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Extended weak order for $\widetilde{S}_n$ and the lattice of torsion classes

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Abstract. The extended weak order is a combinatorial poset associated to a Coxeter group, defined in terms of biclosed sets of roots in a root system. The lattice of torsion classes is an algebraic poset, defined in terms of sets of modules for an algebra. We show that the extended weak order for the affine symmetric group $\widetilde{S}_n$ is in fact a lattice quotient of the lattice of torsion classes for the preprojective algebra of a cycle quiver. We show how this allows one to translate between algebraic and combinatorial perspectives. In particular, we show that the extended weak order on $\widetilde{S}_n$ encodes the exchange graphs of cluster algebras of type $\widetilde{A}$ via its lattice quotients.

Keywords: Coxeter groups, weak order, torsion classes, preprojective algebras

1 Introduction

Given a Coxeter group (W, S), the *extended weak order* of W is a poset $\mathsf{Bic}(W)$ whose elements are the *biclosed sets* of reflections of W. Extended weak order was introduced by Matthew Dyer to study Hecke algebras and Kazhdan–Lusztig theory, and is the subject of many open conjectures [12]. When W is a finite Coxeter group, such as S_n, the extended weak order of W can be identified with the weak order on W. For infinite Coxeter groups, the weak order embeds as an order ideal of $\mathsf{Bic}(W)$, but there are more elements of $\mathsf{Bic}(W)$. If W is an affine Coxeter group of type $\widetilde{A}, \widetilde{B}, \widetilde{C}$, or $\widetilde{D}$, then there are explicit combinatorial models for $\mathsf{Bic}(W)$, introduced in [7]. In particular, if W is the affine symmetric group $\widetilde{S}_n$, which is the Coxeter group of type $\widetilde{A}_{n-1}$, then elements of $\mathsf{Bic}(W)$ can be identified with *translation-invariant total orders* (see Definition 2.1). For finite and affine Coxeter groups, $\mathsf{Bic}(W)$ is known to be a lattice [6], but it is still an open question whether this is the case for any Coxeter group.

There is another family of posets we will also be interested in. Let A be an algebra over $\mathbb{C}$. Then the *lattice of torsion classes* of A is a poset $\mathsf{Tors}(A)$ whose elements are *torsion classes* in the category of A-modules which are finite-dimensional over $\mathbb{C}$. This lattice is studied in the context of τ-tilting and silting theory for algebras. Each quiver

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Q has an associated *preprojective algebra* Π_Q over $\mathbb{C}$. When Q is an oriented Dynkin diagram associated to the finite Coxeter group W, then there is an isomorphism of lattices $\mathsf{Tors}(\Pi_Q) \xrightarrow{\sim} \mathsf{Bic}(W)$ [13]. Our first main result is the analog of this fact when $W = \widetilde{S}_n$[1].

Theorem 1.1. *Let Q be an orientation of the cycle graph on n vertices. Then there is a quotient map of complete lattices*

$$\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathsf{Bic}(\widetilde{S}_n).$$

We prove a stronger version, which allows us to extract information about Π_Q-modules from the lattice theory of $\mathsf{Bic}(W)$. A module M for Π_Q is called a **spherical module** (or a *real brick*) if $\mathrm{Ext}^i(M, M) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}$. In other words, M has the same cohomology groups as a 2-sphere. Our second main result both classifies the spherical modules of Π_Q in terms of join-irreducible elements of $\mathsf{Bic}(W)$, and also describes the existence of maps between spherical modules using the poset structure of $\mathsf{Bic}(W)$. We write $\mathsf{JIrr}(\mathsf{Bic}(W))$ for the *completely join-irreducible elements* of $\mathsf{Bic}(W)$, and we write $\mathsf{Sph}(\Pi_Q)$ for the spherical modules. We will index elements of both sets by diagrams (denoted below by σ) as described in Section 3.

