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Commutative Properties of Schubert Puzzles with Convex Polygonal Boundary Shapes

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Abstract. We generalize classical triangular Schubert puzzles to puzzles with convex polygonal boundary. We give these puzzles a geometric Schubert calculus interpretation and derive novel combinatorial commutativity statements, using both geometric and combinatorial arguments, for puzzles with four, five, and six sides, having various types of symmetry in their boundary conditions.

Keywords: Schubert calculus, Schubert puzzle

1 Introduction

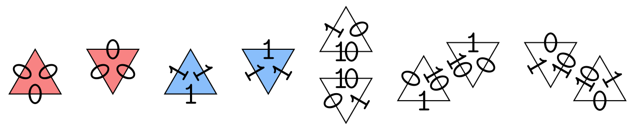
Since their introduction in [3] in 1999, Schubert puzzles have been the subject of continuous research activity, but little work has been done to study non-triangular puzzles up until now. In this extended abstract, we give a survey of the commutative properties of generalized convex polygonal Schubert puzzles of the type that compute structure constants in the ordinary cohomology of the Grassmannian. These properties are analogous to the commutative property of classical triangular puzzles. We remark that these results extend to puzzles that compute the cohomology of 2-step and 3-step flag varieties, as well as puzzles that compute the K-theory of the Grassmannian. We also present an analogue of commutativity for parallelogram-shaped equivariant puzzles.

This work was motivated by the observation that convex polygonal puzzles can be given a geometric interpretation by associating them to classes of triangular puzzles whose boundary conditions follow a certain form. Once translated to this more classical setting, the statements about polygonal puzzles can be proven geometrically or by using basic properties of triangular puzzles. Future research directions include finding interpretations or extensions of these results in other contexts.

1.1 Schubert Puzzles

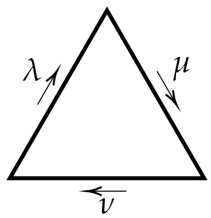
1.1.1 Classical triangular puzzles

Classically, a **Schubert puzzle** is a tiling of an equilateral triangular region, whose boundary we denote $\triangle$, using a set of labeled unit triangles

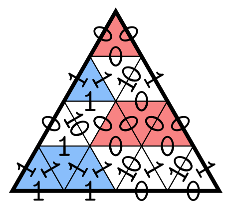


called **puzzle pieces**, so that any two glued edges have the same label, and only 0 and 1 labels appear along the outer boundary $\triangle$, not 10s.

An equilateral triangular boundary whose NW, NE, and South sides are labeled with strings λ, μ, and ν in the orientations shown in Figure 1a will be denoted $\triangle_{\lambda,\mu,\nu}$, and a puzzle with this boundary labeling will be called a $\triangle_{\lambda,\mu,\nu}$**-puzzle**. See Figure 1b for an example.



(a) The labeled boundary $\triangle_{\lambda,\mu,\nu}$



(b) A $\triangle_{1010,0101,0011}$-puzzle

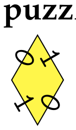
Figure 1

Triangular puzzles have three-fold rotational symmetry, i.e. there are bijections

$$\{\triangle_{\lambda,\mu,\nu}\text{-puzzles}\} \leftrightarrow \{\triangle_{\mu,\nu,\lambda}\text{-puzzles}\} \leftrightarrow \{\triangle_{\nu,\lambda,\mu}\text{-puzzles}\}$$

obtained by rotating puzzles in 120° increments. Triangular puzzles also have a commutative property, i.e. that $\#\{\triangle_{\lambda,\mu,\nu}\text{-puzzles}\} = \#\{\triangle_{\mu,\lambda,\nu}\text{-puzzles}\}$, which is non-obvious from a combinatorial standpoint. A bijective combinatorial proof is given by Purbhoo in [4] using mosaics, which are objects that are in natural bijection with puzzles.

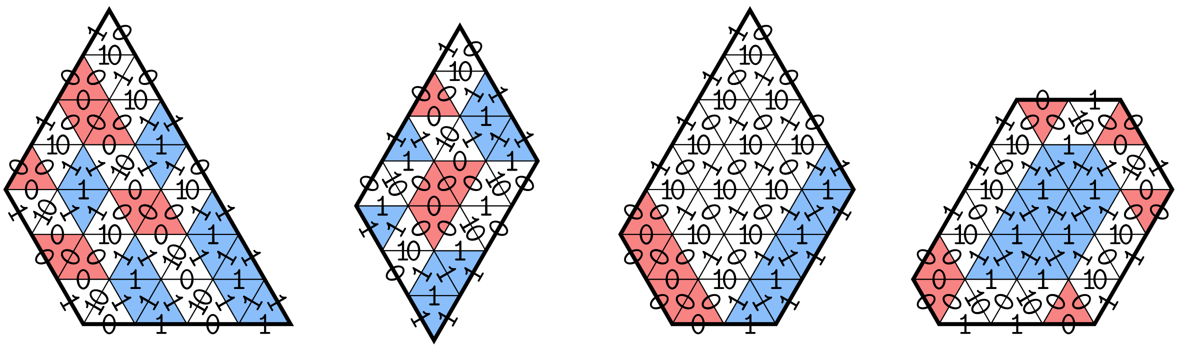
1.1.2 Equivariant puzzles

An **equivariant puzzle**, first described in [2], is one in which we additionally allow the *equivariant piece*  . This piece has a *weight* of the form $y_j - y_i$, where (i, j) is uniquely determined by its position in the puzzle.

