Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #105, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Orbit Structures and Complexity in Schubert and Richardson Varieties

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Abstract. The goal of this paper is twofold. Firstly, we provide a type-uniform formula for the torus complexity of the usual torus action on a Richardson variety, by developing the notion of algebraic dimensions of Bruhat intervals, strengthening a type A result by Donten-Bury, Escobar and Portakal. In the process, we give an explicit description of the torus action on any Deodhar component as well as describe the root subgroups that comprise the component. Secondly, when a Levi subgroup in a reductive algebraic group acts on a Schubert variety, we exhibit a codimension preserving bijection between the Levi–Borel subgroup (a Borel subgroup in the Levi subgroup) orbits in the big open cell of that Schubert variety and torus orbits in the big open cell of a distinguished Schubert subvariety. This bijection has many applications including a type-uniform formula for the Levi–Borel complexity of the usual Levi–Borel subgroup action on a Schubert variety.

Keywords: Schubert varieties, Richardson varieties, orbit complexity, Levi subgroups, toric varieties

1 Introduction

1.1 Group orbits in the full flag variety

Let G be a complex, connected, reductive algebraic group of rank r. Fix a maximal torus T in G, and a Borel subgroup B of G containing T. The homogeneous space G/B is a smooth projective variety known as the *full flag variety*. The study of the flag variety first arose out of the need to formalize and justify the enumerative geometry of H. Schubert as laid out in Hilbert's 15th problem. These varieties are a central theme in much of the mathematics of the 20th century and beyond, with deep and fundamental connections to algebraic geometry, Lie theory, representation theory, algebraic combinatorics, and commutative algebra.

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Of particular importance has been the study of orbits and orbit closures in the flag variety. Famously, the B-orbits for the action of B by left translation yield a cellular filtration of G/B; these orbits, denoted X_w°, are referred to as *Schubert cells* and are indexed by elements w of the Weyl group W of G. In his seminal work [7], C. Chevalley introduced the now ubiquitous Bruhat order to describe the inclusion order of B-orbit closures in G/B. These orbit closures, or as they are more commonly known, *Schubert varieties* X_w for $w \in W$, are well-studied varieties that boast a rich combinatorial structure that encodes many facets of their geometry.

A stratification of G/B via the orbits of the opposite Borel subgroup B^- also exists; the closure of these orbits are the *opposite Schubert varieties* X^w for $w \in W$. The *Richardson variety* $\mathcal{R}_{u,v}$ for $u, v \in W$ is defined to be the intersection $\mathcal{R}_{u,v} := X_v \cap X^u$.

1.2 Torus orbits in Schubert and Richardson Varieties

It is also instructive to study group orbits and their closures within Schubert and Richardson varieties. Of particular interest in this setting is the statistic on the set of orbits of a reductive algebraic group H known as the H-complexity.

If an algebraic group H acts on a variety X by a morphism of algebraic varieties, we say that X is an H-variety. We denote the set of H-orbits in X by $\mathcal{O}_H(X)$. Let H be a reductive algebraic group and B_H a Borel subgroup of H. The *H-complexity* of an H-variety X, denoted $c_H(X)$, is the minimum codimension of a B_H-orbit in X.

The normal H-varieties with H-complexity equal to 0 are the *H-spherical varieties*. Spherical varieties generalize several important classes of algebraic varieties including toric varieties, projective rational homogeneous spaces and symmetric varieties.

Unless otherwise stated, in this paper the action of any subgroup of G on any subvariety of G/B will always be left translation. We will refer to this as the *usual action*. For any $w \in W$, both X_w and X^w are T-varieties for the usual action. We denote the Bruhat order on W by $\leq$, writing $u \leq v$ if and only if $X_u \subseteq X_v$. For $u \leq v \in W$, $\mathcal{R}_{u,v}$ is nonempty and is a normal T-variety for the usual action. The maximal torus T is a reductive algebraic group whose only Borel subgroup is T itself. Hence, a normal T-variety with T-complexity equal to 0 is a T-spherical variety (that is, a toric variety).

P. Karuppuchamy provided a succinct, root-system uniform classification of the toric Schubert varieties [15]. E. Tsukerman and L. Williams introduced Bruhat interval polytopes to study the geometry of Richardson varieties and noted connections to torus orbits [21]. This connection was expanded on by E. Lee, M. Masuda, and S. Park, yielding a classification of toric Richardson varieties in G/B where G is of Dynkin-type A [16], with further results by C. Gaetz [10] . A classification of toric Richardson varieties in any full flag variety was provided by M. Can and P. Saha in [6].

Our first main result continues this line of research, giving a root-system uniform combinatorial formula for the T-complexity of a Richardson variety in any full flag variety. We do so by defining and developing the notion of *algebraic dimensions* for arbitrary Bruhat intervals $[u, v]$ in Section 3, where we provide multiple formulas for $\mathrm{ad}(u, v)$.

