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Colored multiset Eulerian polynomials

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Abstract. This extended abstract is a summary of a paper that studies the colored multiset Eulerian polynomials. These polynomials are a common generalization of MacMahon's multiset Eulerian polynomials and the colored Eulerian polynomials, both of which are known to satisfy well-studied distributional properties including real-rootedness, log-concavity and unimodality. The symmetric colored multiset Eulerian polynomials are characterized and used to prove sufficient conditions for a colored multiset Eulerian polynomial to be self-interlacing. The latter property implies the aforementioned distributional properties as well as others, including the alternatingly increasing property and bi-γ-positivity. To derive these results, multivariate generalizations of an identity due to MacMahon are deduced. The results are applied to a pair of questions, both previously studied in several special cases, that are seen to admit more general answers when framed in the context of colored multiset Eulerian polynomials. The first question pertains to s-Eulerian polynomials, and the second to interpretations of γ-coefficients.

Keywords: Eulerian polynomials; colored permutations; h^*-vectors; interlacing polynomials; γ-positivity; weakly increasing trees

1 Introduction

For a positive integer n, we let $[n] := \{1, 2, ..., n\}$ and $[n]_0 := \{0, 1, ..., n\}$. We denote by $M_{\mathbf{m}} = \{1^{m_1}, 2^{m_2}, ..., n^{m_n}\}$, where $\mathbf{m} := (m_1, m_2, ..., m_n) \in \mathbb{Z}^n_{>0}$, the multiset of cardinality $m := m_1 + \cdots + m_n$ containing m_k copies of k, for $1 \leq k \leq n$. The permutations of the multiset $M_{\mathbf{m}}$, i.e., words $\pi = \pi_1\pi_2 \ldots \pi_m$ in which k appears exactly m_k times, are called *multiset permutations*, the set of which we denote $\mathfrak{S}_{M_{\mathbf{m}}}$. For $\mathbf{r} := (r_1, ..., r_n) \in \mathbb{Z}^n_{>0}$,

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we denote by $\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{r}}}$ the set of *colored multiset permutations* of $M_{\mathbf{m}}$, with color-vector $\mathbf{r} := (r_1, ..., r_n)$; that is, words of the form $\pi^{\mathbf{c}} = \pi_1^{c_1}\pi_2^{c_2}\ldots\pi_m^{c_m}$, where k appears exactly m_k times and $1 \leq c_k \leq r_{\pi_k}$, for each $1 \leq k \leq m$. When $\mathbf{r} = \mathbf{1} := (1, ..., 1)$, the set $\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{r}}}$ coincides with the multiset permutations $\mathfrak{S}_{M_{\mathbf{m}}}$. When $\mathbf{m} = \mathbf{1}$, $\mathfrak{S}_{M_{\mathbf{1}}^{\mathbf{1}}}$ is the permutations of $[n]$, $\mathfrak{S}_{M_{\mathbf{1}}^{2\mathbf{1}}}$ is the *signed permutations* of $[n]$ and more generally $\mathfrak{S}_{M_{\mathbf{1}}^{r\mathbf{1}}}$ for $r \geq 1$ is the *r-colored permutations* of $[n]$, all of which appear frequently in algebraic combinatorics.

Of particular interest are properties of descent statistics defined on $\mathfrak{S}_{M_{\mathbf{1}}^{r\mathbf{1}}}$ and $\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{1}}}$, which admit a common generalization via a descent statistic on $\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{r}}}$. Let k^{c_i} represent the multiset element k with color $c_i \in [r_k]$. We impose the *color ordering* [20]

$$1^1 < 2^1 < \cdots < n^1 < (n+1)^1 < 1^2 < 2^2 < \cdots < n^2 < 1^3 < \cdots < n^{\max\{r_1,\ldots,r_n\}},$$