1.1.3 Puzzles with convex polygonal boundary

Now we will generalize the definition of "puzzles" to include puzzle piece tilings of convex polygonal shapes, where again only 0 and 1 labels are allowed to appear along the outer boundary, not 10s. This allows us to have puzzles with trapezoidal, parallelogram-shaped, pentagonal, and hexagonal boundary as well. (See Figure 2 for examples.)

Similarly to $\triangle_{\lambda,\mu,\nu}$, we will use a shape symbol (where only the number of sides and angles matter, not side lengths) with a subscript sequence of strings, read clockwise starting from the SW, to denote a labeled boundary.



(a) ⏢$_{101,0101,0010111,1010}$-puzzle example (b) ◊$_{101,0101,011,0011}$-puzzle example (c) ⬠$_{00,01111,0001,111,10}$-puzzle example (d) ⬡$_{0,0111,01,01,011,011}$-puzzle example

Figure 2: Examples of puzzles with convex polygonal boundary shapes.

1.2 Schubert calculus

1.2.1 Schubert varieties

Let $\mathrm{Gr}(k;\mathbb{C}^n)$ denote the **Grassmannian** of k-dimensional subspaces of $\mathbb{C}^n$. We will write $\lambda \in \binom{[n]}{k}$ to mean that λ is a **binary string** with a **content** of k 1s and $n-k$ 0s. Let $\lambda = \lambda_1\lambda_2\cdots\lambda_n \in \binom{[n]}{k}$, and let $F_\bullet$ be the standard complete flag $\{\mathbf{0}\} = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n$. Then X_λ is the **Schubert variety** in $\mathrm{Gr}(k;\mathbb{C}^n)$ defined by

$$X_\lambda(F_\bullet) := \{V \in \mathrm{Gr}(k;\mathbb{C}^n) : \dim(V \cap F_i) \geq \lambda_1 + \lambda_2 + \cdots + \lambda_i\}.$$

We will use the superscript $^\vee$ to denote the operation of reversing a string. Then X^λ is the **opposite Schubert variety** defined by $X^\lambda := w_0 \cdot X_{\lambda^\vee}$, where w_0 is the anti-diagonal permutation matrix. The classes $\{[X_\lambda] : \lambda \in \binom{[n]}{k}\}$ form a $\mathbb{Z}$-basis for the cohomology ring $H^*(\mathrm{Gr}(k;\mathbb{C}^n))$, and the classes $\{[X^\lambda] : \lambda \in \binom{[n]}{k}\}$ form a dual basis with $\int_{\mathrm{Gr}(k;\mathbb{C}^n)}[X_\lambda][X^\mu] = \delta_{\lambda,\mu}$. The classes of these Schubert varieties also form dual bases over $\mathbb{Z}[y_1,\ldots,y_n]$ in the T-equivariant cohomology $H_T^*(\mathrm{Gr}(k;\mathbb{C}^n)) := H^*(ET \times^T \mathrm{Gr}(k;\mathbb{C}^n))$, where $T = (\mathbb{C}^\times)^n$ and ET is a contractible space with a free T-action.

1.2.2 Schubert calculus

The goal of **Schubert calculus** is to compute the structure constants in various cohomology theories in the Schubert basis. In $H^*(\mathrm{Gr}(k;\mathbb{C}^n))$, the structure constants

$c_{\lambda,\mu}^{\nu} = \int_{\mathrm{Gr}(k;\mathbb{C}^n)} [X_\lambda][X_\mu][X^\nu]$ are positive, and in fact they are the *Littlewood–Richardson (LR) numbers* that arise in symmetric function theory. (Usually these are indexed by triples of partitions, but we can identify our binary strings with partitions that fit in a $k \times (n-k)$ rectangle via a simple bijection.)

1.2.3 Puzzles compute Schubert calculus

It is a result of Knutson–Tao–Woodward in [3] that puzzles using the basic puzzle piece set compute the structure constants of $H^*(\mathrm{Gr}(k;\mathbb{C}^n))$ in the Schubert basis:

$$c_{\lambda,\mu}^{\nu} = \#\{\triangle_{\lambda,\mu,\nu^\vee}\text{-puzzles}\}.$$

Knutson–Tao later proved in [2] that equivariant puzzles compute the structure constants, which we denote $(c_T)_{\lambda,\mu}^{\nu}$, of $H_T^*(\mathrm{Gr}(k;\mathbb{C}^n))$ according to the following formula:

