j - 1 \right\}.$$

We denote the cardinality of this sets by $A(d, j) = |\mathfrak{A}(d, j)|$.

Remark 1.3. With the previous definition, the Eulerian numbers are given by $a_{d,k} = A(d,k+1)$. We will refer to the numbers $A(d,k)$ as the Eulerian numbers, but there is a necessary word of caution when interpreting them in terms of permutations.

By considering a change of basis, Brenti and Welker showed the following identities involving the cardinalities of the previous sets. We point out that these two equations follow from diagonalizing a matrix and computing the image of an eigenvector, so both $C(r,d,i)$ and $A(d,j)$ appear due to algebraic considerations.

Proposition 1.4 ([2, Proposition 2.3])**.** *Let $d,r \geq 1$. Then*

$$\sum_{j=0}^{d} C(r-1,d+1,ir-j)A(d,j) = r^d A(d,i) \tag{1.1}$$

for $i = 1,2,\dots,d$. In particular, when $i = 1$,

$$\sum_{j=0}^{d} C(r-1,d+1,r-j)A(d,j) = r^d. \tag{1.2}$$

Given the enumerative nature of the quantities involved in the equations, Brenti and Welker asked for a combinatorial proof of these identities. Our main result is to provide one such proof by constructing suitable bijections. The key idea that allows us connect both sides of the equations is to consider dilations of hypersimplices, in particular their *alcoved triangulations* (see Section 2), since the right-hand side of Equation (1.1) can be interpreted as the (normalized) volume of these dilated polytopes (see Section 3).

Theorem 1.5. *Let $\mathcal{A}(r\Delta_{i,d})$ be the set of alcoves of the r-dilated hypersimplex $\Delta_{i,d}$. There exist bijections*

$$\begin{aligned} \text{words}_i \ &: \ \mathcal{A}(r\Delta_{i,d}) \longrightarrow [r]^d \times \mathfrak{A}(d,i) \\ \text{pair}_i \ &: \ \mathcal{A}(r\Delta_{i,d}) \longrightarrow \bigcup_{j=1}^{d} \mathfrak{C}(r-1,d+1,ir-j) \times \mathfrak{A}(d,j) \end{aligned}$$

from which we obtain a combinatorial proof of Equation (1.1).

This document is organized as follows. In Section 2 we review some relevant results about alcoved polytopes. In Section 3.1 we construct the bijection needed to show Equation (1.2) by considering dilated standard simplices; moreover, we describe the dual graph of the triangulation using the combinatorial objects involved in the bijection. We generalize these ideas in Section 3.2 to build the bijections that prove Equation (1.1) combinatorially, and describe the dual graph of the alcoved triangulation of a hypersimplex using permutations. We conclude with some further directions in Section 4.

Remark 1.6. After completion of this document, we found [7, Proposition 9.4] where the authors give a formula for the volume of *thick hypersimplices* and specializes to Equation (1.1) when $\Phi = A_n$ and the parameters are carefully chosen. We point out that their argument relies on considerations of the alcoved triangulations related to affine Weyl groups and, in contrast, we provide a proof of Equations (1.1) and (1.2) by interpreting alcoves in terms of combinatorial objects. Moreover, the equations are related to recent work on slices of prisms (see [4, Corollaries 1.5 and 1.6]).

2 Alcoved polytopes

Given a set of vectors in $\mathbb{R}^n$, their *convex hull* is the smallest convex set containing all of them. A *polytope* is the convex hull of finitely many vectors in $\mathbb{R}^n$. The *vertices* of P is the smallest set of vectors such that their convex hull equals P. A *lattice polytope* is a polytope whose vertices only have integer coordinates. We focus on a particular family of lattice polytopes known as *alcoved polytopes*. There are two ways in which alcoved polytopes usually appear in the literature, depending on the coordinates that are chosen to describe them.

