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The Magic Number Conjecture for the $m = 2$ Amplituhedron and Parke-Taylor Polytopes

Matteo Parisi[*1], Melissa Sherman-Bennett[†2], Ran Tessler[3‡], and Lauren Williams[§4]

[1] *CMSA, Harvard University, Cambridge, MA;*
[2] *Department of Mathematics, UC Davis, Davis, CA;*
[3] *Department of Mathematics, Weizmann Institute of Science, Israel;*
[4] *Department of Mathematics, Harvard University, Cambridge, MA.*

Abstract. The *amplituhedron* $\mathcal{A}_{n,k,m}$ is a geometric object which generalizes the positive Grassmannian (when $n = k + m$) and cyclic polytopes (when $k = 1$). It was originally introduced in the context of *scattering amplitudes*. Of substantial interest are the *tilings* of the amplituhedron, which are analogous to triangulations of a polytope. In [13], it was conjectured that for even m the tilings of $\mathcal{A}_{n,k,m}$ have cardinality the *MacMahon number*, the number of plane partitions which fit inside a $k \times (n-k-m) \times \frac{m}{2}$ box. We refer to this prediction as the *Magic Number Conjecture*. In this paper we prove the Magic Number Conjecture for the $m = 2$ amplituhedron: that is, we show that each tiling of $\mathcal{A}_{n,k,2}$ has cardinality $\binom{n-2}{k}$. We prove this by showing that all positroid tilings of the hypersimplex $\Delta_{k+1,n}$ have cardinality $\binom{n-2}{k}$, then applying *T-duality*. In addition, we give volume formulas for *Parke-Taylor polytopes* and tree positroid polytopes in terms of circular extensions of *cyclic partial orders*; and we prove new variants of the classical *Parke-Taylor identities*.

Keywords: positroid, amplituhedron, hypersimplex, tile, tiling, MacMahon number

1 Introduction

The (tree) *amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of the positive Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$ under the *amplituhedron map* $\tilde{Z} : \mathrm{Gr}_{k,n}^{\geq 0} \to \mathrm{Gr}_{k,k+m}$, a map induced by matrix multiplication by a positive matrix $Z \in \mathrm{Mat}_{n,k+m}^{>0}$. The amplituhedron was introduced by Arkani-Hamed and Trnka [2] in order to give a geometric interpretation of *scattering amplitudes* in $\mathcal{N} = 4$

[*] mparisi@cmsa.fas.harvard.edu.
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super Yang–Mills theory. Central to this interpretation are *tilings* of $\mathcal{A}_{n,k,m}(Z)$, which are decompositions of $\mathcal{A}_{n,k,m}(Z)$ into *tiles* (see Definition 2.2). The notion of tiling can be thought of as a generalization of the notion of triangulation of a polytope.[1]

While the case $m = 4$ is most directly relevant to physics, the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ makes sense for any positive n, k, m such that $k + m \leq n$, and has a very rich geometric and combinatorial structure. It generalizes cyclic polytopes (when $k = 1$), cyclic hyperplane arrangements [12] (when $m = 1$), and the positive Grassmannian (when $k = n - m$), and it is connected to the hypersimplex and the positive tropical Grassmanian [15, 21] (when $m = 2$). This paper will focus on the case $m = 2$.

In [13] it was observed that the known tilings of $\mathcal{A}_{n,k,2}(Z)$ have cardinality $\binom{n-2}{k}$, the known tilings of $\mathcal{A}_{n,k,4}(Z)$ have cardinality the Narayana number $\frac{1}{n-3}\binom{n-3}{k+1}\binom{n-3}{k}$, and all tilings of $\mathcal{A}_{n,1,m}$ for even m have cardinality $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$.[2] Based on these observations, [13, Conjecture 8.1] conjectured that when m is even, $\mathcal{A}_{n,k,m}$ has a tiling with cardinality

$$M_{n,k,m} := M\left(k, n-k-m, \frac{m}{2}\right), \quad \text{where} \quad M(a,b,c) := \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{\ell=1}^{c} \frac{i+j+\ell-1}{i+j+\ell-2}$$

is the *MacMahon number*. This number has many remarkable interpretations, e.g. $M(a,b,c)$ counts the number of *plane partitions* which fit inside an $a \times b \times c$ box, collections of c noncrossing lattice paths inside an $a \times b$ rectangle, perfect matchings of the honeycomb lattice, and more (see [17, 7]). [10] later conjectured that *each* tiling should have this cardinality. We refer to the prediction that tilings of the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ should have cardinality $M_{n,k,m}$ as the *Magic Number Conjecture*.