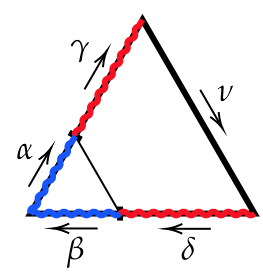
$$(c_T)_{\lambda,\mu}^{\nu} = \sum_{\triangle_{\lambda,\mu,\nu^\vee}\text{-puzzles } P} \mathrm{weight}(P) = \sum_{\triangle_{\lambda,\mu,\nu^\vee}\text{-puzzles } P} \left(\prod_{\substack{\text{equivariant} \\ \text{pieces } p \text{ in } P}} \mathrm{weight}(p) \right).$$

1.3 Main Theorems

Theorems 1.1 (Commutative Properties of Convex Polygonal Puzzles)**.** *For each labeled boundary shape drawn below, we can commute the labels on any pair of sides with matching colored squiggly lines while preserving the number of puzzles filling the boundary. Namely,*

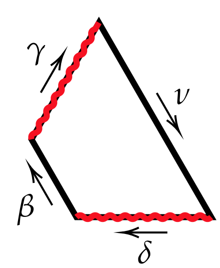
(a) *(Commutative Property of Puzzles with Split Symmetry)*
For $\alpha, \beta \in \binom{[a]}{a_1}$, $\gamma, \delta \in \binom{[c]}{c_1}$, *and* $\nu \in \binom{[c+a]}{a_1+c_1}$,

$$\#\{\triangle_{\alpha\gamma,\nu,\delta\beta}\text{-puzzles}\} = \#\{\triangle_{\beta\gamma,\nu,\delta\alpha}\text{-puzzles}\}$$
$$= \#\{\triangle_{\alpha\delta,\nu,\gamma\beta}\text{-puzzles}\} = \#\{\triangle_{\beta\delta,\nu,\gamma\alpha}\text{-puzzles}\}.$$



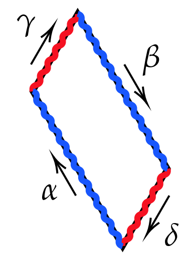
(b) *(Commutative Property of Trapezoidal Puzzles)*
For $\beta \in \binom{[a]}{a_1}$, $\gamma, \delta \in \binom{[c]}{c_1}$, *and* $\nu \in \binom{[c+a]}{a_1+c_1}$,

$$\#\{\text{⏢}_{\beta,\gamma,\nu,\delta}\text{-puzzles}\} = \#\{\text{⏢}_{\beta,\delta,\nu,\gamma}\text{-puzzles}\}.$$



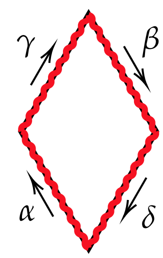
(c) *(Commutative Property of Parallelogram-Shaped Puzzles)*
For $\alpha, \beta \in \binom{[a]}{a_1}$ and $\gamma, \delta \in \binom{[c]}{c_1}$,

$$\#\{\lozenge_{\alpha,\gamma,\beta,\delta}\text{-puzzles}\} = \#\{\lozenge_{\beta,\gamma,\alpha,\delta}\text{-puzzles}\}$$
$$= \#\{\lozenge_{\alpha,\delta,\beta,\gamma}\text{-puzzles}\} = \#\{\lozenge_{\beta,\delta,\alpha,\gamma}\text{-puzzles}\}.$$



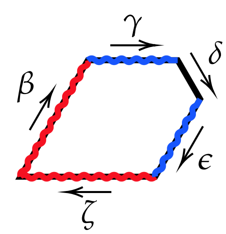
(d) *(Commutative Property of Symmetric Rhombus-Shaped Puzzles)*
For $\alpha, \beta, \gamma, \delta \in \binom{[a]}{a_1}$ and any bijection
$f : \{\alpha, \beta, \gamma, \delta\} \to \{\alpha, \beta, \gamma, \delta\}$,

$$\#\{\lozenge_{\alpha,\gamma,\beta,\delta}\text{-puzzles}\} = \#\{\lozenge_{f(\alpha),f(\gamma),f(\beta),f(\delta)}\text{-puzzles}\}.$$



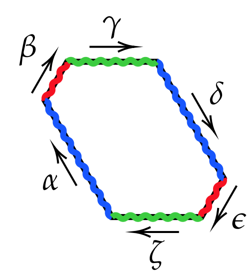
(e) *(Commutative Property of Symmetric Pentagonal Puzzles)*
For $\beta, \zeta \in \binom{[b]}{b_1}$, $\gamma, \epsilon \in \binom{[c]}{c_1}$, and $\delta \in \binom{[d]}{d_1}$,

$$\#\{\pentagon_{\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{\pentagon_{\zeta,\gamma,\delta,\epsilon,\beta}\text{-puzzles}\}$$
$$= \#\{\pentagon_{\beta,\epsilon,\delta,\gamma,\zeta}\text{-puzzles}\} = \#\{\pentagon_{\zeta,\epsilon,\delta,\gamma,\beta}\text{-puzzles}\}.$$