on the elements in the ground set $M_{\mathbf{m}}^{\mathbf{r}}$ of $\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{r}}}$, where we include the extra term $(n+1)^1$ by setting $\pi_{m+1}^{c_{m+1}} := (n+1)^1$ in every colored permutation $\pi_1^{c_1}\pi_2^{c_2}\cdots\pi_m^{c_m} \in \mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{r}}}$. We say that $j \in [m]$ is a *descent* (respectively, *ascent*) of $\pi^{\mathbf{c}} = \pi_1^{c_1}\pi_2^{c_2}\cdots\pi_m^{c_m}\pi_{m+1}^{c_{m+1}}$ if $\pi_j^{c_j} > \pi_{j+1}^{c_{j+1}}$ (respectively, $\pi_j^{c_j} < \pi_{j+1}^{c_{j+1}}$) according to the color ordering. We denote by $\mathrm{DES}(\pi^{\mathbf{c}})$ the set of descents of a colored multiset permutation $\pi^{\mathbf{c}}$, and we let $\mathsf{des}(\pi^{\mathbf{c}}) = |\,\mathrm{DES}(\pi^{\mathbf{c}})|$.

Early in the 20th century, MacMahon proved the following identity for the multiset permutations $\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{1}}}$,

$$\frac{\sum_{\pi\in\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{1}}}} x^{\mathrm{des}(\pi)}}{(1-x)^{m+1}} = \sum_{t\geq 0}\binom{t+m_1}{m_1}\binom{t+m_2}{m_2}\cdots\binom{t+m_n}{m_n}x^t. \tag{1.1}$$

(see, for instance, [15, Volume 2, Chapter IV, page 211].) When $\mathbf{m} = \mathbf{1}$, MacMahon's formula recovers a well-known identity for the *n-th Eulerian polynomial*

$$A_n = \sum_{\pi\in\mathfrak{S}_n} x^{\mathsf{des}(\pi)},$$

which is ubiquitous in enumerative and algebraic combinatorics. The Eulerian polynomials A_n enjoy a variety of sought-after distributional properties including symmetry, unimodality, log-concavity, real-rootedness as well as γ-positivity [6]. When the Eulerian polynomials arise as special cases of larger families of generating polynomials, such as the *P-Eulerian polynomials* [19, Chapter 3] or the *s-Eulerian polynomials* [16], many of these desirable distributional properties are often seen to extend to the larger family. In this paper, we consider the *colored multiset Eulerian polynomial*

$$A_{M_{\mathbf{m}}^{\mathbf{r}}} = \sum_{\pi^{\mathbf{c}}\in\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{r}}}} x^{\mathsf{des}(\pi)}.$$

These polynomials are a common generalization of both the *r-colored Eulerian polynomials* $A_{M_{\mathbf{1}}^{r\mathbf{1}}}$ and MacMahon's *multiset Eulerian polynomials* $A_{M_{\mathbf{m}}^{\mathbf{1}}}$. The distributional properties of both r-colored Eulerian polynomials and MacMahon's multiset Eulerian polynomials have been studied extensively. For instance, the colored Eulerian polynomials were shown to be real-rooted, log-concave and unimodal [18], and they are symmetric if and only if $r \in \{1,2\}$. More recently, they were shown to possess a strong distributional property investigated by Brändén and Solus [8]; namely, they are interlaced by their own reciprocal. This implies additional distributional properties including bi-γ-positivity, and real-rootedness, unimodality and log-concavity of their symmetric decomposition.

The colored Eulerian polynomials as well as the (uncolored) multiset Eulerian polynomial for $\mathbf{m} = 2\mathbf{1}$ were further shown to be s-Eulerian polynomials by Savage and Visontai [18]. Savage and Visontai conjectured that the same is true for the Type-B analog $A_{M_{2\mathbf{1}}^{2\mathbf{1}}}$ [18, Conjecture 3.25] in an effort to more fully describe the combinatorial generating polynomials that are *s-Eulerian*; i.e., equal to some s-Eulerian polynomial. To settle this conjecture, Lin [13, Theorem 6] generalized MacMahon's formula to signed multiset permutations, showing that

$$\frac{\sum_{\pi^{\mathbf{c}} \in \mathfrak{S}_{M_{\mathbf{m}}^{2\mathbf{1}}}} x^{\operatorname{des}(\pi)}}{(1-x)^{m+1}} = \sum_{t \geq 0} \prod_{j=1}^{n} \binom{2t+m_j}{m_j} x^t. \tag{1.2}$$