Main results. The main result of this abstract is the proof of the Magic Number Conjecture for the $m = 2$ amplituhedron (Theorem 3.7). We first prove that each tiling of the hypersimplex $\Delta_{k+1,n}$ by positroid polytopes has cardinality $\binom{n-2}{k}$ (Theorem 3.5). In the special case tilings are finest regular subdivisions of $\Delta_{k+1,n}$ into positroid polytopes, this recovers [26]. We also prove the more general result that if $\Gamma_{\mathcal{M}}$ is a full-dimensional positroid polytope, all of its tilings have the same cardinality (Corollary 3.6). The key ingredient for the proofs is to show that, although tiles do not have the same volume (see Figure 3), there is an additive invariant on $\Delta_{k+1,n}$ defined in terms of *Parke-Taylor functions* (Definition 3.1) that is constant for all tiles (Proposition 3.3). Parke-Taylor functions have numerous connections e.g. with the cohomology of the moduli space $\mathcal{M}_{0,n}$ of n points on the Riemann sphere in relation with *scattering equations* [8] and *Lie polynomials* [9]. Then we use Theorem 3.5 and apply the T-duality theorem from [21, Theorem 11.6] (which appears here as Theorem 2.9) to prove the Magic Number Conjecture for the $m = 2$ amplituhedron (Theorem 3.7). Finally, we introduce *Parke-Taylor polytopes*

[1] We do not require tiles to intersect in a common "face".

[2] Every triangulation of the cyclic polytope $C(n,m)$ contains exactly $\binom{n-1-\frac{m}{2}}{\frac{m}{2}}$ simplices when m is even [5, 24] and for $C(n,m)$ tilings and triangulations coincide [19].

(Definition 3.15), whose volume equals the number of *circular extensions* of a certain partial cyclic order (Proposition 3.18). For each Parke-Taylor polytope we associate a combinatorial identity among Parke-Taylor functions involving such circular extensions (Theorem 3.21). As corollary, we obtain a formula for the volume of the positroid polytopes which are tiles for $\Delta_{k+1,n}$ (Remark 3.19). Complete arguments for these results appear in [20].

2 Background

The *Grassmannian* $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(\mathbb{R})$ is the space of all k-dimensional subspaces of the $\mathbb{R}^n$ vector space. Let $[n]$ denote $\{1, \ldots, n\}$, and $\binom{[n]}{k}$ denote the set of all k-element subsets of $[n]$. We can represent a point $V \in \mathrm{Gr}_{k,n}$ as the row-span of a full-rank $k \times n$ matrix C with entries in $\mathbb{R}$. Then for $I = \{i_1 < \cdots < i_k\} \in \binom{[n]}{k}$, the *Plücker coordinate* $P_I(C)$ is the $k \times k$ minor of C using the columns I. The Plücker coordinates of V are independent of the choice of matrix representative C (up to common rescaling). The *Plücker embedding* $V \mapsto \{P_I(C)\}_{I \in \binom{[n]}{k}}$ embeds $\mathrm{Gr}_{k,n}$ into projective space[3].

Definition 2.1. [16, 22] We say that $V \in \mathrm{Gr}_{k,n}$ is *totally nonnegative* if (up to a global change of sign) $P_I(C) \geq 0$ for all $I \in \binom{[n]}{k}$. Similarly, V is *totally positive* if $P_I(C) > 0$ for all $I \in \binom{[n]}{k}$. We let $\mathrm{Gr}_{k,n}^{\geq 0}$ and $\mathrm{Gr}_{k,n}^{>0}$ denote the set of totally nonnegative and totally positive elements of $\mathrm{Gr}_{k,n}$, respectively. The set $\mathrm{Gr}_{k,n}^{\geq 0}$ is called the *totally nonnegative Grassmannian*, or sometimes just the *positive Grassmannian*.

If we partition $\mathrm{Gr}_{k,n}^{\geq 0}$ into strata based on which Plücker coordinates are strictly positive and which are 0, we obtain a cell decomposition of $\mathrm{Gr}_{k,n}^{\geq 0}$ into *positroid cells* [22] that glue together to form a CW complex [23]. Each positroid cell S gives rise to a matroid $\mathcal{M}$, whose bases are precisely the k-element subsets I such that the Plücker coordinate P_I does not vanish on S; the matroid $\mathcal{M}$ is called a *positroid*.