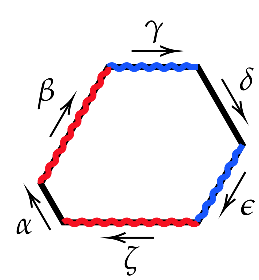
(f) *(Commutative Property of Hexagonal Puzzles with Opposite Sides Symmetry)*
For $\alpha, \delta \in \binom{[a]}{a_1}$, $\beta, \epsilon \in \binom{[b]}{b_1}$, and $\gamma, \zeta \in \binom{[c]}{c_1}$, and any bijections
$f : \{\alpha, \delta\} \to \{\alpha, \delta\}$, $g : \{\beta, \epsilon\} \to \{\beta, \epsilon\}$, and
$h : \{\gamma, \zeta\} \to \{\gamma, \zeta\}$,

$$\#\{\hexagon_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{\hexagon_{f(\alpha),g(\beta),h(\gamma),f(\delta),g(\epsilon),h(\zeta)}\text{-puzzles}\}.$$



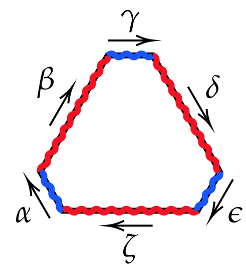
(g) *(Commutative Property of Hexagonal Puzzles with Two-Way Symmetry)*
For $\alpha \in \binom{[a]}{a_1}$, $\beta, \zeta \in \binom{[b]}{b_1}$, $\gamma, \epsilon \in \binom{[c]}{c_1}$, and $\delta \in \binom{[d]}{d_1}$,

$$\#\{\hexagon_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} = \#\{\hexagon_{\alpha,\zeta,\gamma,\delta,\epsilon,\beta}\text{-puzzles}\}$$
$$= \#\{\hexagon_{\alpha,\beta,\epsilon,\delta,\gamma,\zeta}\text{-puzzles}\} = \#\{\hexagon_{\alpha,\zeta,\epsilon,\delta,\gamma,\beta}\text{-puzzles}\}.$$



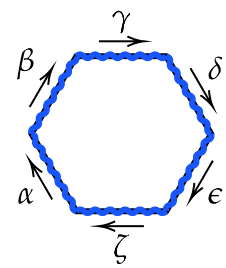
(h) *(Commutative Property of Hexagonal Puzzles with Three-Way Symmetry)*
For $\alpha, \gamma, \epsilon \in \binom{[a]}{a_1}$ and $\beta, \delta, \zeta \in \binom{[b]}{b_1}$, and any bijections $f : \{\alpha, \gamma, \epsilon\} \to \{\alpha, \gamma, \epsilon\}$ and $g : \{\beta, \delta, \zeta\} \to \{\beta, \delta, \zeta\}$,

$$\#\{\hexagon_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-}puzzles\} = \#\{\hexagon_{f(\alpha),g(\beta),f(\gamma),g(\delta),f(\epsilon),g(\zeta)}\text{-}puzzles\}.$$



(i) *(Commutative Property of Hexagonal Puzzles with All-Way Symmetry)*
For $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \binom{[a]}{a_1}$ and any bijection $f : \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\} \to \{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$,

$$\#\{\hexagon_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-}puzzles\} = \#\{\hexagon_{f(\alpha),f(\beta),f(\gamma),f(\delta),f(\epsilon),f(\zeta)}\text{-}puzzles\}.$$



Remark 1.2. Theorems 1.1 have analogous statements for 2-step and 3-step puzzles, as well as puzzles that include the piece 

, which compute stucture constants in the K-theory $K(\mathrm{Gr}(k;\mathbb{C}^n))$ in the Schubert structure sheaf basis $\{[\mathcal{O}_\lambda] : \lambda \in \binom{[n]}{k}\}$. Most proofs go through with little change. Details are given in [1] but are omitted here.

In Section 3.4, we will also give a commutative property for parallelogram-shaped equivariant puzzles.

2 Preliminaries

2.1 A geometric interpretation of convex polygonal puzzles

We can give a convex polygonal puzzle a familiar geometric Schubert calculus interpretation via an operation to "complete" it to a triangular puzzle.

Let **sort** denote the operation of moving all of the 0s in a binary string ahead of all the 1s. So, for $\lambda \in \binom{[n]}{k}$, we have $\mathrm{sort}(\lambda) = 0^{n-k}1^k$.

Proposition 2.1. *By completing polygonal puzzles to triangular puzzles, we obtain the following:*

(a) *For $\beta \in \binom{[a]}{a_1}$, $\gamma, \delta \in \binom{[c]}{c_1}$, and $\nu \in \binom{[c+a]}{a_1+c_1}$, there is a bijection*

$$\{\text{⬠}_{\beta,\gamma,\nu,\delta}\text{-}puzzles\} \leftrightarrow \{\triangle_{\mathrm{sort}(\beta)\gamma,\nu,\delta\beta}\text{-}puzzles\}.$$

(b) *For $\alpha, \beta \in \binom{[a]}{a_1}$ and $\gamma, \delta \in \binom{[c]}{c_1}$, there is a bijection*

$$\{\Diamond_{\alpha,\gamma,\beta,\delta}\text{-}puzzles\} \leftrightarrow \{\triangle_{\mathrm{sort}(\alpha)\gamma,\beta\,\mathrm{sort}(\delta),\delta\alpha}\text{-}puzzles\}.$$

