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Deformations of restricted reflection arrangements

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Abstract. We construct free, non-constant multiplicities and free Shi-like deformations for all restrictions of Weyl arrangements $\mathcal{A}_W$. Both are given in terms of root-theoretic data of W and our proofs are case-free. In type-A we give a bijective proof of resulting product formulas for the number of regions of the deformations, while at the same time proving a wide generalization of Joyal's bijection between labeled trees and functions.

Keywords: multiarrangements, transitive deformations, Weyl groups, Joyal bijection

1 Introduction

A *deformation* of a real, central hyperplane arrangement $\mathcal{A}$ in some space V is an affine arrangement in V each of whose hyperplanes is parallel to some hyperplane of $\mathcal{A}$. If we have fixed a linear form $\alpha_H \in V^*$ for each hyperplane $H \in \mathcal{A}$, any deformation can be encoded by a selection of offsets $\mathfrak{m}(H) \subset \mathbb{R}$ which we record as a function $\mathfrak{m} : \mathcal{A} \to P(\mathbb{R})$. We write $\mathcal{A}^{\mathfrak{m}}$ for the deformation:

$$\mathcal{A}^{\mathfrak{m}} := \big\{\alpha_H(\boldsymbol{x}) \in \mathfrak{m}(H) \mid H \in \mathcal{A}\big\}. \tag{1.1}$$

Deformations of arrangements $\mathcal{A}^{\mathfrak{m}}$ give rise to Ziegler's *multiarrangements* $(\mathcal{A}, \boldsymbol{m})$: abstract structures of arrangements with multiplicities $\boldsymbol{m} : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ that are usually studied in terms of their modules of logarithmic derivations $\mathrm{Der}(\mathcal{A}, m)$ (polynomial vector fields tangent to each hyperplane of $\mathcal{A}$ with order of tangency given by $\boldsymbol{m}$). In more detail, the restriction of the cone of $\mathcal{A}^{\mathfrak{m}}$ on the special hyperplane defines the multiarrangement $(\mathcal{A}, \boldsymbol{m})$ with multiplicities given by the cardinalities of the sets of offsets $\boldsymbol{m}(H) = |\mathfrak{m}(H)|$.

A multiarrangement $(\mathcal{A}, \boldsymbol{m})$ is called *free* when the module $\mathrm{Der}(\mathcal{A}, \boldsymbol{m})$ is free over the polynomial algebra $\mathbb{C}[V]$. In this case, we call $\boldsymbol{m}$ a *free multiciplicity* of $\mathcal{A}$, and we call the (positive, integer) degrees of the generators of $\mathrm{Der}(\mathcal{A}, \boldsymbol{m})$ the *exponents* of $(\mathcal{A}, \boldsymbol{m})$. We call the arrangement $\mathcal{A}$ *free* if $\boldsymbol{m} = \mathbf{1}$ is a free multiplicity of $\mathcal{A}$, in which case the exponents of $(\mathcal{A}, \mathbf{1})$ are the roots of the characteristic polynomial of $\mathcal{A}$. All chordal, supersolvable, and reflection arrangements are free, but it is very difficult, for any given $\mathcal{A}$, to classify its free multiplicities $\boldsymbol{m}$ or even to construct families of free multiplicities.

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Given a free multiplicity $\boldsymbol{m}$ for $\mathcal{A}$, it is even more difficult to determine whether there exist any deformations $\mathcal{A}^{\mathfrak{m}}$, with $\boldsymbol{m}(H) = |\mathfrak{m}(H)|$ for all $H \in \mathcal{A}$, that are themselves (i.e. their cones) free.

The family of reflection arrangements $\mathcal{A}_W$ for a Weyl group W is the most prominent case where we have infinite collections of free multiplicities that all have associated free deformations. All **constant** multiplicities are free with exponents given by Coxeter-theoretic data while the Fuss–Shi and Fuss–Catalan arrangements are compatible deformations (for the even- and odd- constant multiplicities respectively). In particular, the Shi arrangement $\mathrm{Shi}(W)$, given below with its characteristic polynomial, is a free deformation corresponding to the constant $\boldsymbol{m} = \mathbf{2}$ multiplicity:

$$\mathrm{Shi}(W) := \big\{ \langle \rho, \boldsymbol{x} \rangle \in \{0,1\} \mid \rho \in \Phi_W^+ \big\} \qquad \text{has} \qquad \chi\big(\mathrm{Shi}(W); t\big) = (t-h)^n, \tag{1.2}$$

where h and n are the Coxeter number and rank of W and Φ_W^+ its set of positive roots.

Most of the known examples of free multiplicities are constant or almost constant while examples of associated deformations are always subarrangements of affine Weyl arrangements. Only in the case of the braid arrangement $\mathrm{Br}(n)$, Abe–Nuida–Numata [1] have constructed an n-parameter family of free multiplicities *(ANN)* with Nakashima–Tsujie [6] later constructing compatible deformations. In all these cases the offset functions $\mathfrak{m}(H)$ always determine equally-spaced-apart affine hyperplanes.

In this work, we construct free multiplicities for all restricted reflection arrangements $\mathcal{A}_W^X$ of Weyl groups W, for any flat $X \in \mathcal{L}_{\mathcal{A}_W}$, and give associated free deformations. The resulting objects simultaneously generalize the Shi arrangements $\mathrm{Shi}(W)$ to the restricted case and the ANN multiplicities to all Weyl types. The multiplicity functions are non-constant, they are given in terms of relative Coxeter numbers of W, and the associated deformations are not equally-spaced but described in root-theoretic terms.