Theorem 1.2. *There is a bijection*

$$\mathsf{JIrr}(\mathsf{Bic}(W)) \xrightarrow{\sim} \mathsf{Sph}(\Pi_Q)$$
$$J \mapsto \Pi_{\sigma(J)}.$$

Furthermore, the following are equivalent:

- *There exists a non-zero map of Π_Q-modules from $\Pi_{\sigma(J_1)}$ to $\Pi_{\sigma(J_2)}$;*
- *There does not exist $R \in \mathsf{Bic}(W)$ so that $R \geq J_1$ and J_2 covers $R \wedge J_2$.*

The objects put in bijection by Theorem 1.2 are also in bijection with a third, geometric object called *shards*. The bijection between $\mathsf{JIrr}(\mathsf{Bic}(W))$ and shards of the affine braid arrangement was the subject of [2], an extended abstract from FPSAC 2024. In their recent work [10], Dana, Speyer, and Thomas showed that shards inject into $\mathsf{Sph}(\Pi_Q)$; the resulting real bricks are called *shard modules*. In the course of the proof of Theorem 1.2, we show that every spherical module of Π_Q is a shard module. We will also prove [10, Conjecture 6.11] for any quiver.

There are many combinatorial motivations for studying Π_Q-modules. One reason is their connection to canonical bases for the quantum groups and for coordinate rings of flag varieties. For example, each Π_Q-module has an associated polytope, its *Harder–Narasimhan polytope*. The Harder–Narasimhan polytope of a *generic* Π_Q-module is an

[1]Since the original writing of this extended abstract, we have generalized Theorems 1.1, 1.2 and 1.3 from type $\widetilde{A}$ to all affine types. See [4].

MV polytope[2], and MV polytopes give a realization of Kashiwara crystals [8]. Spherical modules are all generic in this sense, so they give rise to MV polytopes. One can use Theorem 1.2 to study these MV polytopes: for example, to deduce vertices corresponding to spherical submodules.

Another combinatorial motivation for studying Π_Q-modules is to understand cluster algebras. Each cluster algebra (with a choice of initial seed) gives rise to a directed graph called the *ordered exchange graph*, which describes the mutations connecting different seeds [9]. For finite-type cluster algebras, these graphs are an orientation of the 1-skeleton of an associahedron of the appropriate type. The following theorem partially resolves a question of David Speyer from OPAC 2020 [15] in the case of type $\widetilde{A}$ cluster algebras.

Theorem 1.3. *Let Q be an orientation of the n-cycle, and let A_Q be the cluster algebra (with principal coefficients) associated to Q. Then there is a lattice congruence $\sim_Q$ of $\mathsf{Bic}(W)$ so that the ordered exchange graph of A_Q embeds as a subgraph of the Hasse diagram of $\mathsf{Bic}(W)/\sim_Q$.*

We deduce Theorem 1.3 essentially formally from Theorem 1.2 together with results from cluster-tilting theory. An interesting question is to describe these quotients combinatorially, which would give a new way of describing type $\widetilde{A}$ cluster algebras. In finite type, this question is answered via the *Cambrian lattices*, which are the source of a lot of interesting Catalan and cluster combinatorics. As a partial answer to the question, in recent work with Colin Defant [5] we describe the combinatorics of $\mathsf{Bic}(W)/\sim_Q$ in the case where Q is the oriented cycle. The resulting lattice quotient is the *affine Tamari lattice*; it can be identified with the subposet of $\mathsf{Bic}(W)$ consisting of 312-avoiding translation-invariant total orders, and has a description using translation-invariant binary trees.

We remark on one more motivation for studying Π_Q modules: the McKay correspondence associates to the cycle quiver an *ADE singularity*; in our case, the singularity is the zero locus X of the polynomial $x^2 + y^2 + z^n$ in $\mathbb{C}^3$. There is a minimal resolution of singularities $Y \to X$, and one version of the McKay correspondence asserts that Π_Q-modules are (derived) equivalent to coherent sheaves on the algebraic variety Y. The homological mirror symmetry conjecture predicts that these objects are further (derived) equivalent to certain Lagrangian submanifolds of a symplectic variety [1, Section 9.2]. Spherical modules correspond to the Lagrangian submanifolds which are diffeomorphic to 2-spheres. The arc diagrams we describe in Section 3 (to parametrize spherical modules) also arise in symplectic geometry as the image of Lagrangian spheres under a Lefschetz fibration. This particular case of homological mirror symmetry seems to be unproven, but we will use intuition motivated by this picture in our argument. It would be an interesting question to interpret our results on the symplectic geometry side of mirror symmetry.

[2]Since we are in the affine case, the right notion is an *affine MV polytope*, which has more data in addition to the polytope itself.