Theorem 1.1. *For $u \leq v \in W$, the T-complexity of the Richardson variety equals*

$$c_T(\mathcal{R}_{u,v}) = \ell(v) - \ell(u) - \mathrm{ad}(u, v).$$

Moreover, we have that

$$\mathrm{ad}(u, v) = \max_{u \leq w \leq v} \ell(v) - \ell(w) \text{ where } \mathcal{R}_{w,v} \text{ is toric.}$$

One highlight of Theorem 1.1 is that our method works uniformly across all Lie types, and our key lemmas work uniformly across geometric realizations for any Coxeter groups, while the theory of Bruhat interval polytopes is well-developed in type A, but not as developed in other types. Moreover, Theorem 1.1 can be seen as a generalization of the main result of [9] to other Lie types (with different arguments). Along the way, in Theorem 3.14, we are also able to explicitly describe the torus action on any *Deodhar component* by enumerating the root subgroups that comprise the component.

In the case of $u = \mathrm{id}$, Theorem 1.1 simplifies to a simple, type-uniform combinatorial formula for the torus complexity of a Schubert variety.

Corollary 1.2. *For $w \in W$, the T-complexity of the Schubert variety X_w equals*

$$c_T(X_w) = \ell(w) - \mathrm{supp}(w),$$

where $\mathrm{supp}(w)$ is the cardinality of the support of w (see Section 2).

1.3 Levi–Borel subgroup orbits in Schubert Varieties

We now investigate the orbits of Levi–Borel subgroups (defined below) in a Schubert variety. Our choice of T and B determine the *root system* Φ and a set of *simple roots* $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, respectively. The Weyl group W of G, is generated by the set of simple reflections $\{s_i := s_{\alpha_i} | \alpha_i \in \Delta\}$. For $I \subseteq \Delta$, W_I is the subgroup of W generated by $\{s_{\alpha_i} | i \in I\}$.

The *standard parabolic subgroups* of G containing B are indexed by subsets of Δ. The standard parabolic subgroup associated to I is $P_I = BW_IB$ with Levi decomposition

$$P_I = L_I \ltimes U_I,$$

where L_I is a reductive subgroup called a *Levi subgroup*, and U_I is the unipotent radical of P_I. Define $B_{L_I} := L_I \cap B$. Then B_{L_I} is a Borel subgroup of L_I, and we shall refer to such subgroups as *Levi–Borel subgroups*. Our second main result concerns the orbits of Levi–Borel subgroups in a Schubert cell.

Theorem 1.3. *Let $w \in W$, $I \subseteq \Delta$, and let $w = {}_Iw\,{}^Iw$ be the left parabolic decomposition of w with respect to I (see Section 2). The map $\mathfrak{O} : \mathcal{O}_T(X^\circ_{{}_Iw}) \to \mathcal{O}_{B_{L_I}}(X^\circ_w)$ given by $\Theta \mapsto B_{L_I}\,{}^Iwx$, where x is any point in Θ, is a surjection. If L_I acts on X_w, then $\mathfrak{O}$ is a codimension preserving bijection. That is,*

$$\dim(X^\circ_w) - \dim(\mathfrak{O}(\Theta)) = \dim(X^\circ_{{}_Iw}) - \dim(\Theta),$$

for every $\Theta \in \mathcal{O}_T(X^\circ_{I_w})$.

Theorem 1.3 allows us to intertwine the study of Levi–Borel orbits with the vast literature on torus orbits and varieties, including our own Corollary 1.2, and has myriad applications which we now detail. Our first application is Proposition 4.9 which provides a lower bound on the codimension of a B_{L_I}-orbit in X_w. In the case where the Levi subgroup L_I acts on the Schubert variety X_w this leads to a closed formula for the minimal codimension of a B_{L_I}-orbit in X_w.

While B_{L_I} acts on any X_w, since $B_{L_I} \subseteq B$, the same is not true for L_I. The stabilizer of X_w in G for the usual action is the standard parabolic subgroup $P_{\mathcal{D}_L(w)}$ [2, Lemma 8.2.3], where $\mathcal{D}_L(w)$ is the left descent set of w defined in Section 2. Thus, the Levi subgroups $L_I \leq P_I \leq P_{\mathcal{D}_L(w)}$ for $I \subseteq \mathcal{D}_L(w)$ are a family of reductive algebraic groups which act on X_w. Our third main result is a root-system uniform, combinatorial formula for the L_I-complexity of any Schubert variety which is an L_I-variety in any full flag variety.

Theorem 1.4. *Let* $w \in W$ *and suppose* L_I *acts on the Schubert variety* X_w *(equivalently,* $I \subseteq \mathcal{D}_L(w)$*). If* $w = {}_I w\, {}^I w$ *is the left parabolic decomposition of* w *with respect to* I*, then*
$$c_{L_I}(X_w) = \ell({}^I w) - \mathrm{supp}({}^I w).$$

Theorem 1.4 is a refinement and generalization of numerous earlier results. The second author and A. Yong initiated a study of L_I-complexity 0 Schubert varieties in [13], and therein conjectured a classification in terms of spherical elements of a Coxeter group. Subsequently, both authors of this paper and A. Yong proved this conjecture first for G of Dynkin-type A [11], and later for any full flag variety [12] (the latter was proved contemporaneously via different methods by M. Can and P. Saha [5]). Using work of R. S. Avdeev and A. V. Petukhov [1], the classification of L_I-complexity 0 Schubert varieties may be interpreted as a generalization of results of P. Magyar, J. Weyman, and A. Zelevinsky [17] and J. Stembridge [19, 20] on spherical actions on (products of) flag varieties. The interested reader may see [13, Theorem 2.4] for the details.