In this extended abstract, we study how the various distributional properties of the Eulerian polynomials and colored Eulerian polynomials extend to the larger family of colored multiset Eulerian polynomials. A characterization of $A_{M_{\mathbf{m}}^{\mathbf{r}}}$ that are symmetric with respect to degree m (the cardinality of $M_{\mathbf{m}}$) is provided. This characterization is applied to show that all $A_{M_{\mathbf{m}}^{\mathbf{r}}}$ of degree m are interlaced by their own reciprocal. This condition implies that these polynomials satisfy a wealth of desirable distributional properties, including real-rootedness, log-concavity, unimodality, the alternatingly increasing property and bi-γ-positivity. To prove these results, an identity of multivariate generating functions, generalizing MacMahon's formula, is proven and utilized. Connections to prior work on s-Eulerian polynomials and combinatorial interpretations of γ-coefficients of certain polynomials are drawn. Some open questions are also proposed.

2 Preliminaries

The main results of this paper describe the distributional properties of the colored multiset Eulerian polynomials. We now recall the definitions of the distributional properties of interest and their basic properties. For a more detailed discussion of these properties and their significance in combinatorics, we recommend the survey article [6].

A polynomial $p(x) = p_0 + p_1x + \cdots + p_dx^d$ of degree d is *unimodal* if its coefficients satisfy $p_0 \leq p_1 \leq \cdots \leq p_s \geq \cdots \geq p_d$ for some $s \in \{0,...,d\}$. It is *log-concave* if

$p_i^2 \geq p_{i-1}p_{i+1}$ for all $i = 1, \ldots, d$. It is well-known that a log-concave polynomial with no internal zeros is also unimodal.

The polynomial p is called *symmetric* with respect to degree n if $p_s = p_{n-s}$ for all $s = 0, ..., n$. The linear space of polynomials that are symmetric with respect to degree n can be expressed in the basis

$$\{x^i(x+1)^{n-2i} : 0 \leq i \leq \lfloor n/2 \rfloor, \}$$

commonly referred to as the *γ-basis*. If a polynomial has nonnegative coefficients in the γ-basis it is called *γ-positive* (or *γ-nonnegative*). If p is γ-positive then it is also unimodal.

The polynomial p is said to be *real-rooted* if p is a constant polynomial or all its zeros are real numbers. When the coefficients of p are nonnegative, it follows that p is log-concave. Moreover, when p is real-rooted and symmetric, p is also γ-positive. Hence, a proof that p with only positive coefficients is real-rooted and symmetric is a proof that p satisfies all of the above distributional properties.

The following distributional property has recently received much attention in the context of algebraic combinatorics and discrete geometry: The polynomial p is *alternatingly increasing* if $p_0 \leq p_d \leq p_1 \leq p_{d-1} \leq \cdots \leq p_{\lfloor \frac{d+1}{2} \rfloor}$. When p is alternatingly increasing, it is also unimodal. The property has received much attention in algebraic combinatorics since the following alternative characterization makes proving alternatingly increasing particularly useful when studying the unimodality of Hilbert polynomials of graded rings expressed in the multinomial basis.

Given $n \geq d$, it can be shown that there exist unique polynomials $a, b \in \mathbb{R}[x]$ such that (1) $p = a + xb$, (2) $\deg(a) \leq n$, (3) $\deg(b) \leq n - 1$, (4) a is symmetric with respect to n, and (5) b is symmetric with respect to $n - 1$. The pair (a, b) is called the *symmetric decomposition* of p with respect to n, or the *$\mathcal{I}_n$-decomposition* of p. A basic observation is that p is alternatingly increasing if and only if both a and b have only nonnegative coefficients and are unimodal. It is therefore of interest to investigate the distributional properties of the polynomials in a symmetric decomposition. We say that p has a real-rooted, unimodal, or log-concave symmetric decomposition whenever both a and b fulfill the specified condition. When a and b are both γ-positive p is said to be *bi-γ-positive*.