Definition 2.1. An *alcoved polytope* P is a polytope that has one of the following hyperplane descriptions:

- An (H, z)-representation

$$P = \left\{ \vec{z} \in \mathbb{R}_z^{n-1} : b_{ij} \leq z_i - z_j \leq c_{ij} \ \text{ for } \ 0 \leq i < j \leq n-1 \right\}$$

  where $z_0 := 0$ and $b_{ij}, c_{ij} \in \mathbb{Z}$ for all i and j.

- An (H, x)-representation

$$P = \left\{ \vec{x} \in \mathbb{R}_x^n \ : \ \begin{array}{c} b_{ij} \leq x_{i+1} + \cdots + x_j \leq c_{ij} \ \text{ for } \ 0 \leq i < j \leq n \\ x_1 + x_2 + \cdots + x_n = k \end{array} \right\}$$

  where $k \in \mathbb{Z}$ and $b_{ij}, c_{ij} \in \mathbb{Z}$ for all i and j.

Example 2.2. The alcoved polytope $P = P(b_{ij}, c_{ij})$ in z-coordinates with parameters $b_{0,1} = -4$, $c_{0,1} = -1$, $b_{0,2} = -3$, $c_{0,2} = -1$, $b_{1,2} = -2$, $c_{1,2} = 1$ is depicted in Figure 1. The map $\psi_2(z_1, z_2) = (z_1, z_2 - z_1, 2 - z_2)$ is an affine equivalence to a polytope in x-coordinates. After translating such polytope by the vector $(1, 1, 1)$, we obtain a polytope that lays on the hyperplane $x_1 + x_2 + x_3 = 5$ in $\mathbb{R}_x^3$.

The most important alcoved polytopes for our purposes are the hypersimplices.

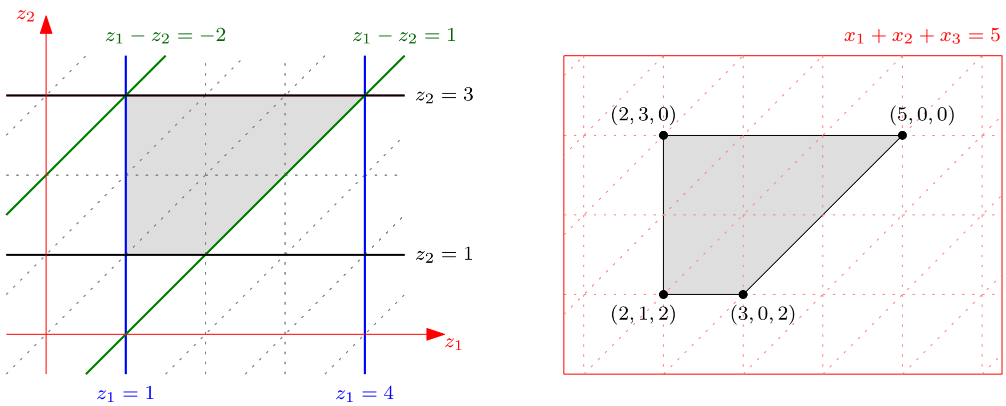


Figure 1: On the left, an alcoved polytope with its explicit (H, z)-representation; the dotted lines represent the elements of the affine Coxeter arrangement of type A_2. On the right, the image of the polytope under the map ψ_2 after translation by $(1, 1, 1)$.

Definition 2.3. The *i-th hypersimplex of dimension d*, denoted by $\Delta_{i,d}$, is the polytope with (H, x)-representation

$$\Delta_{i,d} = \left\{ (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d_x \ : \ \begin{array}{c} 0 \leq x_i \leq 1 \ \text{ for } \ 1 \leq i \leq d \\ x_1 + x_2 + \cdots + x_d = i \end{array} \right\}. \tag{2.1}$$

The *standard simplex of dimension d* is the first hypersimplex of dimension d, that is $\Delta_{1,d}$. It can also be described as the convex hull of the standard basis vectors in $\mathbb{R}^n_x$. A *unimodular simplex* is a polytope S that is affinely equivalent to $\Delta_{1,d}$.