We will be interested in certain images of positroid cells.

Definition 2.2. Let $\phi : \mathrm{Gr}_{k,n}^{\geq 0} \to X$ be a continuous surjective map where $\dim X = d$. The closure $\overline{\phi(S)}$ of the image of a positroid cell $S \subset \mathrm{Gr}_{k,n}^{\geq 0}$ is a *positroid tile* for X if ϕ is injective on S and $\dim S = d$. A *positroid tiling* of X is a collection $\{\overline{\phi(S)}\}_{S \in \mathcal{C}}$ of positroid tiles that cover X and have pair-wise disjoint interiors.

We will focus here on positroid tilings for two different maps ϕ: the moment map μ and the amplituhedron map $\tilde{Z}$, which maps $\mathrm{Gr}_{k,n}^{\geq 0}$ onto the hypersimplex and the amplituhedron, respectively.

[3] We will sometimes abuse notation and identify C with its row-span.

The hypersimplex. Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $\mathbb{R}^n$, and define $e_I := \sum_{i\in I} e_i \in \mathbb{R}^n$. If $x \in \mathbb{R}^n$ and $I \subset [n]$, we set $x_I := \sum_{i\in I} x_i$. For $i \neq j \in [n]$, the cyclic interval from i to j is $[i,j] := \{i, i+1, \ldots, j-1, j\}$.

Definition 2.3. The $(k+1,n)$*-hypersimplex* $\Delta_{k+1,n} \subset \mathbb{R}^n$ is the convex hull of the points e_I where I runs over $\binom{[n]}{k+1}$.

The hypersimplex $\Delta_{k+1,n}$ is the image of the Grassmannian $\mathrm{Gr}_{k+1,n}$ (or the positive Grassmannian $\mathrm{Gr}^{\geq 0}_{k+1,n}$, [28, Proposition 7.10]) under the *moment map*. The hypersimplex $\Delta_{k+1,n}$ is $(n-1)$-dimensional, as it is contained in the hyperplane $x_{[n]} = k+1$.

Remark 2.4. The projected hypersimplex $\pi(\Delta_{k+1,n})$ under $\pi : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ is the slice of the unit hypercube $\square_{n-1}$ contained between $x_{[n-1]} = k$ and $x_{[n-1]} = k+1$.

The moment map images of positroid cells can easily be described using positroids:

Proposition 2.5 ([28, Proposition 7.10])**.** *Let $S \subset \mathrm{Gr}^{\geq 0}_{k+1,n}$ be a positroid cell, $\mathcal{M}$ its corresponding positroid, and $\Gamma_{\mathcal{M}} := conv\{e_I : I \in \mathcal{M}\}$ its associated* positroid polytope. *Then $\overline{\mu(S)} = \Gamma_{\mathcal{M}}$.*

Specializing Definition 2.2, with ϕ being the moment map μ and X being the hypersimplex $\Delta_{k+1,n}$, we denote the positroid tiles for $\Delta_{k+1,n}$ as $\Gamma_S := \overline{\mu(S)}$.

The amplituhedron. Building on [1, 11], Arkani-Hamed and Trnka [2] defined the *(tree) amplituhedron* as the image of the positive Grassmannian under a positive linear map. Let $\mathrm{Mat}^{>0}_{n,p}$ denote the set of $n \times p$ matrices whose maximal minors are positive.

Definition 2.6. Let $Z \in \mathrm{Mat}^{>0}_{n,k+m}$, where $k+m \leq n$. The *amplituhedron map* $\tilde{Z} : \mathrm{Gr}^{\geq 0}_{k,n} \to \mathrm{Gr}_{k,k+m}$ is defined by $\tilde{Z}(C) := CZ$, where C is a $k \times n$ matrix representing an element of $\mathrm{Gr}^{\geq 0}_{k,n}$, and CZ is a $k \times (k+m)$ matrix representing an element of $\mathrm{Gr}_{k,k+m}$. The *amplituhedron* $\mathcal{A}_{n,k,m}(Z) \subset \mathrm{Gr}_{k,k+m}$ is the image $\tilde{Z}(\mathrm{Gr}^{\geq 0}_{k,n})$.