Note that it is possible for a polynomial that is not real-rooted to have a real-rooted symmetric decomposition with respect to its degree. For instance, the polynomial $1 + 5x + 17x^2 + 15x^3 + 2x^4$ has no real zeros but $\mathcal{I}_4$-decomposition $((1 + x)^4, A_3(x))$. In [8], it was shown that a sufficient condition for a symmetric decomposition to be real-rooted is that it is *interlaced* by its own reciprocal. Given two real-rooted polynomials p and q with respective zeros $\alpha_1 \geq \alpha_2 \geq \cdots$ and $\beta_1 \geq \beta_2 \geq \cdots$, we say that p is *interlaced* by q, denoted $q \prec p$ if

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots .$$

In [8], it is shown that a polynomial p of degree d with $\mathcal{I}_d$-decomposition (a, b) having only nonnegative coefficients is interlaced by its own reciprocal $\mathcal{I}_d(p) = x^d p(1/x)$ if and only if $b \prec a$. In this case, it follows that the symmetric decomposition of p is real-rooted, log-concave, and unimodal, but we also recover that p is alternatingly increasing, real-rooted, log-concave, unimodal and bi-γ-positive. Hence, it is of interest to determine when a given combinatorial generating polynomial interlaces its own reciprocal. This property was shown to hold, for example, for colored Eulerian polynomials and then applied to settle several conjectures in algebraic combinatorics [8]. Below, we describe when this condition generalizes to the class of colored multiset Eulerian polynomials.

3 Generating function identities

To derive the desired results on the distributional properties of the polynomial $A_{M_{\mathbf{m}}^{\mathbf{r}}}$, we first give a generalization of MacMahon's formula (1.1) to the colored multiset permutations. The required univariate identity can be recovered from a multivariate identity that also specializes to other previously observed generalizations and q-analogues of MacMahon's formula. Throughout this section, we fix a positive integer n and the positive integral vectors $\mathbf{m} = (m_1, ..., m_n)$ and $\mathbf{r} = (r_1, ..., r_n)$. We let $\pi^{\mathbf{c}} = \pi_1^{c_1} \cdots \pi_m^{c_m} \pi_{m+1}^{c_{m+1}}$ denote a permutation in the set $\mathfrak{S}_{M_{\mathbf{m}}^{\mathbf{r}}}$. We then have the following result.

Theorem 3.1. *For positive integral vectors $\mathbf{m} = (m_1, ..., m_n)$ and $\mathbf{r} = (r_1, ..., r_n)$, we have*

$$\sum_{\pi^c \in \mathfrak{S}_{M_m^{\mathbf{r}}}} \frac{z_{\pi_1}^{c_1 - 1} \cdots z_{\pi_m}^{c_m - 1} \prod\limits_{\substack{i \in [m+1]\setminus\{1\} \\ i-1 \in DES(\pi)}} z_{\pi_i}^{r_{\pi_i}} \cdots z_{\pi_m}^{r_{\pi_m}} z_{m+1}}{\prod\limits_{i \in [m+1]} \left(1 - z_{\pi_i}^{r_{\pi_i}} \cdots z_{\pi_m}^{r_{\pi_m}} z_{m+1}\right)} = \sum_{t \geq 0} z_{m+1}^t \prod_{k \in [n]} \begin{bmatrix} r_k t + m_k \\ m_k \end{bmatrix}_{z_k}. \qquad (3.1)$$

The proof of Theorem 3.1 is combinatorial, and uses the notion of barred permutations popularized by Gessel and Stanley [12], further explored by Lin [13] in the study of signed multiset permutations. While somewhat garish in form, the identity in Theorem 3.1 specializes to several known identities. Setting $z_{m+1} = x$, $z_k = 1$ and $\mathbf{r} = \mathbb{1}$ recovers (1.1). Similarly, (1.2) is recovered by setting $z_{m+1} = x$, $z_k = 1$ and $\mathbf{r} = 2\mathbb{1}$. Equations (1.1) and (1.2) also have q-analogs that are analogously recovered by an appropriate specialization. When $\mathbf{m} = \mathbb{1}$ and $\mathbf{r} = r\mathbb{1}$ for a positive integer r, (3.1) specializes to a formula of Steingrimsson [20, Theorem 17] for $z_{m+1} = x$ and $z_k = 1$ (and its q-analog, [4, Proposition 8.1], for $z_k = q$). Another interesting evaluation arises for $z_{m+1} = x$ and $z_k = q$ for all $k \in [n]$, for $\mathbf{r} = r\mathbb{1}$, where (a flag version of) a permutation statistic similar to the inversion-sequence statistic dmaj considered in [17] appears on the enumerator in (3.1).