Main theorems. When discussing restrictions $\mathcal{A}_W^X$ of the arrangement $\mathcal{A}_W$ associated to a Weyl group W, it is convenient to use the coordinate system of fundamental coweights. Indeed, if $(\alpha_1, \dots, \alpha_n)$ are the simple roots of W so that $\mathcal{A}_W \subset V := \mathrm{Vect}(\alpha_1, \dots, \alpha_n)$, then the fundamental coweights $(\omega_1, \dots, \omega_n)$ are their dual basis (i.e. $\langle \alpha_i, \omega_j \rangle = \delta_{i,j}$) and we write the arbitrary element $\boldsymbol{v} \in V$ as $\boldsymbol{v} = v_1\omega_1 + \ldots + v_n\omega_n$.

For every subset $J \subseteq [n]$, we may define the *standard flat X_J*

$$X_J := \bigcap_{j \in J} \big\{ \boldsymbol{v} \in V : \ \langle \alpha_j, \boldsymbol{v} \rangle = 0 \big\} = \bigoplus_{j \in J^c} \mathbb{R} \cdot \omega_j, \tag{1.3}$$

having basis $\{\omega_j \mid j \in J^c\}$ and coordinates $\boldsymbol{v}_{J^c} := (v_j)_{j \in J^c}$ (where $J^c := [n] \setminus J$). For every root $\rho = \sum a_i \alpha_i \in \Phi_W^+$ we define the *restriction* ρ_{J^c} associated to the standard flat X_J as $\rho_{J^c} := \sum_{j \in J^c} a_j \alpha_j$ (note that ρ_{J^c} does not necessarily belong to X_J). The *restricted reflection arrangement* $\mathcal{A}_W^{X_J} := \{H \cap X_J \mid H \in \mathcal{A}_W,\ H \not\supset X_J\}$ becomes

$$\mathcal{A}_W^{X_J} = \big\{ \langle \rho_{J^c}, \boldsymbol{v}_{J^c} \rangle = 0 \mid \rho \in \Phi_W^+,\ \rho_{J^c} \neq 0 \big\}.$$

We use the auxilliary vector $\omega_J := \sum_{j\in J}\omega_j$ to define the *Shi-deformation of $\mathcal{A}_W^{X_J}$* as:

$$\operatorname{Shi}(\mathcal{A}_W^{X_J}) := \big\{\langle \rho_{J^c}, v_{J^c}\rangle = \epsilon - \langle \rho, \omega_J\rangle \mid \epsilon \in \{0,1\},\, \rho \in \Phi_W^+,\ \rho_{J^c} \neq 0\big\}. \tag{1.4}$$

Notice that many roots $\rho \in \Phi_W^+$ will determine the same hyperplanes in (1.4); those will only be counted once.

Before we present the main Theorem, we need to introduce another notation. We write W_X for the pointwise stabilizer of the flat X in W; it is itself a reflection group, generated by the reflections that fix X, but it may be reducible. For any pair of flats $Z \subseteq X$ in $\mathcal{L}_{\mathcal{A}_W}$ with $\dim(Z) = \dim(X) - 1$, the *relative Coxeter number $h(X, Z)$* is defined as the Coxeter number of the unique irreducible component of W_Z that does not belong to W_X.

The *type* of a flat X is the Coxeter type of W_X. In Figure 1 we see $\mathcal{A}_{D_5}^X$ (given in coweight coordinates) for any flat X of type $A_1 \times A_1 \times A_1$ in the reflection arrangement of $W = D_5$ and the numbers $h(X, Z_i)$ for all $Z_i \in \mathcal{A}^X$.

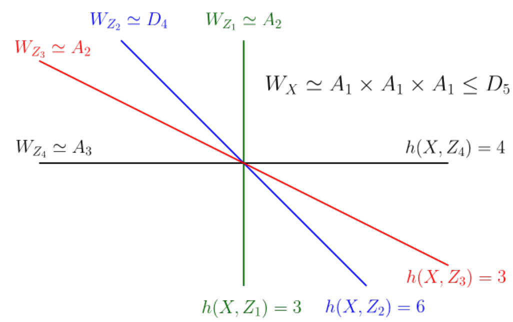

Figure 1: Relative Coxeter numbers in D_5

Theorem 1.1. *Let W be an irreducible Weyl group with rank n and Coxeter number h and $J \subseteq [n]$. Then, the Shi-deformation of $\mathcal{A}_W^{X_J}$ is free with characteristic polynomial*

$$\chi\big(\operatorname{Shi}(\mathcal{A}_W^{X_J}), t\big) = (t-h)^{\dim(X_J)}.$$

Every hyperplane $Z \in \mathcal{A}_W^{X_J}$ appears in $\operatorname{Shi}(\mathcal{A}_W^{X_J})$ with multiplicity $\boldsymbol{m}_J(Z) := h(X_J, Z)$. The multiarrangement $(\mathcal{A}_W^{X_J}, \boldsymbol{m}_J)$ is also free with exponents $(h, h, \ldots, h)$.