Specializing Definition 2.2, with ϕ being the amplituhedron map $\tilde{Z}$ and X being the amplituhedron $\mathcal{A}_{n,k,m}(Z)$, we denote the positroid tiles for $\mathcal{A}_{n,k,m}(Z)$ as $Z_S := \overline{\tilde{Z}(S)}$.

In this work it is convenient to work with "all-Z" tilings, as defined below.

Definition 2.7. We call $\{Z_S\}_{S\in\mathcal{C}}$ an *all-Z tiling* of the amplituhedron $\mathcal{A}_{n,k,m}$ if $\{Z_S\}_{S\in\mathcal{C}}$ is a tiling of $\mathcal{A}_{n,k,m}(Z)$ for all $Z \in \mathrm{Mat}^{>0}_{n,k+m}$.

Correspondence between the hypersimplex and the amplituhedron.

Definition 2.8. Let $\mathbf{P}_n$ be a convex n-gon with vertices labeled from 1 to n in clockwise order. A *tricolored subdivision* τ is a partition of $\mathbf{P}_n$ into black, white, and grey polygons such that two polygons sharing an edge have different colors. We say that τ has *type*

Figure 1: (Left): A tricolored subdivision τ of type $(3,2,8)$. It gives rise to the cyclic order C_τ which is the union of the chains $C_{(2,5,7)}$, $C_{(5,7,6)}$, and $C_{(1,8,7,2)}$. (Right): A bicolored subdivision σ of type $(5,9)$, it corresponds to a tile Γ_σ in $\Delta_{6,9}$.

(k,ℓ,n) if any triangulation of the black (respectively, grey) polygons consists of exactly k black (respectively, ℓ grey) triangles.

A *bicolored subdivision* σ of type (k,n) is a tricolored subdivision of type $(k,0,n)$, that is, with no grey polygons. See Figure 1.

In [21], a subset of the authors of this work showed that positroid tiles for both $\Delta_{k+1,n}$ and $\mathcal{A}_{n,k,2}(Z)$ are in bijection with the bicolored subdivisions of type (k,n)[4]. From a bicolored subdivision σ of type (k,n) one can directly read the inequalities describing the associated positroid tile Γ_σ for $\Delta_{k+1,n}$ and the positroid tile Z_σ for $\mathcal{A}_{n,k,2}(Z)$. In particular, tiles of the hypersimplex $\Delta_{k+1,n}$ and tiles of the amplituhedron $\mathcal{A}_{n,k,2}$ are in bijection with each other, and this bijection induces a bijection on tilings as well:

Theorem 2.9 ([21, Theorem 11.6])**.** *Let $\mathcal{S}$ be a collection of bicolored subdivisions of type (k,n). Then $\{\Gamma_\sigma\}_{\sigma\in\mathcal{S}}$ is a positroid tiling of $\Delta_{k+1,n}$ if and only if $\{Z_\sigma\}_{\sigma\in\mathcal{S}}$ is a tiling of $\mathcal{A}_{n,k,2}$.*

See [21, Table 1] for many other correspondences between the hypersimplex and the $m = 2$ amplituhedron.

The triangulation of the hypersimplex into w-simplices.

Definition 2.10. Let $w \in S_n$. A letter $i < n$ is a *left descent*[5] of w if i occurs to the right of $i+1$ in w, i.e. $w^{-1}(i) > w^{-1}(i+1)$. We say $i \in [n]$ is a *cyclic left descent* of w if either $i < n$ is a left descent of w or if $i = n$ and 1 occurs to the left of n in w. We let $\mathrm{cDes_L}(w)$ be the set of cyclic left descents of w. We often omit the word 'left' in 'cyclic left descent'.

Definition 2.11. Choose $0 \le k \le n-2$. We let D_n be the set of permutations $w \in S_n$ with $w_n = n$ and $D_{k+1,n}$ to be the set of permutations $w \in D_n$ with $k+1$ cyclic descents.

Note that $|D_{k+1,n}|$ equals the *Eulerian number* $E_{k,n-1} := \sum_{\ell=0}^{k+1}(-1)^\ell\binom{n}{\ell}(k+1-\ell)^{n-1}$.

[4] these are enumerated by a refinement of the *large Schröder numbers* [25, A175124] and are in bijection with *separable permutations* on $[n-1]$ with k descents [21, Corollary 12.6].

[5] Left descents, sometimes called *recoils* in the literature, are discussed extensively in [6, Chapter 1].