For our study of the distributional properties of $A_{M_{\mathbf{m}}^{\mathbf{r}}}$, the relevant specialization of the identity in Theorem 3.1 is given by $z_{m+1} = x$, $z_k = 1$, which yields the following.

Corollary 3.2. *For positive integral vectors* $\mathbf{m} = (m_1, ..., m_n)$ *and* $\mathbf{r} = (r_1, ..., r_n)$*, we have*

$$\frac{A_{M_m^r}}{(1-x)^{m+1}} = \sum_{t\geq 0}\left(\prod_{i=1}^{n}\binom{r_i t + m_i}{m_i}\right)x^t.$$

Corollary 3.2 says that the colored multiset Eulerian polynomial $A_{M_{\mathbf{m}}^{\mathbf{r}}}$ is the polynomial $\prod_{i=1}^{n}\binom{r_i x+m_i}{m_i}$ expressed in the multinomial basis for the vector space of univariate polynomials of degree at most d. This observation will allow us to derive real-rootedness and interlacing results on the $\mathcal{I}_m$-decomposition of $A_{M_{\mathbf{m}}^{\mathbf{r}}}$.

Remark 3.3. We note that Corollary 3.2 may alternatively be proven using the identity [7, Proposition 3.5(5)] associated to the s-lecture hall order polytopes of Brändén and Leander. Specifically, Corollary 3.2 arises as a mild generalization of [7, Corollary 3.6].

4 Distributional properties

We now consider the distributional properties of the colored multiset Eulerian polynomials $A_{M_{\mathbf{m}}^{\mathbf{r}}}$ for positive integral vectors $\mathbf{m} = (m_1, \ldots, m_n)$ and $\mathbf{r} = (r_1, \ldots, r_n)$. To do so, we make use of the identity in Corollary 3.2, noting first that this identity has a simple geometric interpretation.

The *Ehrhart polynomial* of a d-dimensional convex lattice polytope $P \subset \mathbb{R}^n$ is $\mathcal{L}(P;t) = |tP \cap \mathbb{Z}^n|$, where $tP = \{tp : p \in P\}$ denotes the *t-th dilate* of P for $t \in \mathbb{Z}_{>0}$. The *(Ehrhart) h^*-polynomial* of P, denoted $h^*(P;x)$, is the Ehrhart polynomial of P when expressed in the multinomial basis for the vector space of polynomials of degree at most d. It is well-known that the $\mathcal{L}(\Delta_d;t) = \binom{t+d}{d}$ when Δ_d is the the standard d-dimensional simplex [3, Theorem 2.2.]. Since $\mathcal{L}(rP;t) = \mathcal{L}(P;rt)$ for any positive integer r and taking a direct product of polytopes corresponds to multiplying the Ehrhart polynomials, we have that the polynomial $\prod_{i=1}^{n}\binom{r_i t+m_i}{m_i}$ appearing in Corollary 3.2 is the Ehrhart polynomial of the product of dilated simplices

$$P_{\mathbf{m}}^{\mathbf{r}} := \prod_{j=1}^{n} r_j \Delta_{m_j}.$$

It follows that $A_{M_{\mathbf{m}}^{\mathbf{r}}}(x)$ is the h^*-polynomial of $P_{\mathbf{m}}^{\mathbf{r}}$.

Lemma 4.1. $A_{M_{\mathbf{m}}^{\mathbf{r}}}(x) = h^*(P_{\mathbf{m}}^{\mathbf{r}};x)$.

Lemma 4.1 provides a generalization of the well-known relationship between the Eulerian polynomials $A_n(x) = A_{M_{\mathbb{1}}^{\mathbb{1}}}$ and the h^*-vector of the n-dimensional cube, which is a product of n one-dimensional simplices. We further note that the corresponding relation for uncolored multisets can arise via the theory of order polytopes for a suitable choice of posets. Lastly, the case $\mathbf{m} = (2, ..., 2)$ and $\mathbf{r} = (2, ..., 2)$ has also been studied (see [1]).

Basic results in Ehrhart theory then imply the following.

Lemma 4.2. *$A_{M^{\mathbf{r}}_{\mathbf{m}}}$ has degree $m+1-\max_{k\in[n]}\left\lceil\frac{m_k+1}{r_k}\right\rceil$.*

Results of Hibi and DeNegri [9] can then be applied to characterize when $A_{M^{\mathbf{r}}_{\mathbf{m}}}$ is symmetric with respect to degree m.