(c) For $\alpha \in \binom{[a]}{a_1}$, $\beta \in \binom{[b]}{b_1}$, $\gamma \in \binom{[c]}{c_1}$, $\delta \in \binom{[d]}{d_1}$, $\epsilon \in \binom{[e]}{e_1}$, and $\zeta \in \binom{[z]}{z_1}$, there is a bijection

$$\{\hexagon_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-}puzzles\} \leftrightarrow \{\triangle_{\mathrm{sort}(\alpha)\beta\gamma,\mathrm{sort}(\gamma)\delta\,\mathrm{sort}(\epsilon),\epsilon\zeta\alpha}\text{-}puzzles\}.$$

Proof. See Figure 3. Given the labels on the triangular boundaries, there exists a unique filling of each grey triangular region with puzzle pieces, which forces the labels around the inner pink regions to replicate those for the polygons on the top row. In each case, a bijection is given by gluing on the uniquely filled grey regions, and the inverse is cutting them off. □

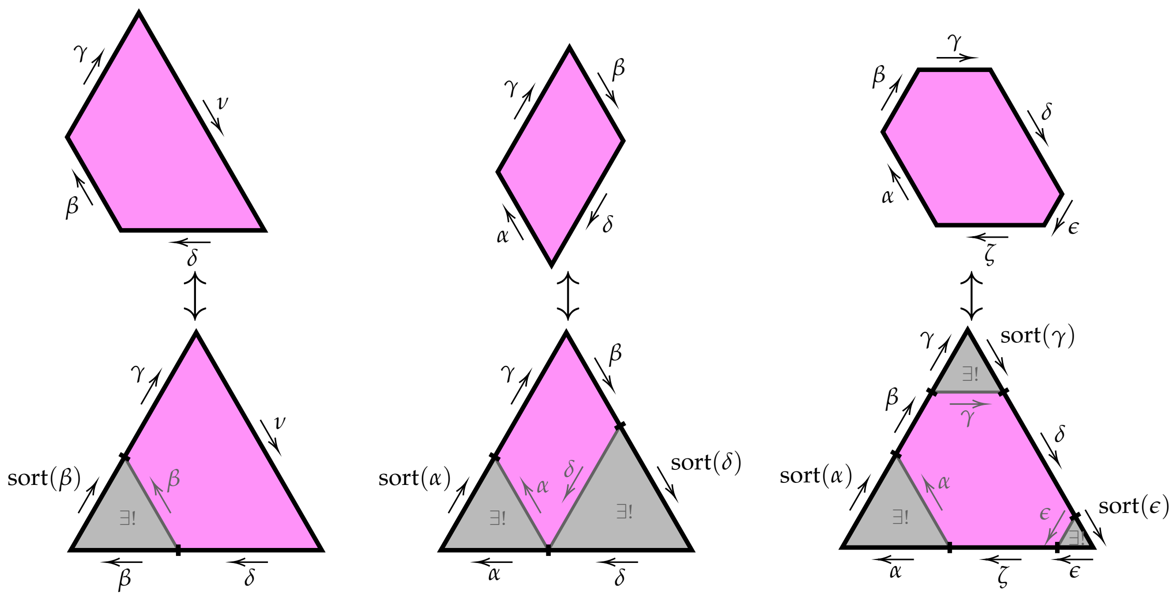


Figure 3: The operations of completing polygonal puzzles to triangular puzzles.

2.2 Geometric lemmas

The following two lemmas form the foundation upon which all of the geometric proofs of the theorems are built. In particular, Lemma 2.3 tells us how we can swap substrings in the labels on two different sides of a triangular boundary when certain symmetries are present, such as the ones seen in the triangular boundary labels in Figure 3.

Lemma 2.2. *Let $F_\bullet$ and $\tilde{F}_\bullet$ be the standard and anti-standard complete flags in $\mathbb{C}^{a+c}$, respectively. Let $T := (\mathbb{C}^\times)^{a+c}$, and define the T-equivariant closed immersion*

$$\Omega : \mathrm{Gr}(a_1; F_a) \times \mathrm{Gr}(c_1; \tilde{F}_c) \hookrightarrow \mathrm{Gr}(a_1 + c_1; \mathbb{C}^{a+c}), \quad (V, W) \mapsto V \oplus W.$$

Then for $\alpha, \beta \in \binom{[a]}{a_1}$ and $\gamma, \delta \in \binom{[c]}{c_1}$, we have

$$X_{\alpha\gamma} \cap X^{(\delta\beta)^\vee} = \Omega((X_\alpha \cap X^{\beta^\vee}) \times (X_\gamma \cap X^{\delta^\vee})), \tag{1}$$

and

$$[X_{\alpha\gamma}][X^{(\delta\beta)^\vee}] = \Omega_*([X_\alpha][X^{\beta^\vee}] \otimes [X_\gamma][X^{\delta^\vee}]) \tag{2}$$

in $H_T^*(\mathrm{Gr}(a_1+c_1;\mathbb{C}^{a+c}))$ and in $H^*(\mathrm{Gr}(a_1+c_1;\mathbb{C}^{a+c}))$.