Example 2.12. The set $D_{2,4}$ contains the following $E_{2,3} = 4$ permutations.

w	1324	3124	2134	2314
$\mathrm{cDes_L}(w)$	$\{2,4\}$	$\{2,4\}$	$\{1,4\}$	$\{1,4\}$

For $w = w_1 w_2 \dots w_n \in S_n$, we write $(w) = (w_1\ w_2 \cdots w_n)$ for the cycle $w_1 \mapsto w_2 \mapsto \cdots \mapsto w_n \mapsto w_1$. The map $w \mapsto (w)$ bijects D_n onto the set of n-cycles in S_n.

Definition 2.13 (w-simplices)**.** For $w = w_1 w_2 \dots w_n \in D_{k+1,n}$, let $w^{(a)}$ denote the cyclic rotation of w ending at a. We denote[6] $I_r = I_r(w) := \mathrm{cDes_L}(w^{(r)})$ and define the *w-simplex* $\Delta_{(w)} \subseteq \Delta_{k+1,n}$ to be the convex hull of $e_{I_1}, \dots, e_{I_n}$; this is an $(n-1)$-dimensional simplex.

Example 2.14. By Example 2.12, $\Delta_{2,4}$ has four w-simplices, see Figure 2. The vertices of $\Delta_{(w)}$ with $w = 1324$ are computed from the cyclic descents of the rotations of w.

r	1	2	3	4
$w^{(r)}$	3241	4132	2413	1324
$I_r = \mathrm{cDes_L}(w^{(r)})$	$\{1,2\}$	$\{2,3\}$	$\{1,3\}$	$\{2,4\}$

Then $\Delta_{(1324)}$ is the convex hull of $e_{12}, e_{23}, e_{13}, e_{24}$ (the top-right red simplex in Figure 2).

Figure 2: (Left): The hypersimplex $\Delta_{2,4}$ projected in $\mathbb{R}^3$. (Right): w-simplices for $\Delta_{2,4}$. We omit parentheses in subscripts for readability.

The following triangulation of the hypersimplex first appeared in [27], though the description in terms of w-simplices appeared first in [14].

Proposition 2.15 ($\Delta_{k+1,n}$ is the union of w-simplices [27])**.** *The w-simplices $\{\Delta_{(w)} : w \in D_{k+1,n}\}$ are the maximal simplices of a triangulation of the hypersimplex $\Delta_{k+1,n}$. Moreover, projecting $\{\Delta_{(w)} : w \in S_n\}$ into $\mathbb{R}^{n-1}$ (see Remark 2.4), we obtain the maximal simplices in a triangulation of the hypercube $\square_{n-1}$ which refines the subdivision of the hypercube into hypersimplices.*

[6] Note I_r depends only on (w) rather than w itself.

The triangulation into w-simplices is the simultaneous refinement of all positroid tilings [14, Theorem 2.7]. It follows that:

Proposition 2.16 ([14]). *Every positroid tile*[7] *for $\Delta_{k+1,n}$ has a triangulation into w-simplices.*

Remark 2.17. The bijection between tiles of the hypersimplex and the amplituhedron can be further refined using w-simplices and *w-chambers*. Each tile Z_σ of the amplituhedron is a union of w-chambers [21, Corollary 10.17]. If $\Delta^Z_{(w)}$ is nonempty, then $\Delta^Z_{(w)} \subset Z_\sigma$ if and only if $\Delta_{(w)} \subset \Gamma_\sigma$ [21, Proposition 11.1].

3 Results

The Magic Number Theorem for the $m = 2$ Amplituhedron. We show that every tiling of the hypersimplex $\Delta_{k+1,n}$ and every all-Z tiling of $\mathcal{A}_{n,k,2}$ consists of $\binom{n-2}{k}$ tiles.

Definition 3.1. The *Parke-Taylor function* of a permutation[8] $w = w_1 \dots w_n \in S_n$ is:

$$\mathrm{PT}(w) := \frac{1}{P_{w_1 w_2} P_{w_2 w_3} \dots P_{w_n w_1}}, \tag{3.1}$$

viewed as a rational function on $\widehat{\mathrm{Gr}}_{2,n}$, the affine cone over the Grassmannian $\mathrm{Gr}_{2,n}$, where the P_{ij} are Plücker coordinates[9] on $\mathrm{Gr}_{2,n}$.