The outline of this paper is as follows. In Section 2, we introduce the necessary background, including Schubert geometry and Coxeter groups. In Section 3, we fully develop the notion of algebraic dimensions and establish Theorem 1.1. In Section 4, we prove Theorem 1.3 and Theorem 1.4 with one of our main tools being an equivariant isomorphism that allows us to use results in Section 3.

2 Preliminaries

2.1 Combinatorics of the Weyl group

Consider the following data associated to elements of the Weyl group W. For $w \in W$, its *Coxeter length* is the smallest ℓ such that w can be written as product of ℓ simple reflections. Such an expression is called a *reduced word* or a *reduced expression* of w.

For $u, v \in W$, the product uv is said to be *length additive* if $\ell(uv) = \ell(u) + \ell(v)$. Let W^I be the set of minimal coset representatives in W of W/W_I. In the same way, let ${}^I W$ be the set of minimal coset representatives in W of $W_I \backslash W$. Given $w \in W$ and $I \subseteq \Delta$, w has a unique *right parabolic decomposition* $w = w^I w_I$ which is length-additive with $w_I \in W_I$ and $w^I \in W^I$. Similarly, w has a *left parabolic decomposition* $w = {}_I w \, {}^I w$ which is length-additive with ${}_I w \in W_I$ and ${}^I w \in {}^I W$. We will denote the longest element of W_I by $w_0(I)$, and the longest element of W by w_0

The *support* of w is

$$\operatorname{Supp}(w) = \{\alpha_i \in \Delta \mid s_i \text{ appears in any/all reduced words of } w\}.$$

The cardinality of $\operatorname{Supp}(w)$ is written as $\operatorname{supp}(w) = |\operatorname{Supp}(w)|$. For $w \in W$, its *left descent set* and the *right descent set* are

$$\mathcal{D}_L(w) = \{\alpha_i \in \Delta \mid \ell(s_i w) < \ell(w)\}, \quad \mathcal{D}_R(w) = \{\alpha_i \in \Delta \mid \ell(w s_i) < \ell(w)\},$$

respectively. And its *left inversion set* and the *right inversion set* are

$$\mathcal{I}_L(w) = \{\alpha \in \Phi^+ \mid w^{-1}(\alpha) \in \Phi^-\}, \quad \mathcal{I}_R(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\},$$

respectively, with $|\mathcal{I}_L(w)| = |\mathcal{I}_R(w)| = \ell(w)$. There are equivalent ways to write down the inversion sets. For example,

$$\mathcal{I}_L(w) = \Phi^+ \cap w(\Phi^-) = \{\alpha \in \Phi^+ \mid \ell(s_\alpha w) < \ell(w)\}.$$

Given a length additive product uv, [4, Chapter VI, Section 1, Corollary 2 of Proposition 17] proves that

$$\mathcal{I}_L(uv) = \mathcal{I}_L(u) \sqcup u(\mathcal{I}_L(v)). \tag{2.1}$$

The *(strong) Bruhat order* is a partial order on W that is generated by

$$w < w s_\alpha \text{ for } \alpha \in \Phi^+, \text{ if } \ell(w) < \ell(w s_\alpha).$$

Write $[u, v] := \{w \in W \mid u \leq w \leq v\}$ for a *Bruhat interval*. A saturated chain from u to v is a sequence of elements $u = w^{(0)} \lessdot w^{(1)} \lessdot \cdots \lessdot w^{(\ell-1)} \lessdot w^{(\ell)} = v$, where $w^{(i)} \in W$ and $\lessdot$ denotes a covering relation in the poset.

2.2 Root subgroups

For $\alpha \in \Phi$, let U_α be the root subgroup corresponding to α. Given $w \in W$, define $U_w := U \cap w U^- w^{-1}$, where U and U^- are the unipotent parts of B and $B^- := w_0 B w_0$, respectively. A sequence of subgroups $H_1, \ldots, H_k$ in an algebraic group H is said to *directly span* H if the product morphism $H_1 \times \cdots \times H_k \to H$ is a bijection. The subgroup U_w of U is closed and normalized by T. Hence [3, Section 14.4] implies that U_w is directly spanned (in any order) by the root subgroups contained in it. Thus

$$U_w := U \cap w U^- w^{-1} = \left(\prod_{\alpha \in \Phi^+} U_\alpha \right) \bigcap \left(w \prod_{\alpha \in \Phi^-} U_\alpha w^{-1} \right) = \left(\prod_{\alpha \in \Phi^+} U_\alpha \right) \bigcap \left(\prod_{\alpha \in \Phi^-} U_{w(\alpha)} \right),$$

where the second equality is [3, Section 14.4] applied to U and U^-, and the last equality is [14, Part II, 1.4(5)]. Thus U_α is contained in U_w if and only if $\alpha \in \mathcal{I}(w)$. Hence

$$U_w = \prod_{\alpha \in \mathcal{I}(w)} U_\alpha. \tag{2.2}$$

Let $w = ad$ be the left parabolic decomposition of w with respect to I, where $a \in W_I$ and $d \in {}^I W$. Define $V_d := aU_d a^{-1}$.