Theorem 4.3. *$A_{M^{\mathbf{r}}_{\mathbf{m}}}$ is symmetric (with respect to its degree) if and only if*

$$\max_{k\in[n]}\left\lceil\frac{m_k+1}{r_k}\right\rceil r_j = m_j+1 \qquad \text{for all } j\in[n]. \tag{4.1}$$

It follows that $A_{M^{\mathbf{r}}_{\mathbf{m}}}$ is symmetric with respect to its degree m if and only if $r_j = m_j+1$ for all $j\in[n]$.

Theorem 4.3 can in turn be applied to describe when the polynomial $A_{M^{\mathbf{r}}_{\mathbf{m}}}$ is interlaced by its own reciprocal, and hence fulfills the numerous distributional properties described in Section 2. To do so, one uses the *subdivision operator*, which is the linear transformation on the space of univariate polynomials given by $\mathcal{E}:\binom{x}{k}\longmapsto x^k$. When applied to an Ehrhart polynomial $\mathcal{L}(P;t)$ that can be expressed as $\mathcal{L}(P;t)=\sum_{i=0}^{d}c_it^i(1+t)^{d-i}$ for nonnegative c_i, the subdivision operator returns a $[-1,0]$-rooted polynomial that is the h^*-polynomial up to a transformation that preserves $(-\infty,0)$-rootedness [5]. Hence, $h^*(P;x)$ is real-rooted, log-concave and unimodal. It can be shown that Ehrhart polynomial $\mathcal{L}(P^{\mathbf{r}}_{\mathbf{m}};t)$ does indeed have nonnegative coefficients in this so-called *magic basis* [10] $t^i(1+t)^{d-i}$, $i=0,\ldots,d$. Since the polynomials satisfying $r_j=m_j+1$ for all $j\in[n]$ are symmetric with respect to their degree m, they are trivially interlaced by their own reciprocal. Applying the results of [8] allows us to deduce the following more general result.

Theorem 4.4. *Suppose that $\mathbf{m}$, $\mathbf{r}$ satisfy $r_j\geq m_j+1$ for all $j=1,\ldots,n$. Then*

$$\mathcal{I}_m(A_{M^{\mathbf{r}}_{\mathbf{m}}})\prec A_{M^{\mathbf{r}}_{\mathbf{m}}}.$$

It follows that if $r_j\geq m_j+1$ for all $j=1,\ldots,n$, then the colored multiset Eulerian polynomial $A_{M^{\mathbf{r}}_{\mathbf{m}}}(x)$ is real-rooted, log-concave, unimodal, alternatingly increasing and bi-γ-positive with a real-rooted, log-concave and unimodal $\mathcal{I}_m$-decomposition. This generalizes the observation of Brändén and Solus [8, Corollary 3.2] for colored Eulerian polynomials to the colored multiset Eulerian polynomial setting.

5 Combinatorial interpretations

We end with a discussion of some alternative interpretations of certain colored multiset Eulerian polynomials as a motivation for future work connecting these polynomials with desirable distributional properties to other lines of combinatorial investigation.

5.1 s-Eulerian polynomials

The s-Eulerian polynomials E_n^s are a large family of combinatorial generating polynomials defined for any sequence of positive integers $s = (s_1, \ldots, s_n)$. They are known to be real-rooted and admit numerous connections to classically studied problems, including the enumeration of lecture hall sequences, as well as connections to discrete geometry where they are interpreted as h^*-polynomials of *s-lecture hall simplices* P_n^s; namely, $E_n^s = h^*(P_n^s; x)$. The s-Eulerian polynomials are surveyed in [16]. It is shown in [18], for instance, that $E_n^{(r,2r,\ldots,rn)} = A_{M_{\mathbb{1}}^{r\mathbb{1}}}$, the r-colored Eulerian polynomial and $A_{M_{2\mathbb{1}}^{\mathbb{1}}} = E_{2n}^s$, where $s = (1,1,3,2,5,3,\ldots,2n-1,n)$.