Definition 3.2. Let the *weight function* of a w-simplex for $\Delta_{k+1,n}$ be $\Omega(\Delta_{(w)}) := \mathrm{PT}(w)$. Let $\Gamma \subset \Delta_{k+1,n}$ be a full-dimensional positroid polytope, or any other subset of $\Delta_{k+1,n}$ which is a union of w-simplices (cf. Proposition 2.16). We define the *weight function* of Γ to be the sum of the weight functions of w-simplices included in Γ:

$$\Omega(\Gamma) := \sum_{\Delta_{(w)} \subset \Gamma} \Omega(\Delta_{(w)}).$$

The main step to prove the magic number conjecture for $\mathcal{A}_{n,k,2}$ is the following:

Proposition 3.3. *Let Γ_σ be a tile of $\Delta_{k+1,n}$. Then the weight function of Γ_σ is*

$$\Omega(\Gamma_\sigma) = (-1)^k \, \mathrm{PT}(\mathbf{I}_n), \tag{3.2}$$

with $\mathbf{I}_n \in S_n$ the identity permutation. Hence, all tiles of $\Delta_{k+1,n}$ have the same weight function.

In contrast, the normalized volume of tiles is far from constant (see Figure 3).

[7] In fact, this statement hold for any full-dimensional positroid polytope.
[8] $\mathrm{PT}(w)$ depends only on the n-cycle (w); that is, if $(u) = (w)$, then $\mathrm{PT}(w) = \mathrm{PT}(u)$
[9] with the convention that Plücker coordinates are antisymmetric in their indices

Proposition 3.4. *Let $R \subset \Delta_{k+1,n}$ be a subset of $\Delta_{k+1,n}$ which admits a positroid tiling, i.e. it can be written as the union of positroid tiles $\{\Gamma_\sigma\}_{\sigma\in\mathcal{S}}$ whose interiors are disjoint. Then all positroid tilings of R have the same cardinality.*

Proof. For *any* tiling $\{\Gamma_\sigma\}_{\sigma\in\mathcal{S}}$ of R, we have

$$\Omega(R) = \sum_{\sigma\in\mathcal{S}} \sum_{\Delta_{(w)}\subset\Gamma_\sigma} \Omega(\Delta_{(w)}) = \sum_{\sigma\in\mathcal{S}} \Omega(\Gamma_\sigma) = |\mathcal{S}|(-1)^k \operatorname{PT}(\mathbf{I}_n),$$

where for the first and second equality we used that $\Gamma_\sigma = \cup_{\Delta_{(w)}\subset\Gamma_\sigma}\Delta_{(w)}$ and that the tiles $\{\Gamma_\sigma\}_{\sigma\in\mathcal{S}}$ have disjoint interiors and cover $\Delta_{k+1,n}$. For the last equality we used Proposition 3.3. It follows that each positroid tiling of R must have the same cardinality. □

By Proposition 3.4, and the fact that $\Delta_{k+1,n}$ admits tilings of size $\binom{n-2}{k}$ [4], we deduce:

Theorem 3.5. *Every positroid tiling of $\Delta_{k+1,n}$ consists of $\binom{n-2}{k}$ tiles.*