Lemma 2.1. *The group V_d is a closed subgroup of U_w normalized by T. Indeed, U_w is directly spanned in any order by the subgroups U_a and V_d, that is, $U_w = U_a V_d = V_d U_a$.*

3 The torus complexity of Richardson varieties

In this section, we establish a formula for the complexity of the usual torus T action on any Richardson variety $\mathcal{R}_{u,v}$ where $u \leq v$ in the Bruhat order.

Definition 3.1. The *(undirected) Bruhat graph* on a Weyl group W is the graph Γ with vertex set W and edges $w \sim s_\alpha w$ for $\alpha \in \Phi^+$. For each edge $w \sim s_\alpha w$, we say that it has *label* or *weight* α, and write $\mathrm{wt}(w, s_\alpha w) = \alpha$. For $u \leq v$, let $\Gamma(u,v)$ be the Bruhat graph restricted to the vertex set $[u,v]$.

Definition 3.2. For $u \leq v$, let $\mathrm{AD}(u,v)$ be the $\mathbb{R}$-span of all edge labels in $\Gamma(u,v)$, i.e.

$$\mathrm{AD}(u,v) = \mathrm{span}_{\mathbb{R}}\{\mathrm{wt}(x,y) \mid u \leq x < y \leq v\}.$$

Let $\mathrm{ad}(u,v) = \dim \mathrm{AD}(u,v)$ be the *algebraic dimension* of the Bruhat interval $[u,v]$.

3.1 The algebraic dimension of Bruhat intervals

We now describe multiple ways to simplify the computation of $\mathrm{AD}(u,v)$.

Lemma 3.3. *$\mathrm{AD}(u,v)$ is spanned by the labels of all cover relations incident to u inside $[u,v]$, and dually, by all cover relations incident to v inside $[u,v]$.*

Proposition 3.4. *Let α_i be a right descent of v. We have the following recursion:*

$$\mathrm{AD}(u,v) = \begin{cases} \mathrm{AD}(us_i, vs_i) & \text{if } us_i < u, \\ \mathrm{AD}(u, vs_i) + \mathbb{R} \cdot \mathrm{wt}(u, us_i) & \text{if } us_i > u. \end{cases}$$

Example 3.5. Denote the symmetric group of degree 4 by S_4. Let $u, v \in S_4$ with $u = 3412$, $v = 1324$ in one-line notation and let $i = 2$. According to the first case of Proposition 3.4, $\mathrm{AD}(1324, 3412) = \mathrm{AD}(1234, 3142)$. In particular, the rank sizes of $[1324, 3412]$ are $1, 4, 4, 1$ while the rank sizes of $[1234, 3142]$ are $1, 3, 3, 1$. The covers from the maximum in these two intervals are highlighted. The weights from 3412 are $e_1 - e_3$, $e_2 - e_3$, $e_1 - e_4$ and $e_2 - e_4$ while the weights from 3142 are $e_1 - e_3$, $e_2 - e_3$ and $e_2 - e_4$ (from left to right). The same linear space is spanned by these two sets of weights.

Proposition 3.6. $\mathrm{AD}(u,v)$ *has the following properties.*

1. *For any* $u \leq w \lessdot v$, $\mathrm{AD}(u,v) = \mathrm{AD}(u,w) + \mathbb{R} \cdot \mathrm{wt}(w,v)$.
2. *For any saturated chain* $u = w^{(0)} \lessdot w^{(1)} \lessdot \cdots \lessdot w^{(\ell-1)} \lessdot w^{(\ell)} = v$,
$$\mathrm{AD}(u,v) = \mathrm{span}_{\mathbb{R}}\{\mathrm{wt}(w^{(i)}, w^{(i+1)}) \mid i = 0, \ldots, \ell-1\}.$$
3. *For any* $w \in [u,v]$, $\mathrm{AD}(u,v)$ *is spanned by the labels of all cover relations incident to* w *inside* $[u,v]$.

Definition 3.7. A Bruhat interval $[u,v]$ is *toric* if $\mathrm{ad}(u,v) = \ell(v) - \ell(u)$.

In light of Proposition 3.6(2), we always have $\mathrm{ad}(u,v) \leq \ell(v) - \ell(u)$.
In fact, knowing which intervals are toric provides us with full control over $\mathrm{ad}(u,v)$.