A *subdivision* of a polytope P is a collection of polytopes $\mathfrak{P}$ such that every face of a polytope in $\mathfrak{P}$ is also in $\mathfrak{P}$, any two polytopes in $\mathfrak{P}$ intersect in a common face and the union of all the polytopes in $\mathfrak{P}$ equals P. If all the polytopes in the collection are (unimodular) simplices, we call the subdivision a *(unimodular) triangulation*. From a triangulation, we can encode the adjacency of the maximal simplices in a graph.

Definition 2.4. Let $\mathcal{T}$ be a triangulation of a polytope P. The *dual graph* of the triangulation $G_{\mathcal{T}}$ has vertex set equal to the maximal simplices of $\mathcal{T}$ and two such simplices S_1 and S_2 form an edge, which we denote by $S_1 \sim S_2$, whenever the dimension of the polytope $S_1 \cap S_2$ is $\dim(P) - 1$.

One of the many interesting features of alcoved polytopes is that they come equipped with a unimodular triangulation. We refer to it as the *alcoved triangulation*. It is induced by the *affine Coxeter arrangement of type* A_{n-1} that subdivides $\mathbb{R}^{n-1}_z$ into unimodular simplices called *alcoves* (see Figure 1 for the case $n = 3$).

Definition 2.5. For an alcoved polytope P, let $\mathcal{A}(P)$ be the set of maximal simplices (with respect to inclusion) in the alcoved triangulation of P. We refer to an element $A \in \mathcal{A}(P)$ as an *alcove of* P, and usually identify it with the set of its vertices.

Lam and Postnikov [6] gave the following a combinatorial description of the alcoves of an alcoved polytope with (H, x)-representation.

Definition 2.6. Let $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ be a collection of r-multisets of $\{1, 2, \ldots, n\}$ where for each multiset we assume $I_j = \{I_{j1} \leq I_{j2} \leq \ldots \leq I_{jr}\}$. We say that the collection $\mathcal{I}$ is *sorted* if

$$I_{11} \leq I_{21} \leq \ldots I_{k1} \leq I_{21} \leq I_{22} \leq \ldots \leq I_{kr}.$$

Denote by $M_{\mathcal{I}}$ the *matrix associated to the collection of multisets* $\mathcal{I}$ constructed by using the (ordered) multisets as rows. Then $\mathcal{I}$ is sorted if the concatenation of columns of $M_{\mathcal{I}}$ from left to right and top to bottom is weakly increasing.

Definition 2.7. For an nonnegative integer vector $\vec{a} \in \mathbb{N}^n$ such that $a_1 + a_2 + \cdots + a_n = r$, let $I_{\vec{a}}$ be the r-multiset of $\{1, 2, \ldots, n\}$ with a_i elements "i" for each i. For a collection of vectors $A = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\} \subseteq \mathbb{N}^n$ such that the coordinates of all of them sum to r, define the *collection of multisets of* A as $\mathcal{I}_A = \{I_{\vec{a}_1}, I_{\vec{a}_2}, \ldots, I_{\vec{a}_k}\}$.

Suppose P has a (H, x)-representation in $\mathbb{R}^n_x$ such that all points of P have nonnegative coordinates. If this is not the case, by translating P using the vector $m\vec{1} = (m, m, \ldots, m) \in \mathbb{R}^n_x$ for a sufficiently large $m \in \mathbb{Z}$ we obtain an affinely equivalent alcoved polytope with the desired property. Denote by $Z_P = P \cap \mathbb{Z}^n \subseteq \mathbb{N}^n$ the set of lattice points of P.

Theorem 2.8 ([6, Theorem 3.1]). *Let $P \subseteq \mathbb{R}^n_x$ be an alcoved polytope lying in the hyperplane $x_1 + x_2 + \cdots + x_n = k$ such that all its points have nonnegative coordinates. A simplex with vertices $A = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \subseteq Z_P$ is an alcove in $\mathcal{A}(P)$ if and only if $\mathcal{I}_A$ is a sorted collection of k-multisets.*

3 Alcoved triangulations of dilated hypersimplices

For an alcoved polytope, its *normalized volume* is given by $|\mathcal{A}(P)|$ (see [1, Section 5.4] for the general discussion on *relative volume* and [6, Theorem 3.2] for the particular case of alcoved polytopes). Thus, in order to compute the normalized volume of an alcoved polytope it is enough to understand the set $\mathcal{A}(P)$.