We now outline the structure of this extended abstract. In Section 2, we define the posets $\mathsf{Bic}(W)$ and $\mathsf{Tors}(\Pi_Q)$ and give relevant background. In Section 3, we construct the bijection $\mathsf{JIrr}(\mathsf{Bic}(W)) \to \mathsf{Sph}(\Pi_Q)$. Then in Section 4, we sketch the proof of Theorems 1.1 and 1.2 and in Section 5 we prove Theorem 1.3.

2 Background

The Coxeter group of type $\widetilde{A}_{n-1}$ is called the **affine symmetric group** $\widetilde{S}_n$. From now on, W will always denote this group.

2.1 Extended weak order of $\widetilde{S}_n$

Extended weak order was introduced by Matthew Dyer and has a uniform description for any Coxeter group. Rather than give this definition, we will focus on the case of $\widetilde{S}_n$, where there is an explicit combinatorial model introduced in [3, 7]. See also [2] for an introduction to this poset, which gives some geometric motivation for its definition.

Definition 2.1. A **translation-invariant total order** (TITO) is a total ordering $\prec$ of $\mathbb{Z}$ satisfying the following two properties:

(a) For any $i, j \in \mathbb{Z}$, we have $i \prec j$ if and only if $i + n \prec j + n$, and

(b) For any $i \in \mathbb{Z}$, if $i + n \prec i$, then there exists some $k \in \mathbb{Z}$ with $i + n \prec k \prec i$.

Just like permutations, TITOs have a one-line notation. This is given by writing the integers in $\prec$-order from left to right. Examples of TITOs for $n = 4$ include

$$\cdots \prec -2 \prec -3 \prec 0 \prec -1 \prec 2 \prec 1 \prec 4 \prec 3 \prec 6 \prec 5 \prec 8 \prec 7 \prec \cdots \tag{2.1}$$

$$\cdots \prec 0 \prec 1 \prec 4 \prec 5 \prec 8 \prec 9 \prec \cdots \prec 10 \prec 11 \prec 6 \prec 7 \prec 2 \prec 3 \prec \cdots \tag{2.2}$$

$$\cdots \prec 0 \prec 4 \prec 8 \prec 12 \prec \cdots \prec -2 \prec -3 \prec -1 \prec 2 \prec 1 \prec 3 \prec 6 \prec 5 \prec 7 \prec \cdots . \tag{2.3}$$

To abbreviate this data, we use **window notation** to encode a TITO. The key observation is that a TITO $\prec$ decomposes into **blocks**, which are subsets of $\mathbb{Z}$ on which $\prec$ restricts to an ordering which is order-isomorphic to $\mathbb{Z}$. For example, the TITO (2.2) has two blocks: the first has the integers congruent to 0 or 1 modulo 4, and the second contains the rest.

Each block has its own *window*: if a block contains k residue classes, then a window for it consists of k consecutive integers from the block. We need one more piece of data to fully encode the block: if i is in the block and $i + n \prec i$, then we underline the window, otherwise we do not underline it. (This is independent of the choice of i.) To get the window notation for the TITO, we list the windows for its blocks in order from left to right. For example, the window notations for (2.1), (2.2), and (2.3) are $[2,1,4,3]$, $[0,1][\underline{2,3}]$, and $[0][2,1,3]$, respectively.

We write $\widetilde{V}$ for the vector space with basis $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$. For each pair of integers $i < j$, we define $\alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$, where for an integer i we define α_i to be the basis vector $\alpha_{i'}$ for the unique $i' \in \{0, \ldots, n-1\}$ congruent to i modulo n. For instance, with $n = 3$ we have that $\alpha_{1,5} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \alpha_0 + 2\alpha_1 + \alpha_2$. The type $\widetilde{A}_{n-1}$ positive real root system is $\Phi^+_{\mathrm{re}} := \{\alpha_{ij} \mid i < j,\ i \not\equiv j \mod n\}$. There are also imaginary roots: $\Phi^+_{\mathrm{im}} := \{\alpha_{ij} \mid i < j,\ i \equiv j \mod n\}$. We write $\delta = \alpha_{0,n}$ for the primitive imaginary root.

