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Generalized Laplacian tilings and root zonotopes

Theo Douvropoulos *[1] and Alex McDonough†[2]

[1]*Department of Mathematics, Brandeis University, Waltham, MA, USA.*
[2] *Department of Mathematics, University of Oregon, Eugene, OR, USA.*

Abstract. Given any two equi-oriented collections A and B of vectors in $\mathbb{R}^r$, we construct a family of tilings of $\mathbb{R}^r$. The tiles themselves are related to the bases of A, while their relative frequency are related to bases of B. Our construction gives a common generalization to Penrose tilings as well as zonotopal space tilings associated to regular matroids.

Keywords: periodic tilings, aperiodic tilings, Laplacian, root systems, zonotopes

1 Introduction

A well known theorem by Shephard [16] states that the zonotope associated with a real matrix M tiles its affine span if and only if the matroid represented by M is regular; this happens in particular if M is unimodular. A less well known generalization by Crapo and Senechal [5] states that for general M, one can still tile space using the bases of M, but some tiles will appear with higher frequency than others. In particular, their tilings consist of translates of the zonotope associated with M along with some additional copies of certain tiles to fill in the gaps. Crapo and Senechal showed further that when the entries of M are rational, the tiling is periodic.

Their work utilizes the cut-and-project method, which uses a lattice $\mathbb{Z}^n$ and a lower dimensional affine space X (that "cuts" $\mathbb{Z}^n$). The tiling is constructed by projecting certain points of $\mathbb{Z}^n$ on to X when these points are sufficiently close to X. This construction goes back to DeBruijn who used it to give an algebraic proof of the properties of the Penrose tilings of the plane with thin and thick rhombi [6]. In the Crapo–Senechal construction, every tile appears with frequency proportional to its volume. In particular, Penrose tilings are aperiodic because the ratio of the volumes of the two types of rhombi is the irrational number $\phi = \frac{1+\sqrt{5}}{2}$.

In cases like the Penrose tiling, where the frequency of each tile is proportional to its volume, it follows that the total volume covered by all tiles of a given type is proportional to the *square* of its volume. When the tilings are periodic, this property essentially implies (via the Cauchy–Binet formula) that the volume of the fundamental domain of the tiling

*tdouvropoulos@brandeis.edu.
†alexmcd@uoregon.edu. Alex McDonough was partially supported by NSF grant DMS-2039316.

must agree with the determinant of the Laplacian matrix $L := M \cdot M^T$. This was already observed in particular when M is unimodular: the zonotope $Z(M)$ has volume equal to $\det(L)$ (see for instance [8, Proposition 2.52]).

The second author first explored tilings as a way to understand a Laplacian result in *high-dimensional chip-firing*, where a matrix is multiplied by its own transpose [12]. However, later work suggested that related constructions can be used to give a geometric interpretation for more general applications of the Cauchy–Binet formula, where one is interested in $A \cdot B^T$ for two possibly distinct matrices A and B [9]. This idea ties in with works of the first author, which suggest that such generalizations often encode nontrivial combinatorics [3, 4]. In this project we unite the two approaches and prove the following main result (see Theorems 2.2 and 2.3 and Corollary 2.4 for more details):

Theorem 1.1. *For a pair A, B of equi-oriented $r \times n$ matrices, there exist tilings of $\mathbb{R}^r$ whose tiles are given by the bases A_I of A, $I \subset [n]$, and appear with frequencies equal to the volumes $\mathrm{Vol}(B_I)$. When all frequencies are integers, the tiling is periodic and its fundamental domain has volume equal to $\det(L)$ for the Laplacian $L := A \cdot B^T$.*

As an application to (irreducible) Weyl groups W, we construct periodic tilings $\mathcal{T}(W)$, whose combinatorics are described by root-theoretic invariants: the Coxeter number h, the rank n, and the connection index $I(W)$ of W. In particular, their fundamental domains have volume h^n, which explains the following well observed [7] phenomenon. The volumes of the root zonotopes in type-A are given by a product formula $\mathrm{vol}\big(Z(A_{n-1})\big) = n^{n-2}$, while for the other root systems they are not, even though the associated Laplacian determinants do factor. Indeed, it is the fundamental domain of $\mathcal{T}(W)$ that behaves well and is related to the Laplacian, as the previous theorem suggests, not the root zonotope $Z(W)$ which is often a strict subset of the fundamental domain.