Example 1.1. We consider in $W = A_4$, with simple roots $(\alpha_i)_{i=1}^4 = e_{i,i+1}$, the set $J = \{1, 3\}$ as in Figure 2. The flat $\mathcal{A}_W^{X_J}$ has basis (ω_2, ω_4), coordinates $\mathbf{v}_{J^c} = (v_2, v_4)$, and we have $\omega_J = \omega_1 + \omega_3$. We group the roots with respect to their restriction on X_J:

- The four roots $e_{23}, e_{13}, e_{24}, e_{14}$ all restrict to the root e_{23} on X_J while the quantities $\langle \rho, \omega_J\rangle$ equal $0, 1, 1, 2$ respectively. We have that $\langle e_{23}, \mathbf{v}_{J^c}\rangle = v_2$, so that in total, they contribute the hyperplanes $v_2 \in \{1, 0, -1, -2\}$.
- The two roots e_{15}, e_{25} restrict to the root $e_{23} + e_{45}$ on X_J and the quantities $\langle \rho, \omega_J\rangle$ equal $2, 1$ respectively. They contribute the hyperplanes $v_2 + v_4 \in \{0, -1, -2\}$.
- The two roots e_{35}, e_{45} restrict to the root e_{45} on X_J and the quantities $\langle \rho, \omega_J\rangle$ equal $1, 0$ respectively. They contribute the hyperplanes $v_4 \in \{1, 0, -1\}$.

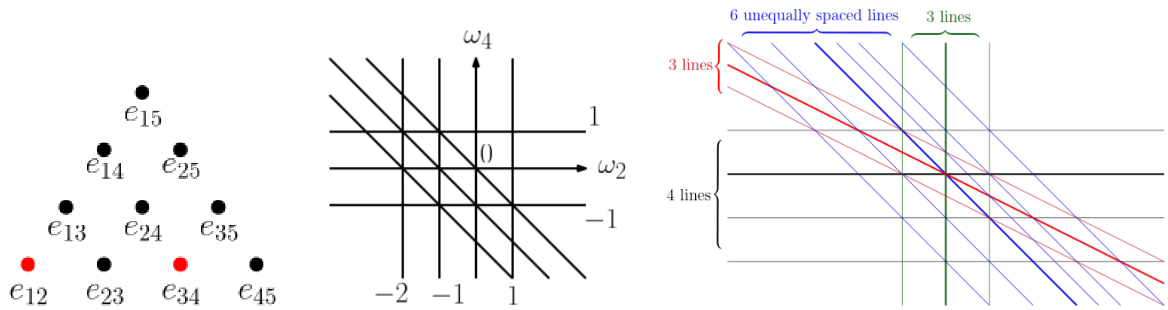

Figure 2: Left and center: The root system for A_4 with $J = \{1,3\}$, and the resulting Shi-deformation $\mathrm{Shi}(\mathcal{A}_{A_4}^{X_J})$. Right: the Shi-deformation associated to the flat X of $\mathcal{A}_{D_5}$ described in Figure 1.

In the center of Figure 2 we see the Shi deformation $\mathrm{Shi}(\mathcal{A}_W^{X_J})$; it has characteristic polynomial $(t-5)^2$.

Remark 1.2. In type-A, as suggested by Example 1.1 the resulting deformations have simpler combinatorial descriptions. The subset $J \subseteq [n]$ naturally determines a partition $\mathbf{m} \vdash (n+1)$: we set $m_i = a_i - a_{i-1}$ where $\{a_0, \dots, a_d\} = \{0\} \cup J^c$ (in Example 1.1, $\mathbf{m} = (2,2,1)$). If $d = n+1-|J|$, it is easy to see that the Shi deformation of $\mathcal{A}_W^{X_J}$ is affinely isomorphic to the deformation of the braid arrangement $\mathrm{Br}(d)$ which is given by the hyperplanes $x_i - x_j \in \{-m_i+1, \dots, m_j\}$ (see more in Section 3).

In this case, the relative Coxeter multiplicities will then equal $m_i + m_j$ for each hyperplane $x_i - x_j = 0$ which recovers the essential family of ANN multiplicities [1].

Remark 1.3. Every flat $X \in \mathcal{L}_{\mathcal{A}_W}$ is conjugate to some standard flat X_I, so indeed Theorem 1.1 holds (with a generalization of the definition of Shi-deformation) for all restrictions of Weyl arrangements. Notice also that the restrictions $\mathcal{A}_W^X$ are usually not isomorphic to Weyl arrangements (outside of types A and B). The existence of deformations as in Theorem 1.1 was surprising: there are no root lattices associated to such $\mathcal{A}_W^X$.

In Section 2 we give a sketch for the proof of Theorem 1.1. The approach is inductive utilizing generalizations of the deletion-restriction formulas to multiarrangements. After Zaslavsky's theorem, the product formula for the characteristic polynomial in Theorem 1.1 implies that the Shi deformations $\mathrm{Shi}(\mathcal{A}_W^{X_J})$ have $(h+1)^{\dim(X_J)}$-many regions. In Section 3 we give in type-A a bijective proof of this result, in fact of a wide generalization, by labeling the regions of the deformation with certain labeled trees which we enumerate via a generalization of Joyal's bijection.

2 Freeness and characteristic polynomial of $\mathrm{Shi}(\mathcal{A}_W^{X_J})$

In this section, we will sketch the proof of Theorem 1.1. We will start with the characteristic polynomial of $\mathrm{Shi}(A_W^{X_J})$ and the case of 2-dimensional flats X_J will also form the base case for the inductive proof of freeness.