Definition 2.2. Let $\prec$ be a TITO. An **inversion** of $\prec$ is a positive real root α_{ij} so that $j \prec i$. Thus

$$\mathrm{inv}(\prec) := \{\alpha_{ij} \mid i < j,\ i \not\equiv j \bmod n,\ j \prec i\}$$

is the **inversion set** of $\prec$. A **biclosed set**[3]for $\widetilde{S}_n$ is a subset of Φ^+_{re} of the form $\mathrm{inv}(\prec)$ for a TITO $\prec$.

We write $\mathsf{Bic}(\widetilde{S}_n)$ for the poset of biclosed sets under inclusion order. This poset is called the **extended weak order** of $\widetilde{S}_n$. It was shown to be a lattice in [6, 7]. The elements of $\widetilde{S}_n$ biject with TITOs having a single non-underlined window. The induced ordering on elements of $\widetilde{S}_n$ is the **weak order**. The map inv sending a TITO to a biclosed set is a bijection, so we could equivalently think of extended weak order as an order on TITOs. (This is the reason for including condition (b) in Definition 2.1; we could drop condition (b) at the price of making inv non-injective.)

Remark 2.3. One cannot define the inversion set to include the imaginary roots α_{ij} with $i \equiv j \bmod n$. The reason is that it is possible that there exist integers $i, j \in \mathbb{Z}$ so that $i \prec i + n$ and $j + n \prec j$. The definition of inversion would tell us that $\alpha_{j,j+n}$ is an inversion while $\alpha_{i,i+n}$ is not an inversion. But these two roots are both equal to δ, the primitive imaginary root. This issue does not arise for real roots. The TITO shown in equation (2.2) with window notation $[0,1][\underline{2,3}]$ gives an example of this phenomenon. There is an explanation for this in terms of brick modules (defined below): the purported "inversion" $\alpha_{2,2+n}$ and "non-inversion" $\alpha_{0,0+n}$ should correspond to *different* sets of bricks, even though they are the same vector. A large part of the proof of Theorem 1.2 is showing that this does not happen for real bricks.

2.2 The preprojective algebra Π_Q

Let Q be the oriented cycle quiver with n vertices, shown below. The **double quiver** $\overline{Q}$ of Q has the same vertices and, in addition to the arrows of Q, has a reversed arrow a^* for each arrow a of Q.

[3]Dyer defines biclosed sets for a general Coxeter group to be sets of positive roots subject to some convexity requirements. One result of [7] is that the biclosed sets for $\widetilde{S}_n$ are exactly the sets described here.

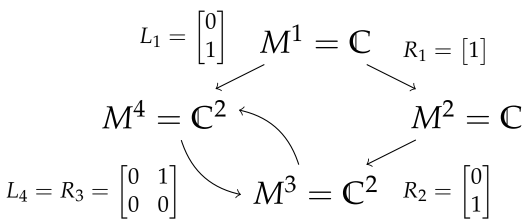


Figure 1: A module for Π_Q. The unlabeled linear maps are 0.

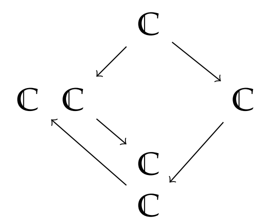


Figure 2: A string module, equivalent to the module in Figure 1.

The **path algebra** of $\overline{Q}$ is the algebra $\mathbb{C}[\overline{Q}]$ whose elements are $\mathbb{C}$-linear combinations of paths in $\overline{Q}$. Multiplication is given by concatenation of paths. The **preprojective algebra** Π_Q is the quotient algebra

$$\Pi_Q := \mathbb{C}[\overline{Q}]/(\sum_{a \text{ an arrow of } Q} aa^* - a^*a).$$

For more details see, e.g., [10]. We can describe modules for Π_Q as certain *quiver representations*. More precisely, the data of a $\mathbb{C}[\overline{Q}]$ module M is equivalently the data of for each $i \in \{0, \dots, n-1\}$: a vector space M^i, a linear map $M^i \xrightarrow{R_i} M^{i+1}$, and a linear map $M^{i-1} \xleftarrow{L_i} M^i$ (here we take the indices to be cyclic modulo n). These data define a module for Π_Q if and only if $R_i L_{i+1} - L_i R_{i-1} = 0$ for all $i \in \{0, \dots, n-1\}$. See Figure 1 for an example of a Π_Q-module.