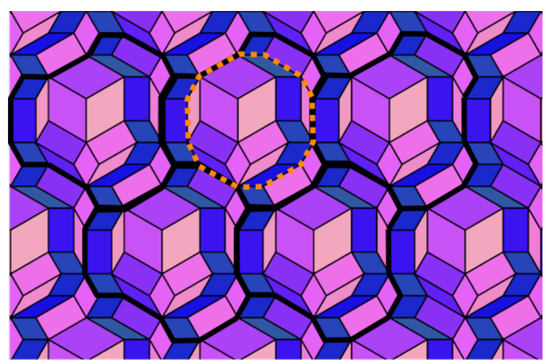

Figure 1: Root-Coroot tiling for $I_2(6)$

Example 1.2. In Figure 1 we see a tiling $\mathcal{T}\big(I_2(6)\big)$; five translates of the fundamental domain are indicated by thick black boundaries, while the central copy contains in orange dashed boundary the root zonotope $Z(\Phi^+)$. There is a total of 9 types of tiles, which form three classes: small rhombi, large rhombi, and parallelograms. Moreover, the little rhombi have volume 1 and relative frequency 3, while the opposite is true for the large rhombi; the parallelograms have volume 2 and relative frequency 2. Theorem 6.1 expresses the probability that a point lies in a translate of some tile Φ^+_σ as $I(W_\sigma)/h^n$, where $I(W_\sigma)$ is the connection index of the reflection subgroup $W_\sigma \leq W$ associated to the tile. Indeed, a point belongs to the translate of a fixed (big or little) rhombus with probability $3/36$ (note that $3 = I(A_2)$) and it belongs to the translate of a fixed parallelogram with probability $4/36$ (note that $4 = I(A_1 \times A_1)$).

2 The Main Construction

Throughout this article, fix positive integers r, k, and n such that $n = r + k$. We will use the notation $\binom{[n]}{r}$ for the r element subsets of the numbers 1 through n. We often use the variable σ for an element of $\binom{[n]}{r}$, and will write $\widehat{\sigma}$ for the set $[n] \setminus \sigma$. For any matrix M with n columns, and any $\sigma \in \binom{[n]}{r}$, we write M_σ for the submatrix of M restricted to columns indexed by σ and $M_{\widehat{\sigma}}$ for the submatrix of M restricted to the remaining columns. Similarly, if $\mathbf{v}$ is an n-dimensional vector, we will write $\mathbf{v}_\sigma$ for the restriction of $\mathbf{v}$ to entries indexed by σ and $\mathbf{v}_{\widehat{\sigma}}$ for the restriction of $\mathbf{v}$ to the remaining entries.

Let A and B be two full rank $r \times n$ matrices with real valued entries. Note that many of our results depend only on the subspaces of $\mathbb{R}^n$ generated by the rows of these matrices, but our construction is easier to describe when specific values are provided. More precisely, row operations on A do not change the combinatorics of the tiling, while row operations on B have no impact at all.

For our construction, we also require that A and B are *equi-oriented*. We say that A and B are *equi-oriented* if for every $\sigma \in \binom{[n]}{r}$, we have

$$\det(A_\sigma)\det(B_\sigma) \geq 0.$$

In addition to equi-oriented matrices A and B, our tiling also depends on a choice of vector $\gamma \in \mathbb{R}^n$ which must satisfy a certain genericity condition.

For a fixed $r \times n$ matrix A and a choice of $\sigma \in \binom{[n]}{r}$, we will write $\mathcal{T}_\sigma$ for the *fundamental parallelepiped* of A_σ. In particular, $\mathcal{T}_\sigma$ is the polytope whose vertices are the sums of all 2^r subsets of columns of A_σ. Equivalently,

$$\mathcal{T}_\sigma := \{A_\sigma \mathbf{x} \mid \mathbf{x} \in [0,1]^r\}.$$

The set of parallelepipeds of the form $\mathcal{T}_\sigma$ for $\sigma \in \binom{[n]}{r}$ are the *prototiles* of our tiling (with possible duplicates and volume 0 tiles). The translates of the prototiles correspond to the vectors in $\mathbb{Z}^r$. More precisely, for every $\sigma \in \binom{[n]}{r}$ and $\mathbf{z} \in \mathbb{Z}^r$, we will construct a translation vector $\mathbf{s}_{(\sigma,\mathbf{z})} \in \mathbb{R}^r$ and then place a copy of $\mathcal{T}_\sigma$ that is translated by this amount. The formula for $\mathbf{s}_{(\sigma,\mathbf{z})}$ is given by (2.2), but it will take some work to define the necessary terminology.