Let $I \subseteq \Phi_W^+$ be a *root ideal*; that is, a set such that if $\alpha \in I$ and $\beta \in \Phi^+$ with $\alpha - \beta \in \Phi^+$, then $\beta \in I$ also. Abe–Terao [2] defined the ideal-Shi arrangements as

$$\mathrm{Shi}(W, I) := \left\{ \begin{array}{ll} \langle \rho, \boldsymbol{x} \rangle = 0, & \text{for } \rho \in I \\ \langle \rho, \boldsymbol{x} \rangle \in \{0, 1\}, & \text{for } \rho \in \Phi^+ \setminus I \end{array} \right\}, \tag{2.1}$$

and showed that they are free and described their exponents. Every root $\rho \in \Phi^+$ has a height given as $\langle \rho, \omega_{[n]} \rangle$ (recall the definition $\omega_{[n]} = \omega_1 + \cdots \omega_n$ from the previous section) which varies between 1 and $h - 1$. In this way, a root ideal I determines a height partition $\lambda \vdash |I|$ (where λ_i counts how many roots have height i), and Abe–Terao showed that the exponents of $\mathrm{Shi}(W, I)$ are precisely the numbers $h - e_I^s$ where $(e_I^s)_{s=1}^n$ is the dual partition to λ (possibly padded with 0's). In particular, they showed that

$$\chi\big(\mathrm{Shi}(W, I), t\big) = \prod_{s=1}^{n} (t - h + e_I^s). \tag{2.2}$$

Proof of Theorem 1.1 regarding the characteristic polynomial. Consider the following parallel translate of the standard flat X_J, $J \subseteq [n]$, defined in (1.3):

$$X_J^1 := \bigcap_{j \in J} \big\{ \boldsymbol{x} \in V : \ \langle \alpha_j, \boldsymbol{x} \rangle = 1 \big\} = \omega_J + \bigoplus_{j \in J^c} \mathbb{R} \cdot \omega_j.$$

Note that X_J^1 is a flat of the Shi arrangement $\mathrm{Shi}(W)$. Moreover, it is not too hard to show that the Shi-deformation of $\mathcal{A}_W^{X_J}$ is precisely the restriction $\mathrm{Shi}(W)^{X_J^1}$. Now we can compute its characteristic polynomial inductively using the deletion-restriction formula. Indeed, if $J = \{j_1 < \ldots < j_d\}$ and $J_r = \{j_1, \ldots, j_r\}$ then each set of (simple) roots $I_r := \{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$ is an order ideal in Φ^+ and has dual height partition (1^r).

After the Abe–Terao result (2.2), the Ideal-Shi arrangements $\mathrm{Shi}(W, I_r)$ have characteristic polynomials equal to $(t - h + 1)^r \cdot (t - h)^{n-r}$ and an inductive application of the deletion-restriction formula shows that $\mathrm{Shi}(W)^{X_{J_r}^1} - \bigcup_{j \in J \setminus J_r} H_{\alpha_j}$ has characteristic polynomial $(t - h + 1)^{d-r} \cdot (t - h)^{n-d}$. In particular,

$$\chi\big(\mathrm{Shi}(W)^{X_J^1}, t\big) = (t - h)^{n-d}.$$

$\square$

Corollary 2.1. *The multiplicities of the hyperplanes $Z \in \mathcal{A}_W^{X_J}$ are given by the relative Coxeter numbers $h(X_J, Z)$.*

Proof. This is in fact a corollary of a localized version of Theorem 1.1. The arrangement $\mathcal{A}_{W_Z}^{X_J}$ is a 1-dimensional restriction of a Weyl arrangement and according to the theorem $\chi\big(\operatorname{Shi}(\mathcal{A}_{W_Z}^{X_J},t)\big) = \big(t-h(X_J,Z)\big)$; i.e. $\operatorname{Shi}(\mathcal{A}_{W_Z}^{X_J})$ has $h(X_J,Z)$-many hyperplanes, which in turn implies that $\operatorname{Shi}(\mathcal{A}_{W}^{X_J})$ has $h(X_J,Z)$-many hyperplanes parallel to Z. $\square$

Sketch for Theorem 1.1. The proof for freeness has the same idea as the construction described above but is more technical. Abe–Terao–Wakefield [3] have developed a version of the deletion-restriction formula for multiarrangements. We still start with known freeness results from [2] about Ideal-Shi arrangements and compute the Euler multiplicity of [3] every time we restrict on a smaller dimensional flat. The recursive definition of the Euler multiplicity is equivalent to the recursive definition of the relative Coxeter numbers.

A base case for the induction is covered by dimension-2 flats in which case knowledge of the characteristic polynomial is almost sufficient to determine the free exponents. After we prove freeness of the multiarrangements, we deduce freeness of the deformations by applying Yoshinaga's method [7]. $\square$

3 A bijection in type A

For a positive integer n, we define $[n] := \{1,2,\dots,n\}$ (for $n \leq 0$, we define $[n] = \emptyset$). For any integers $m \leq n$ we define $[m:n] := \{k \in \mathbb{Z} \mid m \leq k \leq n\}$.

We consider real hyperplane arrangements made of a finite number of affine hyperplanes of the form

$$\{(x_1,\dots,x_d) \in \mathbb{R}^d \ \mid \ x_i - x_j = s\}, \tag{3.1}$$

with $i,j \in \{1,\dots,d\}$ and $s \in \mathbb{Z}$. From now on, we make an abuse of notation and denote by $\{x_i - x_j = s\}$ the hyperplane in (3.1).