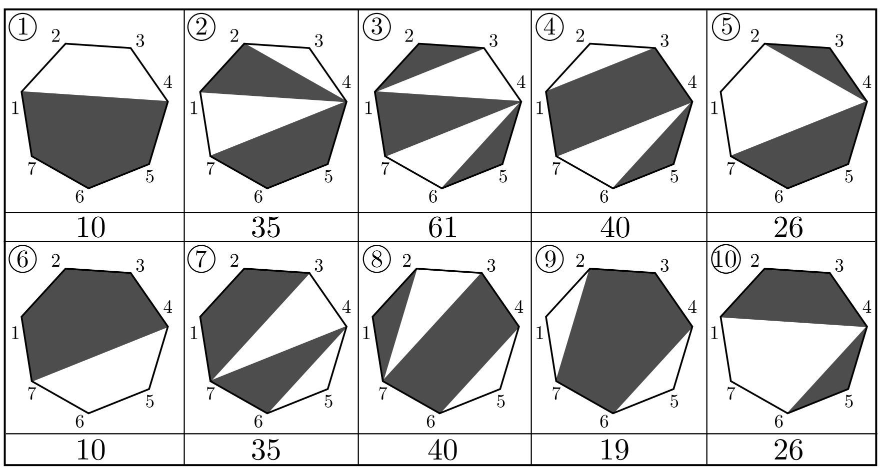

Figure 3: A collection of $10 = \binom{7-2}{3} = M_{7,3,2}$ bicolored subdivisions of type $(3,7)$ (labelled from ① to ⑩) which gives a tiling for $\Delta_{4,7}$ and $\mathcal{A}_{7,3,2}$. The number in the box below each bicolored subdivision σ is the volume of the corresponding positroid polytope Γ_σ in $\Delta_{4,7}$, which equals $\operatorname{Ext}(C_\sigma)$. E.g. for ⑤ we have: $C_\sigma = C_{(1247)} \cup C_{(243)} \cup C_{(4765)}$ and $\operatorname{Ext}(C_\sigma) = 26$. The sum of their volumes is the Eulerian number $E_{3,6} = 302$.

Using results on the positive Dressian, one can show that every full-dimensional positroid polytope has a positroid tiling; see [20, Proposition 5.14]. Therefore, by Proposition 3.4:

Corollary 3.6. *All positroid tilings of a full-dimensional positroid polytope $\Gamma_{\mathcal{M}} \subset \Delta_{k+1,n}$ have the same cardinality.*

It would be interesting to find an explicit combinatorial formula for such cardinality. Using Theorem 2.9 and Theorem 3.5 we can deduce the Magic Number Conjecture:

Theorem 3.7. *Every all-Z positroid tiling of $\mathcal{A}_{n,k,2}$ consists of $M_{n,k,2} = \binom{n-2}{k}$ tiles.*

We can also get the amplituhedron-analogue of Proposition 3.4:

Proposition 3.8. *If R is a full-dimensional subset of $\mathcal{A}_{n,k,2}(Z)$ which admits an all-Z positroid tiling, then every all-Z positroid tiling of R has the same cardinality.*

An interesting case for Proposition 3.8 is when R is (the closure of) the full-dimensional image of a positroid cell S under the amplituhedron map $\tilde{Z}$, also called *Grasstope*.

Circular extensions: Parke-Taylor polytopes and identities. *Circular extensions* and *partial cyclic orders* are analogues of linear extensions and partial orders.

Definition 3.9. A *(partial) cyclic order* on a finite set X is a ternary relation $C \subset X^3$ such that for all $a, b, c, d \in X$:

$$\begin{aligned} (a,b,c) \in C &\implies (c,a,b) \in C && \text{(cyclicity)} \\ (a,b,c) \in C &\implies (a,c,b) \notin C && \text{(asymmetry)} \\ (a,b,c) \in C \text{ and } (a,c,d) \in C &\implies (a,b,d) \in C && \text{(transitivity)} \end{aligned}$$

A cyclic order C is *total* if for all $a, b, c \in X$, either $(a,b,c) \in C$ or $(a,c,b) \in C$.

Informally, a total cyclic order C on $[n]$ is a way of placing $1, \ldots, n$ on a circle, just as a total order is a way of placing $1, \ldots, n$ on a line. Given $w = w_1 \ldots w_n \in S_n$, we denote as C_w the total cyclic order obtained by placing $w_1, w_2, \ldots, w_n$ on the circle clockwise. We also identify C_w with the n-cycle (w).

Definition 3.10. A total cyclic order C is a *circular extension* of a cyclic order C' if $C' \subset C$. We let $\mathrm{Ext}(C)$ denote the set of all[10] circular extensions of a cyclic order C.

Definition 3.11. Let $x_1, \ldots, x_m$ be a sequence of m distinct elements of $[n]$. The *chain* $C_{(x_1, x_2 \ldots, x_m)}$ is the cyclic order in which $x_1, x_2 \ldots, x_m$ appear on a circle ordered clockwise.

We now associate a partial cyclic order to every tricolored subdivision. See Figure 1 for an example.

Definition 3.12 (τ-order)**.** Let τ be a tricolored subdivision of $\mathbf{P}_n$ which includes q non-grey polygons $P_1, \ldots, P_q$. If P_a is white (respectively, black), we let $v_1, \ldots, v_r$ denote its list of vertices read in clockwise (respectively, counterclockwise) order. We then associate the chain $C_a = C_{(v_1, \ldots, v_r)}$ to P_a. Finally, we define the *τ-order* C_τ to be the cyclic order which is the union of the cyclic orders associated to the black and white polygons $C_1, \ldots, C_q$, see Figure 1.

[10] Not all cyclic orders have a circular extension [18], that is, $\mathrm{Ext}(C)$ could be empty.