Proposition 3.8. $\mathrm{ad}(u,v) = \max_{w \in [u,v]} \ell(w) - \ell(u)$ *where* $[u,w]$ *is toric.*

The dual statement of Proposition 3.8:
$$\mathrm{ad}(u,v) = \max_{w \in [u,v]} \ell(v) - \ell(w) \text{ where } [w,v] \text{ is toric,}$$
follows from the exact same argument. It is also clear from Proposition 3.8 that the following more symmetric-looking statement holds:
$$\mathrm{ad}(u,v) = \max_{u \leq x \leq y \leq v} \ell(y) - \ell(x) \text{ where } [x,y] \text{ is toric.}$$

3.2 The torus complexity via Deodhar decompositions

We investigate the torus action on Richardson varieties in detail. Our main technical tool is the *Deodhar decomposition*. We will follow the notation of [18].

Definition 3.9 ([18, Section 3])**.** For an expression $\mathbf{w} = q_{i_1} \cdots q_{i_\ell}$ where each $q_{i_k} \in S \cup \{1\}$, define $w_{(k)} = q_{i_1} \cdots q_{i_k}$ for $k = 1, \ldots, \ell$. Set $w_{(0)} = 1$. Define the following sets:
$$J_{\mathbf{w}}^{+} := \{k \in \{1, \ldots, \ell\} \mid w_{(k-1)} < w_{(k)}\},$$
$$J_{\mathbf{w}}^{\circ} := \{k \in \{1, \ldots, \ell\} \mid w_{(k-1)} = w_{(k)}\},$$
$$J_{\mathbf{w}}^{-} := \{k \in \{1, \ldots, \ell\} \mid w_{(k-1)} > w_{(k)}\}.$$
For a reduced expression $\mathbf{v} = s_{i_1} \cdots s_{i_\ell}$, a *subexpression* is obtained from $\mathbf{v}$ by replacing some of the factors with 1. A subexpression $\mathbf{u}$ of $\mathbf{v}$ is called *distinguished* if $u_{(j)} \leq u_{(j-1)} s_{i_j}$ for all $j = 1, \ldots, \ell$. In other words, if multiplication by s_{i_j} (on the right) decreases the length of $u_{(j-1)}$, then we must have $u_{(j)} = u_{(j-1)} s_{i_j}$ in a distinguished subexpression.

For every simple root $\alpha_i \in \Delta$, there is a corresponding homomorphism $\varphi_i : \mathrm{SL}_2 \to G$. Define the following elements
$$x_{\alpha_i}(m) = \varphi_i \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad y_{\alpha_i}(m) = \varphi_i \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \quad \dot{s}_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For the majority of this work, we abuse notation and refer to the coset representative of $w \in W = N_G(T)/T$ in $N_G(T)$ by w. In their work, Marsch and Rietsch fix specific coset representatives. To stay in line with their notation and constructions, in this self-contained section, we follow their notational conventions. For $s_i \in W$, we fix $\dot{s}_i$ as our coset representative of $s_i \in W$ in $N_G(T)$. For any $w \in W$ with $w = s_{i_1} \cdots s_{i_\ell}$, let $\dot{w} := \dot{s}_{i_1} \cdots \dot{s}_{i_\ell}$ be our coset representative of w in $N_G(T)$.

As introduced in Section 2.2, for each $\alpha \in \Phi$ there is an associated root subgroup U_α in G. The simple root subgroup U_{α_i} for $\alpha_i \in \Delta$ is the image of x_{α_i} in G, namely $\{x_{\alpha_i}(m) \mid m \in \mathbb{C}\}$. For $\alpha \in \Phi$, there exists $w \in W$ and $\alpha_i \in \Delta$ such that $\alpha = w(\alpha_i)$. Then $U_\alpha = \dot{w} U_{\alpha_i} \dot{w}^{-1}$; it is the image of $x_\alpha(m) := \dot{w} x_{\alpha_i}(m) \dot{w}^{-1}$ in G. In a similar way, define $y_\alpha(m) := \dot{w} y_{\alpha_i}(m) \dot{w}^{-1}$. Then $y_\alpha(m) = x_{-\alpha}(m)$. It will be important later, that for any $t \in T$ and $m \in \mathbb{C}$,

$$t x_\alpha(m) = x_\alpha(\alpha(t) m) t. \tag{3.1}$$

Definition 3.10 ([18, Definition 5.1])**.** For a reduced word $\mathbf{v} = s_{i_1} \cdots s_{i_\ell}$ and a distinguished subword $\mathbf{u}$, define the following set

$$G_{\mathbf{u},\mathbf{v}} = \left\{ g = g_1 g_2 \cdots g_\ell \;\middle|\; \begin{array}{lll} g_k = x_{\alpha_{i_k}}(m_k)\dot{s}_{i_k}^{-1} & \text{if } k \in J_{\mathbf{u}}^-, & \\ g_k = y_{\alpha_{i_k}}(p_k) & \text{if } k \in J_{\mathbf{u}}^\circ, & \text{for } p_k \in \mathbb{C}^*,\ m_k \in \mathbb{C} \\ g_k = \dot{s}_{i_k} & \text{if } k \in J_{\mathbf{u}}^+, & \end{array} \right\}.$$