In the following, we fix $s = (1,1,3,2,\ldots,2n-1,n)$, $p = (1,1,3,\ldots,n-1,2n-1)$ and $rs = (r,r,3r,2r,\ldots,r(2n-1),rn)$, $rp = (r,r,3r,\ldots,r(n-1),r(2n-1))$ for $r \geq 1$. It is seen that the s-lecture hall simplex P_{2n}^{rs} is the r-th dilate of P_{2n}^s; that is, $P_{2n}^{rs} = rP_{2n}^s$. Similarly, $P_{2n-1}^{rp} = rP_{2-1n}^p$. In [17, Theorem 14], it is shown that

$$\sum_{t\geq 0}(t+1)^n\left(\frac{t+2}{2}\right)^n x^t = \frac{E_{2n}^s}{(1-x)^{2n+1}} \quad \text{and} \quad \sum_{t\geq 0}(t+1)^n\left(\frac{t+2}{2}\right)^{n-1} x^t = \frac{E_{2n-1}^p}{(1-x)^{2n}}.$$

It follows that the Ehrhart polynomials of P_{2n}^{rs} and P_{2n-1}^{rp} are, respectively,

$$\mathcal{L}(P_{2n}^{rs};t) = (rt+1)^n\left(\frac{rt+2}{2}\right)^n \qquad \text{and} \qquad \mathcal{L}(P_{2n-1}^{rp};t) = (rt+1)^n\left(\frac{rt+2}{2}\right)^{n-1}.$$

It then follows from Equation (3.1) that $A_{M_{2\mathbb{1}}^{r\mathbb{1}}} = E_{2n}^{rs}$ and $A_{M_{(2,\ldots,2,1)}^{r\mathbb{1}}} = E_{2n-1}^{rp}$ for any $r \geq 1$. Hence we have observed the following.

Proposition 5.1. *The colored multiset Eulerian polynomial $A_{M_{\mathbf{m}}^{\mathbf{r}}}$ is an s-Eulerian polynomial for any $\mathbf{m} \in \{(1,\ldots,1),(2,\ldots,2),(2,\ldots,2,1)\}$ and $\mathbf{r} = r\mathbb{1}$ for any $r \geq 1$.*

It follows from Theorem 4.4 that for $r \geq 3$ the s-Eulerian polynomials captured in Proposition 5.1 satisfy the strongest distributional property in Section 2; namely, they are interlaced by their own reciprocal and hence satisfy all other distributional properties of interest. In the case that $r = 1$, the same is true for E_{2n}^s since it is known to be symmetric and real-rooted by prior work [18] as well as in [1, Theorem 6.2] where these polynomials appear in the study of bipermutohedra. E_{2n-1}^p is also known to have a real-rooted symmetric decomposition [14]. Similarly, when $r = 2$, it is only known that E_{2n}^{2s} and E_{2n-1}^{2p} are bi-γ-positive. This connection prompts the following question.

Question 5.2. Characterize the colored multiset Eulerian polynomials that are s-Eulerian polynomials. What are the strongest distributional properties they satisfy?

Such results may give us deeper insights into when an s-Eulerian polynomial interlaces its own reciprocal. To see that such a characterization may be challenging, we note that there is no s-sequence and $n \geq 1$ such that $E_n^s = A_{M_{\mathbf{m}}^{\mathbf{r}}}$ for $\mathbf{r} = (1,1)$ and $\mathbf{m} = (3,3)$.

Remark 5.3. In [7], Brändén and Leander introduced a common generalization of s-lecture hall simplices and order polytopes called *s-lecture hall order polytopes*. If $p = h^*(\mathcal{O}(P,s);x)$, we say that p is (P,s)-*Eulerian*. Question 5.2 asks when $A_{M_{\mathbf{m}}^{\mathbf{r}}}$ is s-Eulerian. We saw above that not all colored multiset Eulerian polynomials have this property. On the other hand, $A_{M_{\mathbf{m}}^{\mathbf{r}}}$ is always (P,s)-Eulerian.

5.2 Interpreting γ-coefficients

The results of Theorem 4.4 and the observations on bi-γ-positivity in the previous subsection yield a large family of relatively unexplored combinatorial generating polynomials that are γ-positive. A natural endeavour is then to explore combinatorial interpretations of these γ-coefficients. Such investigations may prove fruitful in, for instance, the development of a complete understanding of the γ-coefficients for s-Eulerian polynomials. We motivate this investigation with a complete characterization of the γ-coefficients in the symmetric decompositions of $A_{M_{2\mathbb{1}}^{\mathbb{1}}}$ and $A_{M_{(2,\ldots,2,1)}^{\mathbb{1}}}$.