Definition 3.1. Let P be an alcoved polytope. A *labeling* of $\mathcal{A}(P)$ using the elements of a finite set S is a bijection $f : \mathcal{A}(P) \longrightarrow S$.

A famous result from Laplace [8] states that the normalized volume of $\Delta_{i,d}$ is $A(d,i)$. The first triangulation of the hypersimplex that showed this identity combinatorially was constructed by Stanley [9]. We give another combinatorial proof of this result in Section 3.2 by constructing a labeling of $\mathcal{A}(\Delta_{i,d})$ with permutations in $\mathfrak{S}_d$ with $i-1$ descents. Now note that the right-hand side of Equation (1.1) can be rewritten as

$$r^d A(d,i) = r^d \operatorname{vol}(\Delta_{i,d}) = \operatorname{vol}(r\Delta_{i,d})$$

where $r\Delta_{i,d}$ is the dilation of the hypersimplex by a factor of r:

$$r\Delta_{i,d} = \left\{\vec{x} \in \mathbb{R}_x^{d+1} \ : \ 0 \leq x_1\,,\, x_2\,,\,\dots\,,\, x_{d+1} \leq r \quad \text{and} \quad x_1 + x_2 + \cdots + x_{d+1} = ir\right\}.$$

Thus, we want to describe a labelings of the alcoves of $r\Delta_{i,d}$ using suitable objects in order to prove the identity. However, we first consider the case of the standard simplex since it contains important ideas in order to construct the general labelings.

3.1 The dilated standard simplex

3.1.1 Labeling of the alcoves with words

We start by reinterpreting the sorted sets $\mathcal{I}_A$ from Theorem 2.8 using words.

Definition 3.2. Let $\mathcal{I} = \{I_1, I_2, \dots, I_k\}$ be a sorted collection of different r-multisets of $\{1, 2, \dots, n\}$. The *decorated matrix* $\widetilde{M}_{\mathcal{I}}$ is constructed as follows. Arrange the numbers of $M_{\mathcal{I}}$ in a $k \times r$ grid and then (using matrix coordinates)

(a) if $I_{ab} < I_{(a+1)b}$ mark the edge between the cells (a,b) and $(a+1,b)$ in the grid, and

(b) if $I_{nb} < I_{1(b+1)}$ mark the bottom edge of cell (n,b) in the grid.

Example 3.3. Fix $n = 8$ and $r = 6$. Consider the set of points

$$A = \{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4, \vec{a}_5\} = \{(2,1,0,1,1,0,0,1), (2,0,1,1,1,0,0,1), (1,1,0,2,0,1,0,1), (1,1,0,1,1,1,0,1), (1,1,0,1,1,0,1,1)\} \subseteq \mathbb{R}^8$$

The decorated matrix $\widetilde{M}_{\mathcal{I}}$ for $\mathcal{I} = \mathcal{I}_A = \{I_{\vec{a}_1}, I_{\vec{a}_2}, I_{\vec{a}_3}, I_{\vec{a}_4}, I_{\vec{a}_5}\}$ is

1	1	$\underline{2}$	4	5	8
1	$\underline{1}$	$\underline{3}$	4	$\underline{5}$	8
1	2	4	$\underline{4}$	6	8
1	2	4	5	$\underline{6}$	8
1	2	4	5	$\underline{7}$	8

We now examine the case of sorted sets arising from the alcoves of the dilated standard simplex.

Lemma 3.4. *Let $A = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{d+1}\}$ be the set of vertices of an alcove of $r\Delta_{1,d}$ and let $\mathcal{I} = \mathcal{I}_A$ be the associated sorted collection of r-multisets. Then the decorated matrix $\widetilde{M}_{\mathcal{I}}$ satisfies that for each $1 \le i \le d$, there is a unique mark between rows i and $i+1$, and there are no marks in the bottom part of the matrix.*

The previous lemma allows us to define a labeling of $\mathcal{A}(r\Delta_{1,d})$ by recording the position of the unique mark in each of the rows.