Proof idea. In [1], we prove the equality of sets in statement (1) by checking that there is an inclusion in both directions, by directly verifying that the conditions on intersection dimensions encoded by the strings are satisfied. Then statement (2) follows immediately from statement (1). $\square$

Lemma 2.3. *Define block matrices $\Phi_a := \left[\begin{array}{c|c} J_a & \mathbf{0} \\ \hline \mathbf{0} & I_c \end{array}\right]$ and $\Phi_c := \left[\begin{array}{c|c} I_a & \mathbf{0} \\ \hline \mathbf{0} & J_c \end{array}\right]$ in $\mathrm{GL}(\mathbb{C}^{a+c})$, where I_a and J_a (resp. I_c and J_c) denote the $a \times a$ (resp. $c \times c$) identity and anti-diagonal permutation matrices, respectively. Let $\alpha, \beta \in \binom{[a]}{a_1}$ and $\gamma, \delta \in \binom{[c]}{c_1}$. Then,*

(i) in $H_T^*(\mathrm{Gr}(a_1+c_1;\mathbb{C}^{a+c}))$, we have

$$[X_{\alpha\gamma}][X^{(\delta\beta)^\vee}] = \Phi_a \cdot ([X_{\beta\gamma}][X^{(\delta\alpha)^\vee}]) = \Phi_c \cdot ([X_{\alpha\delta}][X^{(\gamma\beta)^\vee}]) = \Phi_c \cdot \Phi_a \cdot ([X_{\beta\delta}][X^{(\gamma\alpha)^\vee}])$$

(ii) in $H^*(\mathrm{Gr}(a_1+c_1;\mathbb{C}^{a+c}))$, we have

$$[X_{\alpha\gamma}][X^{(\delta\beta)^\vee}] = [X_{\beta\gamma}][X^{(\delta\alpha)^\vee}] = [X_{\alpha\delta}][X^{(\gamma\beta)^\vee}] = [X_{\beta\delta}][X^{(\gamma\alpha)^\vee}].$$

Proof idea. Applying Lemma 2.2 and a simple geometric argument gives us that $[X_{\alpha\gamma}][X^{(\delta\beta)^\vee}] = \Omega_*([X_\alpha][X^{\beta^\vee}] \otimes [X_\gamma][X^{\delta^\vee}]) = \Omega_*((J_a, I_c) \cdot ([X_\beta][X^{\alpha^\vee}] \otimes [X_\gamma][X^{\delta^\vee}])) = \Phi_a \cdot \Omega_*([X_\beta][X^{\alpha^\vee}] \otimes [X_\gamma][X^{\delta^\vee}]) = \Phi_a \cdot ([X_{\beta\gamma}][X^{(\delta\alpha)^\vee}])$. A similar argument gives the other two equalities in (i). Then (ii) follows from the fact that, in ordinary cohomology $H^*(\mathrm{Gr}(a_1+c_1;\mathbb{C}^{a+c})$, Φ_a and Φ_c act trivially on the classes. $\square$

3 Proofs of Theorems

3.1 Puzzles with split symmetry and trapezoidal puzzles

3.1.1 Proof of Theorems 1.1(a) (puzzles with split symmetry)

Proof. **Geometric Proof.** The claim is equivalent to

$$\int_{\mathrm{Gr}_{a+c}} [X_{\alpha\gamma}][X^{(\delta\beta)^\vee}][X_\nu] = \int_{\mathrm{Gr}_{a+c}} [X_{\beta\gamma}][X^{(\delta\alpha)^\vee}][X_\nu]$$
$$= \int_{\mathrm{Gr}_{a+c}} [X_{\alpha\delta}][X^{(\gamma\beta)^\vee}][X_\nu] = \int_{\mathrm{Gr}_{a+c}} [X_{\beta\delta}][X^{(\gamma\alpha)^\vee}][X_\nu],$$

which follows immediately from Lemma 2.3.

Proof by the commutative property of triangular puzzles.

In any $\triangle_{\alpha\gamma,\nu,\delta\beta}$-puzzle, due to the symmetries in string content, there can be no 10s appearing along the red line shown in Figure 4a. So we can divide the puzzle into a triangular puzzle and a trapezoidal puzzle. Then we can sum over choices of label λ on the red line and obtain

$$\#\{\triangle_{\alpha\gamma,\nu,\delta\beta}\text{-puzzles}\} = \sum_{\lambda} \left(\#\{\triangle_{\alpha,\lambda,\beta}\text{-puzzles}\}\right) \left(\#\{⏢_{\lambda^\vee,\gamma,\nu,\delta}\text{-puzzles}\}\right).$$

Since $\#\{\triangle_{\alpha,\lambda,\beta}\text{-puzzles}\} = \#\{\triangle_{\beta,\lambda,\alpha}\text{-puzzles}\}$ by the commutative property of triangular puzzles and three-fold rotational symmetry, this implies that α and β commute.