Instead of working directly with B, most of our calculations involve a matrix $\widetilde{B}$ that is in some sense dual to B. To be precise, $\widetilde{B}$ is any full rank $k \times r$ matrix whose row space is dual to the row space of B. In other words,

$$\widetilde{B}B^T = \mathbf{0} \text{ and } \widetilde{B} \text{ is full rank.} \tag{2.1}$$

There is a simple procedure to compute $\widetilde{B}$ from B. First, permute the columns of B until the first r columns are linearly independent (which must be possible since B is full rank). Second, use row operations to put this matrix into reduced row echelon form. This produces a matrix of the form $\begin{pmatrix} I_r & D \end{pmatrix}$, where I_r is the $r \times r$ identity matrix and D is any $r \times k$ real matrix. Next, define a new $k \times n$ matrix $\begin{pmatrix} -D^T & I_k \end{pmatrix}$. Finally, undo the initial row permutation. For straightforward linear algebra reasons, this final matrix must satisfy the conditions for $\widetilde{B}$. Furthermore, there is a nice relationship between the minors of B and $\widetilde{B}$.

Lemma 2.1. *The matrix $\widetilde{B}$ can be chosen such that for every $\sigma \in \binom{[n]}{r}$, we have $\det(B_\sigma) = \det(\widetilde{B}_{\widehat{\sigma}})$.*

Note that Lemma 2.1 can be generalized to any $\widetilde{B}$ satisfying (2.1) to say that the ratio $\det(B_\sigma)/\det(\widetilde{B}_{\widehat{\sigma}})$ is always a fixed value, but is not necessarily 1.

For any choice of $\mathbf{z} \in \mathbb{Z}^r$ and $\sigma \in \binom{[n]}{r}$, the formula for $\mathbf{s}_{(\sigma,\mathbf{z})}$ is given by (2.2). In this formula, we use $\lceil \cdot \rceil$ for the element-wise ceiling function. In other words, we round up each entry to the nearest larger integer. Recall that $\gamma \in \mathbb{R}^n$ is a generic vector, and γ_σ and $\gamma_{\widehat{\sigma}}$ are its restrictions to entries in and out of σ respectively. The precise genericity condition for γ is that for all $\mathbf{z} \in \mathbb{Z}^r$ and $\sigma \in \binom{[n]}{r}$, the vector given by $\widetilde{B}_{\widehat{\sigma}}^{-1}\widetilde{B}_\sigma(\gamma_\sigma - \mathbf{z}) + \gamma_{\widehat{\sigma}}$ does not have any integer entries.

$$\mathbf{s}_{(\sigma,\mathbf{z})} := A_\sigma(\mathbf{z} - \gamma_\sigma) + A_{\widehat{\sigma}}\left(\left\lceil \widetilde{B}_{\widehat{\sigma}}^{-1}\widetilde{B}_\sigma(\gamma_\sigma - \mathbf{z}) + \gamma_{\widehat{\sigma}} \right\rceil - \gamma_{\widehat{\sigma}}\right). \tag{2.2}$$

We are now ready to state the two main theorems of this article.

Theorem 2.2. *For each $\sigma \in \binom{[n]}{r}$ and $\mathbf{z} \in \mathbb{Z}^r$, place a copy of $\mathcal{T}_\sigma$ translated by $\mathbf{s}_{(\sigma,\mathbf{z})}$. This gives a tiling of $\mathbb{R}^r$ with no gaps or overlaps (other than the volume 0 overlap at the boundaries of the tiles).*

In Section 3, we show how to obtain this tiling by taking a slice of a periodic tiling of $\mathbb{R}^n$. One appealing property of the tiling is the following probabilistic result.

Theorem 2.3. *Consider the tiling described in Theorem 2.2. The probability that a randomly chosen point of the tiling lies in a translate of $\mathcal{T}_\sigma$ is equal to*

$$\det(A_\sigma)\det(B_\sigma)/\det(AB^T). \tag{2.3}$$

Corollary 2.4. *If $\det(B_\sigma)$ is an integer for all $\sigma \in \binom{[n]}{r}$, then the tiling is periodic. If the determinants do not all have a common factor, then the fundamental domain of the tiling contains $\det(B_\sigma)$ copies of the tile $\mathcal{T}_\sigma$ for each $\sigma \in \binom{[n]}{r}$.*

Remark 2.5. *To make the statement of Theorem 2.3 precise, fix any convex polytope P in $\mathbb{R}^r$ and consider the uniform probability density when restricting to a dilation of P. Theorem 2.3 says that as the dilations increase in volume, the limiting probability of choosing a point in a translate of $\mathcal{T}_\sigma$ is given by (2.3).*

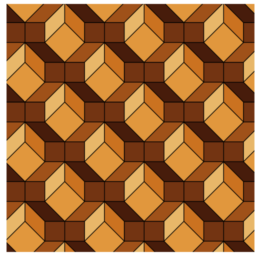

Figure 2: This is the tiling obtained in Example 2.6. Note that 5 of the 6 prototiles have the same area, while the sixth has double the area. This corresponds to the fact that $|\det(A_{\{2,4\}})| = 2$, but $|\det(A_\sigma)| = 1$ for every other $\sigma \in \binom{[n]}{r}$. Furthermore, $|\det(B_{\{1,3\}})| = 2$, but $|\det(B_\sigma)| = 1$ for every other $\sigma \in \binom{[n]}{r}$. This means that $\mathcal{T}_{\{1,3\}}$ (the brown square) appears twice as often as every other prototile.