Given a d-tuple of integers $\mathbf{m} = (m_1,\dots,m_d) \in \mathbb{N}^d$, we define the ***m**-Shi arrangement* as

$$\mathcal{A}_{\mathbf{m}} := \bigcap_{\substack{1\leq i<j\leq d\\ s\in[-m_i:m_j+1]}} \{x_i - x_j = s\}.$$

Our goal is to describe a bijection explaining the product formula for the number of regions in the **m**-Shi arrangement (see Theorem 1.1 and Remark 1.2).

Our method actually applies to a larger family of arrangements, which includes both the **m**-Shi arrangements and the **m**-Catalan arrangements. For a positive integer d we denote

$$R_d^+ := \{(i,j) \in [d]^2 \mid i < j\}.$$

Given a set $I \subseteq R_d^+$ we define the arrangement

$$\mathcal{A}_{\mathbf{m}}^I := \bigcap_{\substack{1 \le i < j \le d \\ s \in [-m_i : m_j + a_{i,j}]}} \{x_i - x_j = s\},$$

where $a_{i,j} = 1$ if $(i,j) \in I$, and $a_{i,j} = 0$ otherwise. We say that the set $I \subseteq R_d^+$ is *down-closed* if for any pair $(i,j) \in I$ and any integer $i' \in [i]$, the pair (i',j) is also in I.

We will show that the regions of $\mathcal{A}_{\mathbf{m}}^I$ are in bijection with a set of (rooted plane labeled) trees, via the general bijective framework of [4], and that, in turn, this set of trees is in bijection with a set of (plane directed) graphs which are easy to enumerate whenever the set I is down-closed.

3.1 Bijection with plane labeled graphs

We need to recall some vocabulary and results from [4].

Definition 3.1. *A braid-type arrangement $\mathcal{A} \subset \mathbb{R}^d$ is called* transitive *if for all distinct indices $i, j, k \in [d]$ and integers $s, t \ge 0$ one has:*

$$\text{if } \{x_i - x_j = s\} \notin \mathcal{A} \text{ with } s > 0 \text{ or } i > j, \text{ and } \{x_j - x_k = t\} \notin \mathcal{A} \text{ with } t > 0 \text{ or } j > k,$$
$$\text{then } \{x_i - x_k = s + t\} \notin \mathcal{A}.$$

It is easy to see that $\mathcal{A}_{\mathbf{m}}^I$ is transitive for any set $I \subseteq R_d^+$. Indeed, if $\{x_i - x_j = s\} \notin \mathcal{A}_{\mathbf{m}}^I$ and $\{x_j - x_k = t\} \notin \mathcal{A}_{\mathbf{m}}^I$ with $s, t \ge 0$, then $s \ge m_j + 1$ and $t \ge m_k + 1$; hence $s + t > m_k + 1$ and $\{x_i - x_k = s + t\} \notin \mathcal{A}_{\mathbf{m}}^I$.

For any transitive arrangement, a bijection is described in [4] between the regions of $\mathcal{A}$ and a set of trees. Recall that a *rooted k-ary tree* is a rooted plane tree such that every node has exactly k children, which are ordered.

Definition 3.2. *Let $m, d \in \mathbb{N}$. We define $\mathcal{T}_m^d$ as the set of rooted $(m+1)$-ary trees with d nodes labeled with distinct labels in $[d]$ (the leaves have no labels).*

For a tree $T \in \mathcal{T}_m^d$ we identify the nodes with their labels in $[d]$ (so that the node set of T in $[d]$). By definition, a node $j \in [d]$ of T has exactly $m+1$ (ordered) children, which are denoted by 0-child(j), 1-child(j), …, m-child(j) respectively.

*The node $i \in [d]$ is the s-*cadet *of the node $j \in [d]$ if $i = s$-child(j) and t-child(j) is a leaf for all $s < t \le m$. In this case, we write $i = s$-cadet(j).*

Definition 3.3. *Let $m, d \in \mathbb{N}$, and let*

$$\mathcal{A}_m^d := \bigcup_{\substack{1 \le i < j \le d \\ s \in [-m : m]}} \{x_i - x_j = s\} \subseteq \mathbb{R}^d.$$

For any sub-arrangement $\mathcal{A} \subseteq \mathcal{A}_m^d$ we define

$$\mathcal{T}_{\mathrm{m}}^{\mathrm{d}}(\mathcal{A}) := \{T \in \mathcal{T}_m^d \mid \text{if } i,j \in [d] \text{ and } s \in \{0,1,\ldots,m\} \text{ are such that } \{x_i - x_j = s\} \notin \mathcal{A} \text{ then } i \neq s\text{-cadet}(j) \text{ unless } s = 0 \text{ and } i > j\}.$$

Theorem 3.4 ([4, Theorem 8.8])**.** *If an arrangement $\mathcal{A} \subseteq \mathcal{A}_m^d$ is transitive, then the regions of $\mathcal{A}$ are in bijection with the trees in $\mathcal{T}_m^d(\mathcal{A})$.*

We want to apply Theorem 3.4 to the arrangement $\mathcal{A}_{\mathbf{m}}^I$. Let us describe more explicitly the set of trees associated to the regions of $\mathcal{A}_{\mathbf{m}}^I$.