Definition 3.5. Let $\mathrm{word}_1 : \mathcal{A}(r\Delta_{1,d}) \to [r]^d$ be the map defined as follows. If A is the set of vertices of an alcove of $r\Delta_{1,d}$ with associated collection of multisets $\mathcal{I} = \mathcal{I}_A$ then $\mathrm{word}_1(A)$ is obtained by reading the column label of the marks of $\widetilde{M}_{\mathcal{I}}$ from top to bottom.

Theorem 3.6. *The map*

$$\mathrm{word}_1 \; : \; \mathcal{A}(r\Delta_{1,d}) \longrightarrow [r]^d$$

is a bijection.

Example 3.7. Fix $d = 4$ and $r = 6$. The set of points

$$A = \{(3,1,1,0,1), (2,2,1,0,1), (2,2,0,1,1), (2,1,1,1,1), (2,1,1,0,2)\}$$

defines an alcove of $6\Delta_{1,4} \subseteq \mathbb{R}^5_x$. The decorated matrix in this case is

$$\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & \underline{1} & 2 & 3 & 5 \\
\hline
1 & 1 & 2 & 2 & \underline{3} & 5 \\
\hline
1 & 1 & 2 & \underline{2} & 4 & 5 \\
\hline
1 & 1 & 2 & 3 & \underline{4} & 5 \\
\hline
1 & 1 & 2 & 3 & 5 & 5 \\
\hline
\end{array}$$

and then $\mathrm{word}_1(A) = 3545 \in [6]^4$.

3.1.2 Labeling of the alcoves with pairs of compositions and permutations

We start by associating a composition to each collection of nonnegative integer vectors.

Definition 3.8. Let $V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\} \subseteq \mathbb{N}^{d+1}$ be a collection of vectors with nonnegative integer coordinates. Define $\mathrm{comp}(V) = (c_1, c_2, \ldots, c_{d+1})$ to be the composition with parts $c_k = \min\left\{(\vec{v}_j)_k \mid j \in [m]\right\}$ where $(\vec{v}_j)_k$ denotes the k-th entry of the vector $\vec{v}_j$.

If we restrict to collections of vertices of alcoves of a dilated standard simplex, we can give a description of the composition using decorated matrices.

Lemma 3.9. *Let $A = \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_k\} \in \mathcal{A}(r\Delta_{1,d})$. Then* $\mathrm{comp}(A) = (c_1, c_2, \dots, c_{d+1})$ *satisfies the following conditions: $c_1 = k$ if the first mark is in column $k+1$ and $c_{d+1} = \ell$ if the last mark is in column $r - \ell$. For $2 \leq i \leq d$, suppose the i-th mark is in column b_i. Then*

$$c_i = \begin{cases} b_i - b_{i-1} - 1 & \text{if the i-th mark is higher than the } (i-1)\text{-th mark} \\ b_i - b_{i-1} & \text{if the i-th mark is higher than the } (i-1)\text{-th mark} \end{cases}$$

Now we associate a permutation to each of the alcoves of the dilated hypersimplices.

Definition 3.10. Let $A = \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_{d+1}\} \in \mathcal{A}(r\Delta_{1,d})$. Define $\sigma_A \in S_d$ to be the permutation such that its one-line notation of is the word obtained from $\widetilde{M}_{\mathcal{I}_A}$ by recording the position of the marks reading the columns from top to bottom and from left to right in the matrix.

Example 3.11. For the alcove $A \in \mathcal{A}(6\Delta_{1,4})$ from Example 3.7, $\mathrm{comp}(A) = 2\,1\,0\,0\,1$, and $\sigma_A = 1\,3\,2\,4$ in one-line notation.

With these two objects we can construct the second labeling that allows us to prove Equation (1.2) combinatorially.