We have $G_{\mathbf{u},\mathbf{v}} \simeq (\mathbb{C}^*)^{J_{\mathbf{u}}^\circ} \times \mathbb{C}^{J_{\mathbf{u}}^-}$. Now, the *Deodhar component* is $D_{\mathbf{u},\mathbf{v}} := G_{u,v}B/B \subset G/B$. We have the isomorphism $G_{u,v} \simeq D_{\mathbf{u},\mathbf{v}}$ via $g \mapsto gB/B$ ([18, Proposition 5.2]).

A distinguished subword $\mathbf{u}$ is called *positive* if $u_{(j-1)} \leq u_{(j)}$ for all j, i.e. $J_{\mathbf{u}}^- = \emptyset$. Given a fixed reduced word $\mathbf{v}$ of v and an element $u \leq v$, there is a unique positive distinguished subword $\mathbf{u}_+$ of u, whose Deodhar component $D_{\mathbf{u}_+,\mathbf{v}}$ has the same dimension as $\mathcal{R}_{u,v}$ by [18, Lemma 3.5].

Theorem 3.11 ([8, Theorem 1.1])**.** *Assume $u \leq w$ and fix a reduced word $\mathbf{v}$ of v. The Richardson cell $\mathcal{R}_{u,v}^\circ := X_v^\circ \cap X_\circ^u$ has a decomposition $\mathcal{R}_{u,v}^\circ = \bigsqcup_{\mathbf{u}} D_{\mathbf{u},\mathbf{v}}$ into a disjoint union over all distinguished subwords $\mathbf{u}$ of $\mathbf{v}$ that evaluate to u.*

Definition 3.12. Fix a reduced word $\mathbf{v}$ of v and a distinguished subword $\mathbf{u}$. For each $k \in J_{\mathbf{u}}^\circ$, define $\beta_k = \mathrm{wt}(u_{(k-1)}, u_{(k-1)}s_{i_k})$; for each $k \in J_{\mathbf{u}}^-$, define $\beta_k = \mathrm{wt}(u_{(k)}, u_{(k-1)})$ (cf. Definition 3.1). Note that in both cases, $s_{\beta_k} = u_{(k-1)} s_{i_k} u_{(k-1)}^{-1}$ so $\beta_k = \pm u_{(k-1)}(\alpha_{i_k})$. To be precise, when $k \in J_{\mathbf{u}}^\circ$, $u_{(k-1)} < u_{(k-1)}s_{i_k}$ and when $k \in J_{\mathbf{u}}^-$, $u_{(k-1)} > u_{(k-1)}s_{i_k}$, so we have that $\beta_k = u_{(k-1)}(\alpha_{i_k})$ when $k \in J_{\mathbf{u}}^\circ$ and $\beta_k = -u_{(k-1)}(\alpha_{i_k})$ when $k \in J_{\mathbf{u}}^-$. Also define

$$U_k = \begin{cases} U_{-\beta_k}^* & \text{if } k \in J_{\mathbf{u}}^\circ \\ U_{-\beta_k} & \text{if } k \in J_{\mathbf{u}}^- \end{cases}.$$

Define $\mathrm{TD}(\mathbf{u},\mathbf{v}) = \mathrm{span}_{\mathbb{R}}\{\beta_k \mid k \in J_{\mathbf{u}}^\circ \cup J_{\mathbf{u}}^-\}$ and $\mathrm{td}(\mathbf{u},\mathbf{v}) = \dim \mathrm{TD}(\mathbf{u},\mathbf{v})$.

For any $x \in D_{\mathbf{u},\mathbf{v}}$, there is a unique $g \in G_{\mathbf{u},\mathbf{v}}$ such that the coset gB equals x. In this case, call g the *standard form* of x. For a $g \in G_{\mathbf{u},\mathbf{v}}$ and $k \in J^\circ_{\mathbf{u}} \cup J^-_{\mathbf{u}}$, let

$$g^{(k)} = \begin{cases} p_k & \text{if } k \in J^\circ_{\mathbf{u}} \\ m_k & \text{if } k \in J^-_{\mathbf{u}} \end{cases}.$$

We say that x is a *general* point in $D_{\mathbf{u},\mathbf{v}}$ if $g^{(k)}$ is nonzero for all $k \in J^-_{\mathbf{u}}$.