Definition 3.5. *Given a tuple $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, and a set $I \subseteq R_d^+$, we set $m = 1 + \max(m_i \mid i \in [d])$ and define $\mathcal{T}_{\boldsymbol{m}}^I$ as the set of trees in $\mathcal{T}_{\mathrm{m}}^{\mathrm{d}}$ such that for every node $j \in [d]$,*

- *s-child(j) is a leaf for all $s > m_j + 1$,*
- *(m_j+1)-child(j) is either a leaf or a node i such that (i,j) is in I.*

It is easy to see that the set of trees $\mathcal{T}_{\mathbf{m}}^I$ defined above coincides with the set of trees $\mathcal{T}_m^d(\mathcal{A}_{\mathbf{m}}^I)$ given by Definition 3.3. Therefore, applying Theorem 3.4 to the transitive arrangement $\mathcal{A}_{\mathbf{m}}^I$ gives the following result.

Proposition 3.6. *Let d be a positive integer, let $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, and let $m = 1 + \max(m_i \mid i \in [n])$. For any set $I \subseteq R_d^+$ the regions of the arrangement $\mathcal{A}_{\boldsymbol{m}}^I$ are in bijection with the trees in $\mathcal{T}_{\boldsymbol{m}}^I$.*

3.2 Bijection with plane functional graphs, and enumeration

Definition 3.7. *A* fun-graph[1] *is a directed graph such that every vertex v has outdegree at most one, together with a total ordering of the incoming edges at each vertex.*

Two fun-graphs are represented in Figure 3.

Note that rooted plane trees, oriented toward the root-vertex, are fun-graphs. We will therefore (somewhat foolhardily) extend some common vocabulary about rooted plane trees to fun-graphs:

- If (u,v) is a directed edge of the fun-graph F we say that v is the *parent* of v and u is the *child* of v.
- A vertex v is a *leaf* if is has indegree 0, and a *node* otherwise. A vertex v is a *root* if it has outdegree 0.
- The children of a node v are ordered according to the total ordering of the incoming edges at v. If v has $m+1$ children we will denote them by 0-child$(v), \ldots, m$-child(v) respectively.

[1]The term fun-graph is short for "functional plane graph": *functional* because the edges can be thought as a function from a subset of the vertices to the set of vertices, and *plane* because the total ordering of the incoming edges at each vertex can be used to define a planar embedding of the graph.

Definition 3.8. *Let $m, d \in \mathbb{N}$. We define $\mathcal{F}_m^d$ as the set of all fun-graphs with d nodes labeled with distinct labels in $[d]$ (the leaves have no labels), such that every node has $m+1$ children, and the only root is the node labeled d.*

We will now describe a bijection Ψ between the set $\mathcal{F}_m^d$ of fun-graphs, and the set $\mathcal{T}_m^d$ of trees. This bijection is a variation on a construction due to Joyal for proving the Cayley formula [5]. Figure 3 illustrates this bijection (for $m = 3$ and $d = 15$).

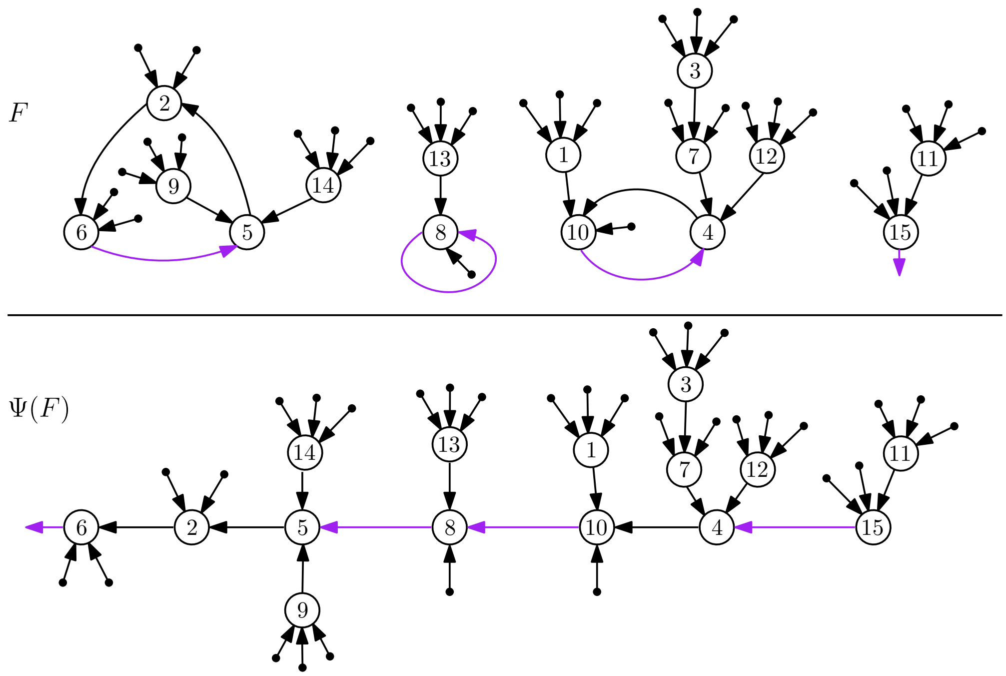

Figure 3: Top: A fun-graph F in $\mathcal{F}_3^{15}$. Bottom: the tree $\Psi(F)$, which is in $\mathcal{T}_3^{15}$.