Theorem 3.12. *The map*

$$\mathrm{pair}_1 \; : \; \mathcal{A}(r\Delta_{1,d}) \longrightarrow \bigcup_{j=1}^{d} \mathfrak{C}(r-1, d+1, r-j) \times \mathfrak{A}(d, j)$$

given by $\mathrm{pair}_1(A) = (\mathrm{comp}(A), \sigma_A)$ *is a bijection.*

3.1.3 Dual graph of the triangulation

From Theorem 3.6, the maximal simplices of the alcoved triangulation of $r\Delta_{1,d}$ are labeled by words in $[r]^d$. The following theorem gives a description of the dual graph of this triangulation in terms of words. We show the graph for $r = 2$ and $d = 3$ in Example 4.2.

Definition 3.13. For $r, d \geq 1$, let $G_{r,d}$ be the graph on vertex set $[r]^d$ and edges given by

1. $w_1\, w_2\, \dots\, w_d \;\sim\; (w_d + 1)\, w_1\, w_2\, \dots\, w_{d-1}$ whenever $1 \leq w_d < r$, and
2. $w_1\, \dots\, w_i\, w_{i+1}\, \dots\, w_d \;\sim\; w_1\, \dots\, w_{i+1}\, w_i\, \dots\, w_d$ for any $1 \leq i \leq d-1$ such that $w_i \neq w_{i+1}$.

Theorem 3.14. *Let $\mathcal{T}$ be the alcoved triangulation of $r\Delta_{1,d}$. Then $G_{\mathcal{T}}$ is isomorphic to $G_{r,d}$.*

3.2 The dilated hypersimplex

The map $\sigma_\bullet$ from Definition 3.10 can be defined for alcoves of $\Delta_{i,d}$ in an analogous way. Moreover, this map gives a new combinatorial proof of the fact the Eulerian numbers coincide with the volumes of the hypersimplices.

Theorem 3.15. *For $1 \leq i \leq d$, the map*

$$\sigma_\bullet^{(i)} \ : \ \mathcal{A}(\Delta_{i,d}) \longrightarrow \mathfrak{A}(d,i)$$

is a bijection, and then $\mathrm{vol}(\Delta_{i,d}) = A(d,i)$.

Let $G_{\mathcal{A},i,d}$ be the dual graph of the alcoved triangulation of $\Delta_{i,d}$. We give a description of this graph using permutations. We illustrate $G_{\mathcal{A},2,3}$ in Example 4.2.

Proposition 3.16. *The graph $G_{\mathcal{A},i,d}$ is isomorphic to the graph with vertex set $\mathfrak{A}(d,i)$ where $\{\sigma,\tau\}$ is an edge if and only if the permutations satisfy that*

1. *$\sigma = s_k\tau$ for some $k = 1,2,\ldots,d-1$ (where s_k is the kth simple transposition), or*
2. *if $\sigma = \sigma_1\,\sigma_2\,\ldots\,\sigma_d$ and $\tau = \tau_1\,\tau_2\,\ldots\,\tau_d$ are the one-line notations, and ℓ and m are the indices such that $\sigma_\ell = 1$ and $\sigma_m = d$, then either $\tau_\ell = d$ and $\tau_j = \sigma_j + 1$ for $j \neq \ell$, or $\tau_m = 1$ and $\tau_j = \sigma_j - 1$ for $j \neq m$.*

3.2.1 Labeling of the alcoves with pairs of words and permutations

Theorem 3.6 can be used to label triangulations of dilated polytopes in general as follows. This construction was considered in [5, Section 4] where the authors study the properties of the resulting triangulations.

Observation 3.17. Let P be a polytope of dimension d that has a unimodular triangulation $\mathcal{T} = \{S_i \mid i \in I\}$. Consider the (non-unimodular) triangulation $r\mathcal{T} = \{rS \,|\, S \in \mathcal{T}\}$ of rP. Then we can use the map word_1 from Section 3.1 to construct a labeling $f : \Delta \to I \times [r]^d$ of the unimodular triangulation Δ that arises from alcove-triangulating each simplex $rS \simeq r\Delta_{1,d}$.