Example 2.6. Consider the matrices

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}.$$

It is straightforward to show that these are full rank and equi-oriented matrices. Thus, the construction from Theorem 2.2 will produce a tiling that satisfies the conditions of Theorem 2.3. This tiling is given in Figure 2.

The first step is to choose a matrix $\widetilde{B}$ which satisfies (2.1) and a sufficiently generic vector γ. In particular, we can choose

$$\widetilde{B} = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \gamma = (\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})^T.$$

Applying the construction will give a tiling of $\mathbb{R}^2$ with $\binom{4}{2} = 6$ different tiles. For example, the tile $\mathcal{T}_{\{1,3\}}$ a unit cube such that the proportion of the plane covered is

$$\frac{\det(A_{\{1,3\}})\det(B_{\{1,3\}})}{\det(AB^T)} = \frac{\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \det\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}}{\det\begin{pmatrix} 2 & 4 \\ -2 & 0 \end{pmatrix}} = \frac{2}{8}.$$

This means that the brown squares in Figure 2 cover a quarter of $\mathbb{R}^2$ when the tiling is repeated.

To calculate the position where each copy of $\mathcal{T}_\sigma$ is placed, we use the translation vector from (2.2). For example, the translation vector corresponding to $\mathbf{z} = (1,2)^T$ is given by

$$\mathbf{s}_{(\{1,3\},\mathbf{z})} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1-\sqrt{2}\\ 2-\sqrt{5}\end{pmatrix} + \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\left(\left\lceil\begin{pmatrix}0 & -2\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}\sqrt{2}-1\\ \sqrt{5}-2\end{pmatrix} + \begin{pmatrix}\sqrt{3}\\ \sqrt{7}\end{pmatrix}\right\rceil - \begin{pmatrix}\sqrt{3}\\ \sqrt{7}\end{pmatrix}\right)$$

$$= \begin{pmatrix}1-\sqrt{2}\\ 2-\sqrt{5}\end{pmatrix} + \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\left(\left\lceil\frac{1}{2}\begin{pmatrix}1+\sqrt{2}+2\sqrt{3}-\sqrt{5}\\ 3-\sqrt{2}-\sqrt{5}+2\sqrt{7}\end{pmatrix}\right\rceil - \begin{pmatrix}\sqrt{3}\\ \sqrt{7}\end{pmatrix}\right)$$

$$= \begin{pmatrix}1-\sqrt{2}\\ 2-\sqrt{5}\end{pmatrix} + \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}2-\sqrt{3}\\ 3-\sqrt{7}\end{pmatrix} = \begin{pmatrix}2-\sqrt{2}+\sqrt{3}-\sqrt{7}\\ 7-\sqrt{3}-\sqrt{5}-\sqrt{7}\end{pmatrix}$$

This calculation shows that one of the squares in the tiling must have its bottom left vertex at this coordinate.

3 Why the tiling works

Our construction is based on a periodic tiling of $\mathbb{R}^n$ that was introduced by the second author and then generalized by Doolittle and the second author [13, 12, 9]. In the original construction, the tiles correspond to *bases* of an *oriented arithmetic matroid*, while the lattice points in the fundamental domain correspond to elements of the matroid's *sandpile group*. When this tiling is intersected with a plane that is perpendicular to the last k coordinates, this produces the tiling of $\mathbb{R}^r$ that is defined in Section 2.

As before, we begin with a pair of equi-oriented matrices A and B, and then find a matrix $\widetilde{B}$ whose row space is orthogonal to the row space of A. We also need a new $n \times n$ matrix M which is given by

$$M = \begin{pmatrix}A\\ -\widetilde{B}\end{pmatrix}.$$

For each $\sigma \in \binom{[n]}{r}$, let $S_\sigma(M)$ be the $n \times n$ matrix whose i^{th} column is given by:

1. If $i \in \sigma$, then the first r entries of the i^{th} column of $S_\sigma(M)$ match the i^{th} column of A and the last k entries are all 0.
2. If $i \notin \sigma$, then the first r entries of the i^{th} column of $S_\sigma(M)$ are all 0 and the last k entries match the i^{th} column of $\widetilde{B}$.