To prove that γ and δ commute, we first apply the commutative property of triangular puzzles to the NW and NE sides of the outer boundary to get $\#\{\triangle_{\alpha\gamma,\nu,\delta\beta}\text{-puzzles}\} = \#\{\triangle_{\nu,\alpha\gamma,\delta\beta}\text{-puzzles}\}$, which leads to the picture in Figure 4b. Then we follow a similar argument as the one used to commute α and β above. ☐

(a) α and β commute via the commutative property of triangular puzzles.

(b) γ and δ commute via the commutative property of triangular puzzles.

Figure 4

3.1.2 Proof of Theorems 1.1(b) (trapezoidal puzzles)

Proof. **Proof by the commutative property of puzzles with split symmetry.** After completing to a triangle as in Figure 3a, we can immediately apply Theorems 1.1(a). ☐

3.2 Parallelogram-shaped puzzles

3.2.1 Proof of Theorems 1.1(c) (parallelogram-shaped puzzles)

Proof. For all of the proofs below, we begin by completing to a triangle as in Figure 3b.

Geometric proof. The claim is equivalent to

$$\int_{\mathrm{Gr}_{a+c}} [X_{\mathrm{sort}(\alpha)\gamma}][X_{\beta\,\mathrm{sort}(\delta)}][X^{(\delta\alpha)^\vee}] = \int_{\mathrm{Gr}_{a+c}} [X_{\mathrm{sort}(\alpha)\gamma}][X_{\alpha\,\mathrm{sort}(\delta)}][X^{(\delta\beta)^\vee}]$$
$$= \int_{\mathrm{Gr}_{a+c}} [X_{\mathrm{sort}(\alpha)\delta}][X_{\beta\,\mathrm{sort}(\delta)}][X^{(\gamma\alpha)^\vee}] = \int_{\mathrm{Gr}_{a+c}} [X_{\mathrm{sort}(\alpha)\delta}][X_{\alpha\,\mathrm{sort}(\delta)}][X^{(\gamma\beta)^\vee}],$$

which can be easily obtained through applications of Lemma 2.3.

Proof by the commutative property of triangular puzzles. We first apply the commutative property of triangular puzzles to the NW and NE sides of the outer boundary to get $\#\{\triangle_{\beta\,\mathrm{sort}(\delta),\mathrm{sort}(\alpha)\gamma,\delta\alpha}\text{-puzzles}\} = \#\{\triangle_{\nu,\alpha\gamma,\delta\beta}\text{-puzzles}\}$. From there, we can immediately commute the pairs α, β and γ, δ in a manner similar to that in Figure 4.

Proof by the commutative property of puzzles with split symmetry. This property (given in Theorems 1.1(a)) combined with the three-fold rotational symmetry of puzzles proves the statement. □

3.2.2 Proof of Theorems 1.1(d) (symmetric rhombus-shaped puzzles)

Proof. **Geometric proof.** A geometric proof is given in [1] but is omitted here.

Proof by the commutative properties of triangular and parallelogram-shaped puzzles. This special case of a parallelogram-shaped puzzle can be seen as two triangular puzzles glued together. We apply Theorems 1.1(c) along with the commutative property of triangular puzzles to obtain all permutations of the rhombus's boundary labels. □

3.3 Hexagonal puzzles

Proof of Theorems Theorems 1.1(f) (hexagonal puzzles with opposite sides symmetry)

3.3.1 Proof of Theorems Theorems 1.1(f) (hexagonal puzzles with opposite sides symmetry)

Proof. **Geometric proof.** A geometric proof is given in [1] but is omitted here.

Proof by the commutative property of parallelogram-shaped puzzles. We can complete a hexagonal puzzle to a parallelogram-shaped puzzle to obtain a bijection $\{\hexagon_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta}\text{-puzzles}\} \leftrightarrow \{\lozenge_{\mathrm{sort}(\zeta)\alpha,\beta\gamma,\mathrm{sort}(\gamma)\delta,\epsilon\zeta}\text{-puzzles}\}$. We can then use Theorems 1.1(c) to swap the SW and NE labels, i.e. $\mathrm{sort}(\zeta)\alpha$ and $\mathrm{sort}(\gamma)\delta$, which effectively swaps α and δ, as $\mathrm{sort}(\gamma) = \mathrm{sort}(\zeta)$. Combining this with three-fold rotational symmetry yields the other swaps. □

3.3.2 Proof of Theorems 1.1(g) (hexagonal puzzles with two-way symmetry)

Proof. **Geometric proof.** A geometric proof is given in [1] but is omitted here.

Proof by the commutative property of trapezoidal puzzles: A hexagonal puzzle with this type of symmetry can be seen as two trapezoidal puzzles glued together. Thus, the claim follows easily from Theorems 1.1(b). □

3.3.3 Proof of Theorems 1.1(e),(h),(i)

Proof. These statements can be derived from Theorems 1.1(f) and Theorems 1.1(g), as they are just more specialized cases of those types of symmetries. □

3.4 Commutative property of parallelogram-shaped equivariant puzzles

The equivariant structure constants in the theorem below are the ones associated to parallelogram-shaped equivariant puzzles, after completing to a triangle as in Figure 3b.