Using this observation we can give a labeling of the alcoves of a dilated hypersimplex as follows. Consider $A \in \mathcal{A}(r\Delta_{i,d})$. It satisfies $A \subseteq rB$ where $B \in \mathcal{A}(\Delta_{i,d})$. Define the *permutation associated to* A to be $\tau_A = \sigma_B^{(i)}$. Moreover, there is an affine equivalence $\varphi_B : B \to \Delta_{1,d}$. Through this map, $\varphi_B(A)$ is an alcove of $r\Delta_{1,d}$, and we can compute its word_1. Define the *word of* A to be $\mathrm{word}'_i(A) = \mathrm{word}_1(\varphi_B(A))$. Using these objects, together with Theorem 3.15, we obtain the first labeling of $\mathcal{A}(r\Delta_{i,d})$. We show these calculations explicitly in Example 3.20.

Theorem 3.18. *The map*

$$\mathrm{words}_i \ : \ \mathcal{A}(r\Delta_{i,d}) \longrightarrow [r]^d \times \mathfrak{A}(d,i)$$

defined by $\mathrm{words}_i(A) = (\mathrm{word}'_i(A), \tau_A)$ *is a bijection.*

3.2.2 Labeling of the alcoves with pairs of compositions and permutations

Now we describe a more direct labeling of $\mathcal{A}(r\Delta_{i,d})$ by extending the maps from Section 3.1.2. For alcoves $A \in \mathcal{A}(r\Delta_{i,d})$ we can still construct a composition and a permutation in a similar way as we did for $r\Delta_{1,d}$. We denote by $\mathrm{comp}'(A)$ and σ'_A the composition and permutation (respectively) obtained from $\widetilde{M}_{\mathcal{I}_A}$.

Theorem 3.19. *The map*

$$\mathrm{pair}_i : \mathcal{A}(r\Delta_{i,d}) \longrightarrow \bigcup_{j=1}^{d} \mathfrak{C}(r-1, d+1, ir-j) \times \mathfrak{A}(d, j)$$

given by $\mathrm{pair}_i(A) = (\mathrm{comp}'(A), \sigma'_A)$ *is a bijection.*

The maps from Theorems 3.18 and 3.19 are precisely the bijections mentioned in Theorem 1.5.

Example 3.20. Fix $d = 5$, $r = 4$, $i = 2$ and $j = 3$ as parameters in the previous theorems. The set of points

$$A = \{(2,3,0,1,2,0), (2,2,1,1,2,0), (2,2,0,2,2,0), \\ (2,2,0,2,1,1), (1,3,0,2,1,1), (1,3,0,1,2,1)\} \subseteq \mathbb{R}^6_x$$

defines an alcove in $\mathcal{A}(4\Delta_{2,5})$. It satisfies $\mathrm{conv}(A) \subseteq 4A^{(2)}_{31245}$ where $A^{(2)}_{\bullet}$ is the inverse of $\sigma^{(2)}_{\bullet}$ from Theorem 3.15; therefore $\tau_A = 31245$. Moreover, $A^{(2)}_{31245} = \mathrm{conv}(\mathfrak{B})$ with

$$\mathfrak{B} = \{(1,1,0,0,0,0), (1,0,1,0,0,0), (1,0,0,1,0,0), \\ (0,1,0,1,0,0), (0,1,0,0,1,0), (0,1,0,0,0,1)\}.$$

Using $\mathfrak{B}$ as a basis for $\mathbb{R}^6_x$, the elements of A can be rewritten as

$$A \cong \{(1,0,1,0,2,0), (1,1,0,1,1,0), (0,0,2,0,2,0), \\ (0,0,2,0,1,1), (0,0,1,1,1,1), (0,0,1,0,2,1)\}$$

where $\cong$ denotes the change of basis from the standard basis to $\mathfrak{B}$. From this description we see that the decorated matrix of A relative to $4A^{(2)}_{31245}$ is

1	3	5	5
2	3	5	5
3	3	5	5
3	3	5	6
3	4	5	6
3	5	5	6

Hence we obtain $\text{words}_2(A) = (1\,1\,4\,2\,2\,,\,3\,1\,2\,4\,5) \in [4]^5 \times \mathfrak{A}(5,2)$.