A fun-graph F has two types of connected components: some connected components are rooted plane trees (oriented toward their root), and the other components have exactly one directed cycle, to which some rooted plane trees are attached. We call these types of connected components the *tree components* and *tree-cycle components* of F respectively. Note that the number of tree components is equal to the number of roots of F.

Let F be a fun-graph in $\mathcal{F}_m^d$, and let $k \geq 0$ be its number of tree-cycle components. For each tree-cycle component C we call *head* the node of C with maximal label among the nodes on the directed cycle of C. Let $a_1, \ldots, a_k$ be the heads of the tree-cycle components of F with the convention $a_1 < a_2 < \cdots < a_k$. Let also $b_1, \ldots, b_k$ be the parents in F of $a_1, \ldots, a_k$ respectively. We define $\Psi(F)$ as the fun-graph obtained as follows:

- For all $i \in [k]$, cut the edge (a_i, b_i) in two halves: an outgoing half-edge h_i incident to a_i and an incoming half-edge h_i' incident to b_i.

- For all $i \in [k-1]$, glue the half edge h'_i to the half-edge h_{i+1} (hence, if $a_i = s$-child(b_i) in F, then $a_{i+1} = s$-child(b_i) in $\Psi(F)$).
- If $k > 0$, then delete the half-edge h_1 (so that a_1 becomes a root of $\Psi(F)$), and attach an outgoing half-edge to node d and glue it to h'_k (hence, if $a_k = s$-child(b_k) in F, then $d = s$-child(b_k) in $\Psi(F)$).

An example is given in Figure 3.

It is easy to see that $\Psi(F)$ is a rooted plane tree (with root a_1 if $k > 0$ and root d if $k = 0$). We also claim that Ψ is a bijection between $\mathcal{F}^d_m$ and $\mathcal{T}^d_m$. Before formally stating this result, we define some parameters which are preserved by Ψ. Given a fun-graph F with exactly d nodes labeled with distinct labels in $[d]$, we say that a triple (i, j, s) is a *descent triple* of F if $i = s$-child(j) for some nodes $i, j \in [d]$ such that $i < j$ and some $s \in \mathbb{N}$. We say that (j, s) is a *weak-ascent pair* of F if $j \in [d]$ and s-child(j) is a node i such that $i \geq j$.

Theorem 3.9. *The map Ψ is a bijection between $\mathcal{F}^d_m$ and $\mathcal{T}^d_m$. Moreover, for any fun-graph $F \in \mathcal{F}^d_m$, the set of descent triples (resp. weak-ascent pairs) of F is equal to the set of descent triples (resp. weak-ascent pairs) of $\Psi(F)$.*

Proof. It is easy to see that the image, under Ψ, of any fun-graph in $\mathcal{F}^d_m$ is a tree in $\mathcal{T}^d_m$. In order to see that Ψ is a bijection we describe the inverse map. Given a rooted tree $T \in \mathcal{T}^d_m$ (oriented toward the rood vertex) we consider the sequence of nodes $i_1, \ldots, i_\ell = d$ between the root i_1 and the node d. We consider the left-to-right maximum $a_1, a_2, \ldots, a_k, a_{k+1} = d$ in this sequence. For $i \in [k]$, let b_i be the parent of a_{i+1} We define $\Gamma(T)$ as the fun-graph obtained as follows:

- For all $i \in [k]$, cut the edge (a_{i+1}, b_i) in two halves: an outgoing half-edge h_{i+1} incident to a_{i+1} and an incoming half-edge h'_i incident to b_i.
- Delete the half-edge h_{k+1} (so that the node d becomes a root of $\Gamma(T)$), attach an outgoing half-edge h_1 to a_1, and for all $i \in [k]$ glue the half edge h'_i to the half-edge h_i.

It is easy to see that Ψ and Γ are inverse mappings (this is essentially the "Foata correspondence" for permutations, which maps the number of cycles to the number of left-to-right maxima). Thus, the map Ψ is a bijection between $\mathcal{F}^d_m$ and $\mathcal{T}^d_m$.

Lastly, note that for a fun-graph $F \in \mathcal{F}^d_m$ all the edges which are different in F and $\Psi(F)$ (the edges that are cut by Ψ or Γ) are edges of the form (i, j) where i, j are nodes with $i \geq j$ (because, in the above notation, $a_i \geq b_i$ and $a_{i+1} > b_i$). This implies that the descent triples and weak-ascent pairs are all preserved by Ψ. $\square$

Definition 3.10. *Given a tuple $\boldsymbol{m}=(m_1, \ldots, m_d) \in \mathbb{N}^d$, and a set $I \subseteq R^+_d$, we set $m = 1 + \max(m_i \mid i \in [n])$ and define $\mathcal{F}^I_{\boldsymbol{m}}$ as the set of fun-graphs in $\mathcal{T}^{\mathrm{d}}_{\mathrm{m}}$ such that for every node $j \in [d]$,*

- *s-child(j) is a leaf for all $s > m_j + 1$,*
- *$(m_j{+}1)$-child(j) is either a leaf or a node i such that (i, j) is in I.*

Proposition 3.11. *Let $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, and let $I \subseteq R_d^+$. The trees in $\mathcal{T}_{\boldsymbol{m}}^I$ are in bijection with the fun-graphs in $\mathcal{F}_{\mathbf{m}}^I$ via the bijection Ψ.*

Proof. Let $m = 1 + \max(m_i \mid i \in [n])$. It is easy to see that a tree in $\mathcal{T}_m^d$ is in $\mathcal{T}_{\mathbf{m}}^I$ if and only if

(a) it has no weak-ascent pairs of the form (j, s) for $s > m_j$,
(b) it has no descent triple of the form (i, j, s) for $s > m_j$ except maybe if $s = m_j + 1$ and $(i, j) \in I$.