A *plane tree* is a rooted tree in which the children of each node are ordered (according to any ordering).

Definition 5.4 (Weakly increasing trees [14])**.** *Let $M = \{1^{m_1}, 2^{m_2}, \cdots, n^{m_n}\}$ be a multiset. A* weakly increasing tree *on M is a plane tree that:*

(i) contains $|M|+1$ nodes, labeled by the elements in $M \cup \{0\}$,

(ii) its root is labeled by 0 and each node receives a label weakly greater than its parent,

(iii) the labels of the children of each node are weakly increasing from left to right.

A leaf or an internal node (other than the root) of a weakly increasing tree is *old* if it is the rightmost child of its parent; otherwise, it is *young*. The polynomial $A_{M_{2\mathbb{1}}^{\mathbb{1}}}$ is known to be symmetric, and hence it is its own symmetric decomposition. Since it is also real-rooted, it is γ-positive. In [14] it was shown that its γ-coefficient γ_i counts the number of weakly increasing trees on $M_{2\mathbb{1}}^{\mathbb{1}}$ with $i+1$ leaves and no young leaves. The polynomial $A_{M_{(2,\ldots,2,1)}^{\mathbb{1}}}$ is not symmetric with respect to its degree, but it is known to be bi-γ-positive [14]. The b-polynomial in its symmetric decomposition is a scalar multiple of $A_{M_{2\mathbb{1}}^{\mathbb{1}}}$ [14], and hence we recover a combinatorial description of the γ-coefficients by the result above. For the a-polynomial, one can extend the methods of [14] to prove the following.

Theorem 5.5. *For $n \geq 1$, the a-polynomial in the symmetric decomposition of $A_{M^{\mathbb{1}}_{(2,\ldots,2,1)}}$ satisfies*

$$a(x) = \sum_{i=0}^{(n-1)} \gamma_{n,i} x^i (1+x)^{2(n-1)+1-2i}, \tag{5.1}$$

where $\gamma_{n,i}$ is the number of weakly increasing trees on $\{1^2, 2^2, \cdots, (n-1)^2, n, n+1\}$ with $i+1$ leaves and no young leaves.

Theorem 4.4 yields a large family of relatively unexplored combinatorial generating polynomials that are bi-γ-positive. A natural endeavour is to explore combinatorial interpretations of these γ-coefficients. Such investigations may prove fruitful in, for instance, the development of a complete understanding of the γ-coefficients for s-Eulerian polynomials when interpreted through the lens of Proposition 5.1. Hence, in this section, we provide results pertaining to the following general question:

Question 5.6. Let (a, b) be the $\mathcal{I}_d$-decomposition of a bi-γ-positive colored multiset Eulerian polynomial $A_{M^{\mathbf{r}}_{\mathbf{m}}}$, where d is the degree of $A_{M^{\mathbf{r}}_{\mathbf{m}}}$. What is a combinatorial interpretation of the coefficients of a and b when expressed in the γ-basis?

Theorem 4.4 provides a general family of colored multiset Eulerian polynomials that are bi-γ-positive for which Question 5.6 may be considered. This family includes the previously studied colored Eulerian polynomials $A_{M^{r\mathbf{1}}_{\mathbf{1}}}$ for $r \geq 1$. In [2, Question 2.27], Athanasiadis considers a special case, asking for a solution to Question 5.6 for $A_{M^{r\mathbf{1}}_{\mathbf{1}}}$ for $r \geq 1$. A solution to Athanasiadis's question would extend the well-known interpretation of the γ-coefficients for the Eulerian polynomial $A_n = A_{M^{\mathbf{1}}_{\mathbf{1}}}$ due to Foata and Strehl [11] (see for instance [6, Section 3.1]), to all colored Eulerian polynomials. Hence, one could more generally solve Athanasiadis's question by solving Question 5.6 for all $A_{M^{\mathbf{r}}_{\mathbf{m}}}$ captured in Theorem 4.4.

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