For example, if A and $\widetilde{B}$ are the same as in Example 2.6, then

$$M = \begin{pmatrix}1 & -1 & 0 & 1\\ 0 & 1 & 1 & 1\\ -1 & 0 & -1 & 2\\ 0 & -1 & 1 & -1\end{pmatrix} \qquad \text{and} \qquad S_{\{1,3\}}(M) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -2\\ 0 & 1 & 0 & 1\end{pmatrix}.$$

Theorem 3.1 ([9, Corollary 3.10], [13, Corollary 9.2.8]). *Suppose that* $\det(S_\sigma(M))$ *is non-negative for all* $\sigma \in \binom{[n]}{r}$ *or non-positive for all* $\sigma \in \binom{[n]}{r}$. *Then, every point in* $\mathbb{R}^n$ *can be expressed as a sum of the form* $M\mathbf{z} + S_\sigma(M)\mathbf{x}$ *for some* $\mathbf{z} \in \mathbb{Z}^n$, $\mathbf{x} \in [0,1]^n$, *and* $\sigma \in \binom{[n]}{r}$. *For all but a measure 0 set of points, such an expression is unique.*

Another way to state Theorem 3.1 is to say that the fundamental parallelepipeds of each matrix $S_\sigma(M)$ form a periodic tiling of $\mathbb{R}^n$ with the columns of M as translation vectors. Note that the "measure 0 set overlap" caveat can be removed by working with *half-open parallelepipeds*. See [9] for more details.

The condition required by the first sentence of Theorem 3.1 can be shown to be equivalent to our requirement for $\widetilde{B}$ in Section 2. In particular, $\det(S_\sigma(M))$ is always non-negative or always non-positive if and only if the row space of $\widetilde{B}$ is dual to the row space of a full rank matrix B where A and B are equi-oriented.

3.1 Proof of Theorem 2.2

To obtain the construction from Section 2, we intersect the tiling of $\mathbb{R}^n$ from Theorem 3.1 with a hyperplane that is perpendicular to the last k coordinates. This idea was partially explored in [12, Section 7], but the technique used is only applicable for integer matrices, and cannot be used to produce an aperiodic tiling.

Before intersecting the tiling with a hyperplane, we first translate all of the tiles by $-M\gamma$ for some vector γ. Note that since M is the fundamental domain of the n-dimensional tiling, the entries of γ only matter mod 1. In particular, we can think of γ as a vector in $[0,1]^n$.

Each tile of the n-dimensional tiling corresponds to a choice of $\sigma \in \binom{[n]}{r}$ and $\mathbf{z} \in \mathbb{Z}^n$. In particular, this tile is made up of all points of the form $M(\mathbf{z} - \gamma) + S_\sigma(M)\mathbf{x}$, where $\mathbf{x} \in [0,1]^n$. From the structures of M and $S_\sigma(M)$, it is straightforward to show that the first r values of this point are given by

$$A_\sigma(\mathbf{z}_\sigma - \gamma_\sigma + \mathbf{x}_\sigma) + A_{\widehat{\sigma}}(\mathbf{z}_{\widehat{\sigma}} - \gamma_{\widehat{\sigma}}),$$

while the last k values are given by

$$\widetilde{B}_\sigma(\gamma_\sigma - \mathbf{z}_\sigma) + \widetilde{B}_{\widehat{\sigma}}(\gamma_{\widehat{\sigma}} - \mathbf{z}_{\widehat{\sigma}} + \mathbf{x}_{\widehat{\sigma}}).$$

Observe that for every choice of $\mathbf{z}_\sigma$, there is a unique choice of $\mathbf{z}_{\widehat{\sigma}}$ and $\mathbf{x}_{\widehat{\sigma}}$ such that these last k entries are zero. Routine algebraic manipulations show that

$$\mathbf{z}_{\widehat{\sigma}} = \widetilde{B}_{\widehat{\sigma}}^{-1}\left(\widetilde{B}_\sigma(\gamma_\sigma - \mathbf{z}_\sigma)\right) + \gamma_{\widehat{\sigma}} + \mathbf{x}_{\widehat{\sigma}}.$$

Additionally, since all of the values of $\mathbf{x}_{\widehat{\sigma}}$ are between 0 and 1, and we choose $\gamma_{\widehat{\sigma}}$ such that they cannot be exactly 0 or 1, we can conclude that

$$\mathbf{z}_{\widehat{\sigma}} = \left\lceil \widetilde{B}_{\widehat{\sigma}}^{-1}\left(\widetilde{B}_\sigma(\gamma_\sigma - \mathbf{z}_\sigma)\right) + \gamma_{\widehat{\sigma}} \right\rceil,$$

where $\lceil \cdot \rceil$ is the entry-wise ceiling function. Plugging back into the formula for the first r coordinates gives (2.2).