To compute $\text{pair}_2(A)$ we consider the decorated matrix of $\mathcal{I}_A$ with respect to the canonical basis of $\mathbb{R}^6_x$. That is,

1	1	2	2	2	4	5	5
1	1	2	2	3	4	5	5
1	1	2	2	4	4	5	5
1	1	2	2	4	4	5	6
1	2	2	2	4	4	5	6
1	2	2	2	4	5	5	6

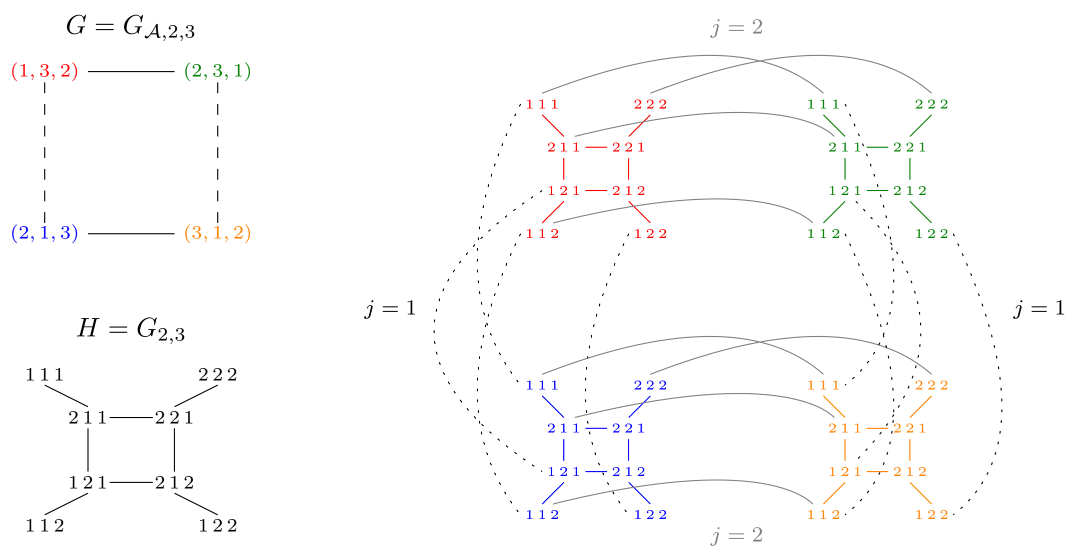
Then $\text{pair}_2(A) = ((1,2,0,1,1,0),\, 4\,1\,2\,5\,3) \in \mathfrak{C}(3,6,5) \times \mathfrak{A}(5,3)$.

4 Final Remarks

We conjecture the structure of the dual graph of the alcoved triangulation of $r\Delta_{i,d}$ based on Theorem 3.14 and Observation 3.17 (see [10, Section 3.2.3] for the definitions).

Conjecture 4.1. *Let $G = G_{\mathcal{A},i,d}$ and $H = G_{r,d}$. The edge-coloring of G determined by the hyperplane types prescribes a choice of connecting sets that make $G\langle H\rangle$ isomorphic to the dual graph of the alcoved triangulation of $r\Delta_{i,d}$.*

Example 4.2. Consider the case $i = 2$, $d = 3$ and $r = 2$. The graphs G, H, and $G\langle H\rangle$ in Conjecture 4.1 are



The graph composition, shown on the right, is obtained from the connecting sets labeled by the index j (see [10, Definition 3.36] for the relevant definitions).

Further, we believe that Observation 3.17 can lead to combinatorial interpretations of unimodular triangulations of dilation of polytopes arising from combinatorial objects such as order polytopes and special cases of flow polytopes. Indeed, these polytopes admit unimodular triangulations labeled by combinatorial objects, so it is natural to investigate the structure of each individual dilated simplex comprising the triangulation in terms of "folded" combinatorial objects.

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