Similarly, a fun-graphs in $\mathcal{F}_m^d$ is in $\mathcal{F}_{\mathbf{m}}^I$ if and only if it satisfies (a) and (b). Hence Theorem 3.9 immediately implies that Ψ induces a bijection between $\mathcal{F}_{\mathbf{m}}^I$ and $\mathcal{T}_{\mathbf{m}}^I$. □

We can now establish a product formula for the number of regions of the arrangement $\mathcal{A}_{\mathbf{m}}^I$, provided that I is down-closed. Recall that $I \subseteq R_d^+$ is called *down-closed* if for any pair $(i, j) \in I$ and any $i' \in [i]$, the pair (i', j) is also in I.

Corollary 3.12. *Let $\boldsymbol{m} = (m_1, \ldots, m_d) \in \mathbb{N}^d$, and let $I \subseteq R_d^+$. If I is down-closed, then the number of regions of $\mathcal{A}_{\boldsymbol{m}}^I$ is*

$$\prod_{i=1}^{d-1} (n + 1 - d + i + c_i),$$

where $n := \sum_{j=1}^{d} (m_j + 1)$ for all $i \in [d]$, $c_i = \#\{j \in [d] \mid (i, j) \in I\}$.

Note that the set R_d^+ is down-closed, hence Corollary applies to the $\mathbf{m}$-Shi arrangement $\mathcal{S}hi_{\mathbf{m}} = \mathcal{A}_{\mathbf{m}}^{R_d^+}$ and shows that the number of regions of this arrangement is n^{d-1}, where $n = \sum_{j=1}^{d} (m_j + 1)$. Note also that applying Corollary to the set $I = \varnothing$ (which is down-closed) shows that the $\mathbf{m}$-Catalan arrangement $\mathcal{A}_{\mathbf{m}}^{\varnothing} = \bigcap_{\substack{1 \le i < j \le d \\ s \in [-m_i : m_j]}} \{x_i - x_j = s\}$ has $n!/(n - d + 1)!$ regions.

Proof. By Propositions 3.6 and 3.11, the number of regions of $\mathcal{A}_{\mathbf{m}}^I$ is equal to $|\mathcal{F}_{\mathbf{m}}^I|$. By definition, fun-graphs in $\mathcal{F}_{\mathbf{m}}^I$ have exactly $d - 1$ edges between nodes: exactly one edge out of each node labeled in $[d - 1]$. Let us count the number of possibilities for choosing the edge out of node i, starting with the node $i = d - 1$ and going down.

- For the edge out of the node $i = d - 1$, there are $n + c_{d-1}$ possibilities: the term n accounts for the possibilities where i is s-child(j) with $s \in \{0, \ldots, m_j\}$ for some $j \in [d]$, and the term c_{d-1} accounts for the possibilities where v is $(m_j{+}1)$-child(j) for some $j \in [d]$.
- For the edge out of the node $i = d - 2$, there are $n + c_{d-2} - 1$ possibilities: the term n accounts for the possibilities where i is s-child(j) with $s \in \{0, \ldots, m_j\}$ for some $j \in [d]$, the term c_{d-2} accounts for the possibilities where v is $(m_j{+}1)$-child(j) for some $j \in [d]$, while the -1 term accounts for the fact that 1 of the above possibilities has been taken by the node $d - 1$ (the condition on I implies that any of the choices available to the node $d - 1$ is also among the choices available to the node $d - 2$).

- For the edge out of the node $i = d-3$, there are $n + c_{d-3} - 2$ possibilities: the term n accounts for the possibilities where i is s-child(j) with $s \in \{0, \ldots, m_j\}$ for some $j \in [d]$, the term c_{d-2} accounts for the possibilities where v is (m_j+1)-child(j) for some $j \in [d]$, while the term -2 accounts for the fact that 2 of the above possibilities has been taken by the nodes $d-1$ and $d-2$ (the condition on I implies that any of the choices available to the nodes $d-1$ and $d-2$ is also available to the node $d-3$).
- $\ldots$
- For the edge out of the node $i = 1$, there are $n + c_1 - (d-2)$ possibilities: the term n accounts for the possibilities where i is s-child(j) with $s \in \{0, \ldots, m_j\}$ for some $j \in [d]$, the term c_1 accounts for the possibilities where v is (m_j+1)-child(j) for some $j \in [d]$, while the term $-(d-2)$ accounts for the fact that $(d-2)$ of the above possibilities has been taken by the nodes $d-1, d-2, \ldots, 2$ (the condition on I implies that any of the choices available to the nodes $d-1, d-2, \ldots, 2$ is also available to the node 1).

Multiplying these number of possibilities together gives the claimed counting formula. □

Remark 3.13. Observe that the above self-contained proof implies the formula for the number of regions in the Shi and Catalan arrangements (without assuming the Cayley or Catalan formulas). In fact, Theorem 3.9 gives a new easy way to prove various counting formulas for trees (plane trees, k-ary trees, labeled plane trees according to the degree distribution etc.).

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