3.2 Proof sketch of Theorem 2.3

Let P be a convex polytope with volume $\mathrm{vol}(P)$. If P is sufficiently large relative to the tiles, we get a good approximation for the number of copies of $\mathcal{T}_\sigma$ that lie inside of P by considering the number of $\mathbf{z} \in \mathbb{Z}^r$ such that the translation vector $\mathbf{s}_{(\sigma,\mathbf{z})}$ lies inside of P. We can also ignore tiles that lie partially inside of P since these cases will be sparse in the limit.

For a given $\mathbf{z}$ and σ, let $\epsilon_{\sigma,\mathbf{z}}$ be the vector added when the ceiling function is applied in (2.2). This lets us rewrite (2.2) as

$$A_\sigma(\mathbf{z} - \gamma_\sigma) + A_{\widehat{\sigma}}\left(\widetilde{B}_{\widehat{\sigma}}^{-1}\left(\widetilde{B}_\sigma(\gamma_\sigma - \mathbf{z})\right) + \epsilon_{\sigma,\mathbf{z}}\right) =$$
$$\left(A_\sigma - A_{\widehat{\sigma}}\widetilde{B}_{\widehat{\sigma}}^{-1}\widetilde{B}_\sigma\right)\mathbf{z} + A_{\widehat{\sigma}}\epsilon_{\sigma,\mathbf{z}} + A_{\widehat{\sigma}}\widetilde{B}_{\widehat{\sigma}}^{-1}\widetilde{B}_\sigma\gamma_\sigma.$$

The vectors $\epsilon_{\sigma,\mathbf{z}}$ and γ_σ always have entries between 0 and 1. Thus, the first term is the only relevant term in the limit.

The number of integer points in a region is approximately equal to the volume of the region, and the determinant of a linear transformation describes how this volume changes. In particular, the number of $\mathbf{s}_{(\sigma,\mathbf{z})}$ that lie within P is approximately $\mathrm{vol}(P)/|\det(A_\sigma - A_{\widehat{\sigma}}\widetilde{B}_{\widehat{\sigma}}^{-1}\widetilde{B}_\sigma)|$. Furthermore, by the Schur complement formula,

$$\det(\widetilde{B}_{\widehat{\sigma}})\det(A_\sigma - A_{\widehat{\sigma}}\widetilde{B}_{\widehat{\sigma}}^{-1}\widetilde{B}_\sigma) = \det\begin{pmatrix} A_\sigma & A_{\widehat{\sigma}} \\ \widetilde{B}_\sigma & \widetilde{B}_{\widehat{\sigma}} \end{pmatrix}$$

Notice that the matrix on the right has the same determinant up to sign for every $\sigma \in \binom{[n]}{r}$ (the columns will just be rearranged). This implies that the number of translates of $\mathcal{T}_\sigma$ contained in P is (in the limit) proportional to $\det(\widetilde{B}_{\widehat{\sigma}})$. Since each tile has volume $\det(A_\sigma)$, the total volume of points contained in a translate of $\mathcal{T}_\sigma$ is proportional to $\det(A_\sigma)\det(\widetilde{B}_{\widehat{\sigma}})$.

By Lemma 2.1, we can choose $\widetilde{B}_{\widehat{\sigma}}$ so that $\det(B_\sigma) = \det(\widetilde{B}_{\widehat{\sigma}})$ for all $\sigma \in \binom{[n]}{r}$. Thus, the total volume of points contained in a translate of $\mathcal{T}_\sigma$ is proportional to $\det(A_\sigma)\det(B_\sigma)$. Finally, it follows from the Cauchy–Binet formula that the sum of $\det(A_\sigma)\det(B_\sigma)$ over all $\sigma \in \binom{[n]}{r}$ is equal to $\det(AB^T)$.