Theorem 3.1 (Commutative Property of Structure Constants Associated to Parallelogram-Shaped Equivariant Puzzles)**.** *Let Φ_a and Φ_c be defined as in Lemma 2.3. Let $\alpha, \beta \in \binom{[a]}{a_1}$ and $\gamma, \delta \in \binom{[c]}{c_1}$. Then in $H_T^*(\mathrm{Gr}(a_1 + c_1; \mathbb{C}^{a+c}))$, we have*

$$\begin{aligned}(c_T)^{(\delta\alpha)^\vee}_{\mathrm{sort}(\alpha)\gamma,\beta\,\mathrm{sort}(\delta)} &= \Phi_a \cdot (c_T)^{(\delta\beta)^\vee}_{\mathrm{sort}(\beta)\gamma,\alpha\,\mathrm{sort}(\delta)} \\ &= \Phi_c \cdot (c_T)^{(\gamma\alpha)^\vee}_{\mathrm{sort}(\alpha)\delta,\beta\,\mathrm{sort}(\gamma)} = \Phi_c \cdot \Phi_a \cdot (c_T)^{(\gamma\beta)^\vee}_{\mathrm{sort}(\beta)\delta,\alpha\,\mathrm{sort}(\gamma)}.\end{aligned}$$

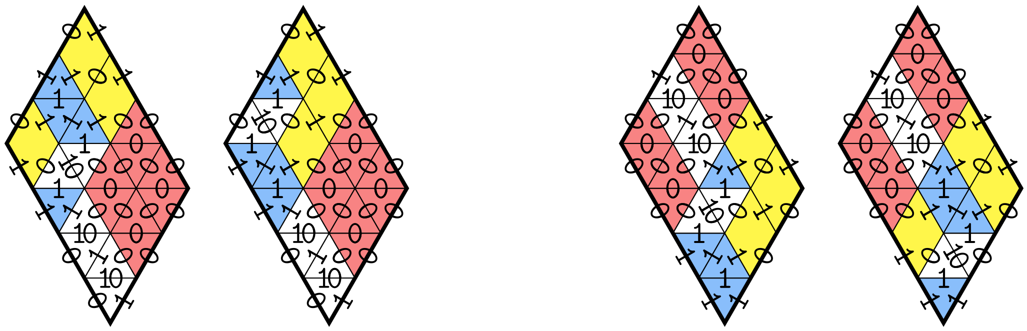
In other words commuting the pair α, β reverses the $y_1, \ldots, y_a$, and commuting the pair γ, δ reverses the $y_{a+1}, \ldots, y_{a+c}$, in the structure constant.

Proof idea. It is not difficult to show that $\Phi_a \cdot X_{\mathrm{sort}(\alpha)\gamma} = X_{\mathrm{sort}(\alpha)\gamma}$, and this along with Lemma 2.3 gives

$$\begin{aligned}\Phi_a \cdot (c_T)^{(\delta\beta)^\vee}_{\mathrm{sort}(\beta)\gamma,\alpha\,\mathrm{sort}(\delta)} &= \int_{\mathrm{Gr}_{a+c}} \left(\Phi_a \cdot [X_{\mathrm{sort}(\alpha)\gamma}]\right) \left(\Phi_a \cdot ([X_{\alpha\,\mathrm{sort}(\delta)}][X^{(\delta\beta)^\vee}])\right) \\ &= \int_{\mathrm{Gr}_{a+c}} [X_{\mathrm{sort}(\alpha)\gamma}][X_{\beta\,\mathrm{sort}(\delta)}][X^{(\delta\alpha)^\vee}] = (c_T)^{(\delta\alpha)^\vee}_{\mathrm{sort}(\alpha)\gamma,\beta\,\mathrm{sort}(\delta)}.\end{aligned}$$

A similar argument proves the other two equalities in the claim. □

Example 3.2. See Figure 5.



(a) The set of all $\lozenge_{0011,010,1100,001}$-puzzles. Sum of the puzzle weights: $(y_5 - y_1)(y_7 - y_1)(y_7 - y_2) + (y_6 - y_2)(y_7 - y_1)(y_7 - y_2)$

(b) The set of all $\lozenge_{1100,010,0011,001}$-puzzles. Sum of the puzzle weights: $(y_6 - y_4)(y_7 - y_3)(y_7 - y_4) + (y_5 - y_3)(y_7 - y_3)(y_7 - y_4)$

Figure 5: The effect of commuting the labels on the SW and NE sides is reversing the y_1, y_2, y_3, y_4 in the structure constant.

4 Further Questions

So far, we relate polygonal puzzles to special classes of triangular puzzles and their associated structure constants, but it is unknown if polygonal puzzles hold a more intrinsic meaning geometrically, combinatorially, or representation-theoretically. It would be interesting to explore this question and to find any illuminating manifestations or proofs of these commutative properties in the various contexts where LR numbers arise.

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