Lemma 3.13. *Fix $u \le v$ and a reduced word $\mathbf{v} = s_{i_1} \cdots s_{i_\ell}$ of v. Let $\mathbf{u} = u_1 \cdots u_\ell$ be a distinguished subword of $\mathbf{v}$ for u. Let $J^\circ_{\mathbf{u}} \cup J^-_{\mathbf{u}} = \{j_1 < j_2 < \cdots < j_z\}$ and $g \in G_{\mathbf{u},\mathbf{v}}$. Then*

$$g = \prod_{k=1}^{z} x_{-\beta_{j_k}}(g^{(j_k)})\dot{u}, \qquad D_{\mathbf{u},\mathbf{v}} = \prod_{k=1}^{z} U_{j_k}\dot{u}B.$$

Theorem 3.14. *Fix $u \le v$ and fix a reduced word $\mathbf{v}$ of v. Let $\mathbf{u}$ be a distinguished subword of $\mathbf{v}$ for u. For a general point $x \in D_{\mathbf{u},\mathbf{v}}$ the span of the weights of the torus action on the orbit Tx equals* $\mathrm{TD}(\mathbf{u},\mathbf{v})$. *And if $x \in D_{\mathbf{u},\mathbf{v}}$ is not general, then the torus weights on Tx are contained in* $\mathrm{TD}(\mathbf{u},\mathbf{v})$. *Moreover, we have*

$$\mathrm{TD}(\mathbf{u},\mathbf{v}) \subset \mathrm{TD}(\mathbf{u}_+,\mathbf{v}) = \mathrm{AD}(u,v).$$

Corollary 3.15. *For $u \le v$, the Bruhat interval $[u,v]$ is toric if and only if the Richardson variety $\mathcal{R}_{u,v}$ is toric.*

4 The Levi complexity of Schubert varieties

4.1 An equivariant isomorphism

The content in the first portion of this section is similar in broad form to our argument from [12], but here our results apply to any $w \in W$ and $I \subseteq \Delta$.

Let X be a T-variety with action denoted by $\cdot$. For each $u \in W$ we define a T-action $\cdot_u$ on X by $t \cdot_u x = utu^{-1} \cdot x$ for all $x \in X$ and $t \in T$. The action $\cdot_u$ does not depend on the coset representative chosen for u and so is well-defined. Henceforth, $\cdot$ will always denote the usual action of T on G/B by left multiplication.

For a positive integer n let $\mathbb{A}^n$ be the *affine n-space*. We use the following well-known fact [3, Sections 14.12, 14.4] repeatedly in what follows. For all $v \in W$,

$$X^\circ_v = U_v vB \cong \mathbb{A}^{\ell(v)} \text{(as varieties)}. \tag{4.1}$$

From now on, to simplify our notation let the left parabolic decomposition of w with respect to $I \subseteq S$ be $w = ad$ where $a \in W_I$ and $d \in {}^I W$. Recall that $V_d := aU_d a^{-1}$.

Definition 4.1. The *I-heart* of the Schubert variety X_w, denoted $\mathrm{Heart}_I(X_w)$, is the subvariety $V_d wB \subseteq U_w wB = X^\circ_w$.

Proposition 4.2. $\mathrm{Heart}_I(X_w)$ *is T-stable for the action* $\cdot$. *Additionally,* $\mathrm{Heart}_I(X_w)$ *is T-equivariantly isomorphic to X°_d equipped with the T-action $\cdot_{a^{-1}}$ under the map ϕ :* $\mathrm{Heart}_I(X_w) \to X^\circ_d$ *given by $hB \mapsto a^{-1}hB$.*

Define $B_{L_I} := L_I \cap B$ and U_{L_I} to be the unipotent radical of B_{L_I}. Then $B_{L_I} = T \ltimes U_{L_I}$ is a Borel subgroup of L_I, and $U_{L_I} = B_{L_I} \cap U = \prod_{\alpha \in \Phi^+(I)} U_\alpha$ [3, Section 14].

Set $U_{\bar{a}} = \prod_{\alpha \in \phi^+(I) \setminus \mathcal{I}(a)} U_\alpha$. If $\phi^+(I) \setminus \mathcal{I}(a) = \emptyset$, then we define $U_{\bar{a}}$ to be the trivial subgroup of G. Then $U_{\bar{a}}$ is a closed subgroup of U normalized by T and

$$B_{L_I} = T \ltimes U_{L_I} = T \ltimes \prod_{\alpha \in \phi^+(I)} U_\alpha = T \ltimes (U_{\bar{a}} U_a).$$

Lemma 4.3. *The following are equivalent. (1) $U_{\bar{a}}$ is the trivial subgroup of G. (2) $\phi^+(I) \setminus \mathcal{I}_L(a) = \emptyset$, where $\phi^+(I)$ is the set of positive roots corresponding to the Levi subgroup L_I (3) L_I acts on X_w. (4) $I \subseteq \mathcal{D}_L(w)$. (5) $a = w_0(I)$.*

Lemma 4.4. $\mathrm{Heart}_I(X_w)$ *is $U_{\bar{a}}$-stable for the usual action.*

Lemma 4.5. *Let $x \in X_w^\circ \setminus \mathrm{Heart}_I(X_w)$ and $h_1, h_2 \in \mathrm{Heart}_I(X_w)$.*