4 The Penrose Tiling and other Special cases

One simple condition to guarantee that A and B are equi-oriented is to simply choose $A = B$. This condition is used to construct one of the most famous tilings in mathematics:

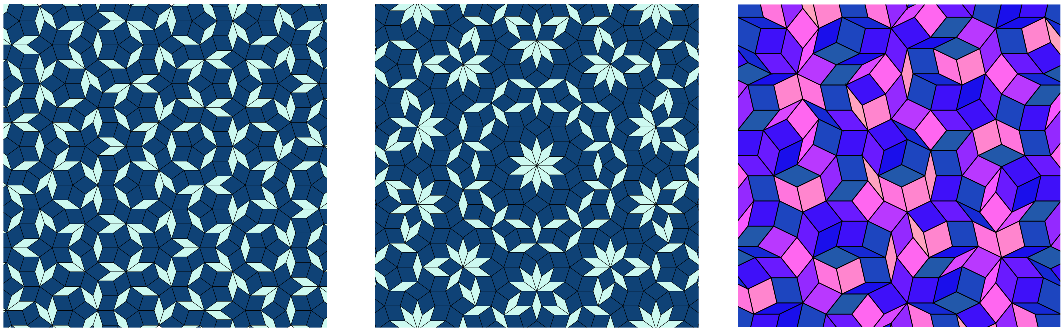

Figure 3: The first two tilings are generalized Penrose tilings that come from our construction. On the left is a tiling obtained from $\gamma = (.5, .5, .5, .00001, .5)^T$, which satisfies the matching rules for a traditional Penrose tiling within this region. In the center is the tiling obtained from $\gamma = (.5, .5, .5, .5, .5)^T$, which was studied in [11, Section 7.4]. Finally, the image on the right is from an analogous construction using seventh roots of unity and $\gamma = (.5, .5, .00001, .00001, .00001, .00001, .00001)^T$.

the Penrose tiling. The first step of the tiling is to set the columns of A and B to be the 5th roots of unity. In other words,

$$A = B = \begin{pmatrix} 1 & \cos(\zeta) & \cos(2\zeta) & \cos(3\zeta) & \cos(4\zeta) \\ 0 & \sin(\zeta) & \sin(2\zeta) & \sin(3\zeta) & \sin(4\zeta) \end{pmatrix},$$

where $\zeta = 2\pi/5$.

All of the maximal minors of A are nonzero, which means that there are a total of $\binom{5}{2} = 10$ prototiles. However, the symmetry of the vectors actually means that there are actually just two kinds of tiles up to rotation. In particular, the parallelepiped formed by v_i and v_j will have an area of $\sin(\zeta)$ if $i - j \equiv \pm 1 (\mathrm{mod} 5)$ and will have an area of $\sin(2\zeta)$ if $i - j \equiv \pm 2 (\mathrm{mod} 5)$.

By Theorems 2.2 and 2.3, any choice of $\gamma \in [0, 1]^5$ will produce a tiling of $\mathbb{R}^2$ using the 10 prototiles. Furthermore, each of the 5 possible rotations of each tile appears with the same frequency. In particular, the probability that a randomly chosen point is contained in a large tile is

$$\frac{5 \sin^2(\zeta)}{\det(AB^T)} = \frac{5 \sin^2(\zeta)}{5(\sin^2(2\zeta) + \sin^2(\zeta))} = \frac{\sin^2(\zeta)}{\sin^2(2\zeta) + \sin^2(\zeta)} = \frac{5 + \sqrt{5}}{10}$$

Such a tiling is called a *generalized Penrose tiling*. However, the precise tiling depends on γ. See Figure 3 for a few possibilities. For an excellent reference on Penrose tilings and other related tilings, we recommend [15].

We learned while researching Penrose tilings that our method of generating tilings is closely related to something called an *oblique tiling* [14]. For our tilings, we begin with a somewhat complicated tiling of $\mathbb{R}^n$ and then slice along a plane that is perpendicular to the last k coordinates (see Section 3). Conversely, the oblique tiling begins with a simple tiling of unit cubes in $\mathbb{R}^n$ and then slices along a more complicated hyperplane. While the two constructions work similarly for the $A = B$ case, the oblique tiling perspective does not generalize as naturally to the case where A and B are equi-oriented but not equal. In the last two sections, we will explore a few applications where the equi-oriented generality is necessary.

5 The Laplacian of two equi-oriented matrices

Burman et al. [2] define a Laplacian operator $L(A, B)$ that depends on *two* collections of vectors $A := (a_i)_{i=1}^n$ and $B := (b_i)_{i=1}^n$, from some vector space $V \cong \mathbb{R}^r$ as follows:

$$L(A,B)(x) := \sum_{i=1}^{N} \langle b_i, x\rangle \cdot a_i \quad \text{or equivalently} \quad L(A,B)(x) = AB^T \cdot x. \tag{5.1}$$

Notice that $L(A, B)$ is a sum of rank-1 operators ($x \to \langle b_i, x\rangle \cdot a_i$ has its image in the line spanned by a_i). The following proposition gives the characteristic polynomial of $L(A, B)$ in terms of the Grammian determinants of associated sets of columns of A and B. Recall here that given two equal sized collections of vectors $K = (k_i)_{i\in I}$ and $P = (p_i)_{i\in I}$, the *Grammian determinant* $\mathrm{GD}(K, P)$ is defined as

$$\mathrm{GD}(K,P) := \det\Big(\langle k_i, p_j\rangle\Big)_{i,j\in I}, \tag{5.2}$$

as the determinant of the (generalized) Grammian matrix formed by K and P. The following is essentially [4, Lemma 8.2]. In this abstract we are only interested in the determinant $\det\big(L(A, B)\big)$ for which case the following proposition is the Cauchy–Binet formula.