For each tricolored subdivision, we now introduce a polytope in $\mathbb{R}^{n-1}$. We will need to work with the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ from Remark 2.4.

Definition 3.13. Given a bicolored subdivision σ, let $\tilde{\Gamma}_\sigma$ denote the projected polytope $\pi(\Gamma_\sigma)$. Given a w-simplex $\Delta_{(w)}$, let $\tilde{\Delta}_{(w)}$ denote the projected simplex $\pi(\Delta_{(w)})$.

Definition 3.14. Let τ be a tricolored subdivision. Given a pair of vertices i, j of $\mathbf{P}_n$, we say that the arc $i \to j$ is *compatible* with τ if the arc either is an edge of a black, white, or grey polygon, or lies entirely inside a black or white polygon of τ. If $i \to j$ is compatible with τ, the *area to the left of* $i \to j$ (respectively, *grey area to the left of* $i \to j$), denoted by $\text{area}(i \to j)$ (respectively, $\text{gr-area}(i \to j)$), is the number of black triangles (respectively, grey triangles) to the left of $i \to j$ in any triangulation of the black (respectively, grey) polygons of τ.

Definition 3.15. Let τ be a tricolored subdivision of $\mathbf{P}_n$ of type (k, ℓ, n). We define the *Parke-Taylor* polytope $\tilde{\Gamma}_\tau \subset \mathbb{R}^{n-1}$ by the following inequalities: for any compatible arc $i \to j$ with $i < j$,

$$\text{area}(i \to j) \leq x_{[i,j-1]} \leq \text{area}(i \to j) + \text{gr-area}(i \to j) + 1.$$

Example 3.16. If τ is the tricolored subdivision of $\mathbf{P}_n$ which is just a grey polygon on n vertices, then the Parke-Taylor polytope $\tilde{\Gamma}_\tau$ is the unit hypercube $\square_{n-1} \subset \mathbb{R}^{n-1}$.

Remark 3.17. In the case when τ has no grey polygons ($\ell = 0$), i.e. τ is a bicolored subdivision of type (k, n), Parke-Taylor polytopes are (the projected) tree positroid polytopes, which are the tiles for $\Delta_{k+1,n}$. This follows from Definition 3.15 and [21, Proposition 9.5].

Parke-Taylor polytopes admit positroid tilings, for example via the *kermit subdivisions* (see [20, Definition 2.23])). Each $\tilde{\Gamma}_\tau$ can also be triangulated into projected w-simplices using the combinatorics of circular extensions of the cyclic τ-order:

Proposition 3.18. *Let τ be a tricolored subdivision of type (k, ℓ, n). Then*

$$\tilde{\Gamma}_\tau = \bigcup_{(w) \in \text{Ext}(C_\tau)} \tilde{\Delta}_{(w)}.$$

It follows that the volume of $\tilde{\Gamma}_\tau$ is the number of circular extensions of the cyclic order C_τ. That is, $\text{Vol}(\tilde{\Gamma}_\tau) = |\text{Ext}(C_\tau)|$, *see Figure 3.*

Remark 3.19. By Remark 3.17, Proposition 3.18 describes exactly which w-simplices triangulate the positroid polytope Γ_σ, which is a tile for $\Delta_{k+1,n}$. Moreover, it provides a formula for the volume of Γ_σ as the number of circular extensions of the cyclic order C_σ.

Remark 3.20. Part of Proposition 3.18 bears some analogies with a result from [3]. The two results agree in a very special case, but otherwise deal with different contexts.

For each tricolored subdivision, using Proposition 3.3, we also prove an identity among Parke-Taylor functions (cf. Definition 3.1).

Theorem 3.21 (Parke-Taylor identities from tricolored subdivisions)**.** *Let τ be a tricolored subdivision of $\mathbf{P}_n$ which contains at least one grey polygon, and let C_τ be the corresponding cyclic partial order. Recall that $\mathrm{Ext}(C_\tau)$ is the set of cyclic extensions of C_τ. Then we have that*

$$\sum_{(w)\in \mathrm{Ext}(C_\tau)} \mathrm{PT}(w) = 0.$$

By specializing to particular tricolored subdivisions, we can use Theorem 3.21 to prove some well-known Parke-Taylor identities from physics [9], such as the *shuffle* and *$U(1)$-decoupling identities* [20, Proposition 7.17, Corollary 7.16].

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