1. *$u_2 h_1 \notin \mathrm{Heart}_I(X_w)$ for all $u_2 \in U_a$ with $u_2 \neq e$.*
2. *$t u_1 x \notin \mathrm{Heart}_I(X_w)$ for all $t \in T$ and $u_1 \in U_{\bar{a}}$.*
3. *Let $b = t u_1 u_2 \in B_{L_I}$ with $t \in T$, $u_1 \in U_{\bar{a}}$, and $u_2 \in U_a$. If $b h_1 = h_2$, then $u_2 = e$ and $t u_1 h_1 = h_2$.*

Lemma 4.6. *Let $x \in X_w^\circ$. Then $(B_{L_I} \cdot x) \cap \mathrm{Heart}_I(X_w) \neq \emptyset$.*

We now define our surjection from T-orbits in $\mathrm{Heart}_I(X_w)$ to B_{L_I}-orbits in X_w°, which is a codimension preserving bijection in the case where L_I acts on the Schubert variety .

Theorem 4.7. *The map $\beta : \mathcal{O}_T(\mathrm{Heart}_I(X_w)) \to \mathcal{O}_{B_{L_I}}(X_w^\circ)$ given by $\Theta \mapsto B_{L_I} x$ where x is any point in Θ is a surjection. If L_I acts on X_w, then β is a codimension preserving bijection. That is,*

$$\dim(X_w^\circ) - \dim(\beta(\Theta)) = \dim(\mathrm{Heart}_I(X_w)) - \dim(\Theta),$$

for every $\Theta \in \mathcal{O}_T(\mathrm{Heart}_I(X_w))$.

We are now able to prove Theorem 1.3.

Corollary 4.8. *If L_I acts on X_w, the minimal codimension of a B_{L_I}-orbit in X_w° equals the torus complexity of X_d° for the usual torus action. That is, $c_{L_I}(X_w^\circ) = c_T(X_d^\circ)$.*

4.2 Orbits in the Schubert Variety

In this section let $w \in W$ and $I \subseteq \Delta$. For any $u \in W$, let $u = {}_I u\, {}^I u$ be the left parabolic decomposition of u with respect to I. Our goal is to extend Corollary 4.8 to a formula for the Levi complexity of a Schubert variety. To do this, we will need to lower bound the codimension of a B_{L_I}-orbit in X_u° for $u \leq w$. The difficulty in this case is that L_I may not act on X_u, though of course B_{L_I} acts, and so $U_{\bar{a}}$ will not be the trivial subgroup of G. For a subvariety X of a variety Y, write $\mathrm{codim}_Y(X)$ for the codimension of X in Y.

Proposition 4.9. *Let $u \leq w \in W$. Let $\Xi \in \mathcal{O}_{B_{L_I}}(X_u^\circ)$. Let $h \in \Xi \cap \mathrm{Heart}_I(X_u)$ and $\Theta = T \cdot h$. Then*

$$\mathrm{codim}_{X_u^\circ}(\Xi) \geq \ell(u) - \mathrm{supp}({}^I u) - \ell(w_0(I))$$

and

$$\mathrm{codim}_{X_w}(\Xi) \geq \ell(w) - \mathrm{supp}({}^I u) - \ell(w_0(I)).$$

We can now prove Theorem 1.4.

Proof of Theorem 1.4. The equivalence of L_I acting on X_w and $I \subseteq \mathcal{D}_L(w)$ is Lemma 4.3.

The Bruhat decomposition tells us that the Schubert variety X_w is the disjoint union of the Schubert cells for $u \leq w$, $X_w = \bigsqcup_{u \leq w} X_u^\circ$, and hence

$$\mathcal{O}_{B_{L_I}}(X_w) = \bigsqcup_{u \leq w} \mathcal{O}_{B_{L_I}}(X_u^\circ).$$

Thus

$$\begin{aligned} c_{B_{L_I}}(X_w) &= \min_{\substack{u \leq w \\ \Xi \in \mathcal{O}_{B_{L_I}}(X_u^\circ)}} \mathrm{codim}_{X_w}(\Xi) \\ &\geq \min_{u \leq w} \Big(\ell(w) - \mathrm{supp}({}^I u) - \ell(w_0(I)) \Big) \\ &= \ell(w) - \ell(w_0(I)) + \min_{u \leq w} \Big(-\mathrm{supp}({}^I u) \Big) \\ &= \ell({}^I w) - \max_{u \leq w} \Big(\mathrm{supp}({}^I u) \Big) \\ &= \ell({}^I w) - \mathrm{supp}({}^I w), \end{aligned}$$

where the inequality is Proposition 4.9, the third equality follows from Lemma 4.3, and the final equality follows from the fact that $u \leq w$ implies ${}^I u \leq {}^I w$.

Corollary 4.8 and Corollary 1.2 imply that this lower bound on $c_{B_{L_I}}(X_w)$ is in fact achieved. We conclude that $c_{B_{L_I}}(X_w) = \ell({}^I w) + \mathrm{supp}({}^I w)$. □

Acknowledgements

We thank Alexander Yong for many helpful conversations.

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