Proposition 5.1. *The characteristic polynomial of the Laplacian operator* (5.1) *is given via:*

$$\det\big(t\cdot \mathbf{I}_r + L(A,B)\big) = \sum_{\substack{\sigma\subseteq[n]\\ |\sigma|\le r}} \mathrm{GD}(A_\sigma, B_\sigma)\cdot t^{r-|\sigma|}.$$

When two matrices A, B are equi-oriented, all grammian determinants $\mathrm{GD}(A_\sigma, B_\sigma)$ are non-negative and equal the product of the volumes of the fundamental parallelepipeds for bases A_σ and B_σ. After applying Corollary 2.4, we have proven the following.

Corollary 5.2. *The volume of the fundamental domain for a periodic tiling $\mathcal{T}(A, B)$ equals the determinant of the Laplacian $L(A, B)$.*

6 Tiling space with root zonotopes and their relatives

A particularly nice application of Theorems 2.2 and 2.3 is for the case that the collections of vectors A and B are respectively the (positive) roots Φ^+ and coroots $\widehat{\Phi}^+$ of a Weyl group W (we will mostly represent the roots in terms of a basis of the ambient space V of W instead of the root or weight basis because it leads to more appealing pictures but this is not necessary). We will denote such a tiling by $\mathcal{T}(W) := \mathcal{T}\big(\Phi^+, \widehat{\Phi}^+\big)$.

Theorem 6.1. *Let W be an irreducible Weyl group of rank n and Coxeter number h, and $\mathcal{T}(W)$ its associated periodic tiling described above. Then, any fundamental domain of $\mathcal{T}(W)$ has volume equal to h^n. Moreover, for every tile $\mathcal{T}_\sigma$ of $\mathcal{T}(W)$ the probability that a random point belongs to a translate of $\mathcal{T}_\sigma$ is equal to $\dfrac{I(W_\sigma)}{h^n}$ where $I(W_\sigma)$ is the connection index of the reflection subgroup W_σ associated to $\mathcal{T}_\sigma$.*

Proof. Let $N = hn/2 = |\Phi^+|$ denote the number of roots of W, and Q and $\widehat{Q}$ its root and coroot lattice. Then, any $\mathbb{R}$-basis of roots corresponding to some $\sigma \in \binom{[N]}{n}$ generates a rank-n reflection subgroup $W_\sigma \leq W$. Furthermore, the roots Φ^+_σ and coroots $\widehat{\Phi}^+_\sigma$ generate respectively the root and coroot lattices Q_σ and $\widehat{Q}_\sigma$ of W_σ (see [1, Corollary 1.2]). This means that $\mathrm{Vol}(\Phi^+_\sigma) = \mathrm{Vol}(Q_\sigma)$ and $\mathrm{Vol}(\widehat{\Phi}^+_\sigma) = \mathrm{Vol}(\widehat{Q}_\sigma)$. Now, the connection index $I(W_\sigma)$ is defined as the determinant of the Cartan matrix of W_σ, which is the Laplacian of the two collections of simple roots and simple coroots of W_σ, and hence agrees with the product of the volumes of the two lattices:

$$I(W_\sigma) = \mathrm{Vol}(Q_\sigma) \cdot \mathrm{Vol}(\widehat{Q}_\sigma).$$

Finally the determinant of the Laplacian $L(\Phi^+, \widehat{\Phi}^+)$ is equal to h^n after [3, Proposition 3.3] and the proof is complete. □

Remark 6.2. *The theorem above gives a geometric exegesis to the formula [10, Section 7.3]*

$$h^n = \sum_{W' \leq W} \#\,\mathrm{RGS}(W') \cdot I(W'),$$

where the sum is over all rank-n reflection subgroups of W and $\mathrm{RGS}(W')$ denotes the set of minimum generating sets of reflections of W'. Indeed, it corresponds to the partition of the fundamental domain of $\mathcal{T}(W)$ with respect to the distinct tiles $\mathcal{T}_\sigma$.

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