In terms of this data, a map $\phi : M \to M'$ between Π_Q modules corresponds to a linear map $\phi_i : M^i \to (M')^i$ for each $i \in \{0, \dots, n-1\}$, such that $\phi_{i+1} R_i = R'_i \phi_i$ and $\phi_{i-1} L_i = L'_i \phi_i$. We write $\mathbf{mod}(\Pi_Q)$ for the category of finite-dimensional Π_Q-modules. (These are the modules so that $\sum_{i=0}^{n-1} \dim(M^i)$ is finite.) Unless otherwise specified, "module" refers to an object of $\mathbf{mod}(\Pi_Q)$.

Given a module M, we define its **dimension vector** to be the vector $\underline{\dim}(M) := \sum_{i=0}^{n-1} \dim(M_i)\alpha_i$ in $\widetilde{V}$.

Definition 2.4. A module M is called a **brick** if $\mathrm{Hom}_{\Pi_Q}(M, M) = \mathbb{C}$ (equivalently, if every nonzero map $M \to M$ is invertible). A brick M is called **real** if $\underline{\dim}(M)$ is in Φ^+_{re}, and **imaginary** if $\underline{\dim}(M)$ is in Φ^+_{im}. We write $\mathsf{Bricks}(\Pi_Q)$ for the set of isomorphism classes of bricks of Π_Q.

The following lemma is a strengthening of [10, Proposition 4.14] (using that

$\mathbf{mod}(\Pi_Q)$ is a 2-Calabi–Yau category, we are over $\mathbb{C}$, and a standard argument to deduce non-real bricks are imaginary). It implies that spherical modules and real bricks coincide.

Lemma 2.5. *Every brick is either real or imaginary. A module M is a real brick if and only if*

$$\mathrm{Ext}^i(M,M) = \begin{cases} \mathbb{C} & \text{if } i = 0 \text{ or } 2 \\ 0 & \text{otherwise} \end{cases}.$$

For each $i \in \{0,\dots,n-1\}$ there is a simple module S_i defined uniquely by $S_i^i = \mathbb{C}$ and $S_i^j = 0$ if $j \neq i$. The modules $S_0,\dots,S_{n-1}$ are exactly the simple spherical modules. We will be most interested in **string modules**. These are modules that can be depicted as in Figure 2. We interpret this picture in the following way: each copy of $\mathbb{C}$ over a vertex i corresponds to a basis vector of M^i. The map R_i sends a basis vector e in M^i to the basis vector of M^{i+1} that the clockwise-leaving arrow from e points to, or to 0 if there is no clockwise-leaving arrow from e. We define L_i similarly. Hence the modules depicted in Figures 1 and 2 are the same. Not every string module is a spherical module, but we will show below that every spherical module is a string module.

2.3 The lattice of torsion classes $\mathsf{Tors}(\Pi_Q)$

Definition 2.6. A **torsion class** in $\mathbf{mod}(\Pi_Q)$ is a collection of modules $\mathcal{T}$ satisfying the following properties:

- if M is in $\mathcal{T}$ and M' is isomorphic to M then M' is in $\mathcal{T}$, and
- if M is in $\mathcal{T}$ and Q is a quotient module of M then Q is in $\mathcal{T}$, and
- if A is a submodule of M and A and M/A are both in $\mathcal{T}$ then M is in $\mathcal{T}$.

We denote the poset of torsion classes under inclusion order by $\mathsf{Tors}(\Pi_Q)$.

The collection of torsion classes is closed under intersection, so any set of modules $\mathcal{S}$ is contained in a unique minimal torsion class $\overline{\mathcal{S}}$. In particular, $\mathsf{Tors}(\Pi_Q)$ is a complete lattice. Torsion classes are determined by the bricks that they contain, in the sense that if $\mathcal{T}$ is a torsion class, then $\mathcal{T} = \overline{\mathcal{T} \cap \mathsf{Bricks}(\Pi_Q)}$ [11]. Thus understanding bricks is the first step to understanding torsion classes, and this is what we do next.

3 The bijection $\mathsf{JIrr}(\mathsf{Bic}(W)) \to \mathsf{Sph}(\Pi_Q)$

An element J of a complete lattice is called a **complete join-irreducible** (JI) if there is an element J_* covered by J so that whenever $X < J$, also $X \leq J_*$. Equivalently, J is not the join of any set of lattice elements not containing J. We write $\mathsf{JIrr}(\mathsf{Bic}(W))$ for the set

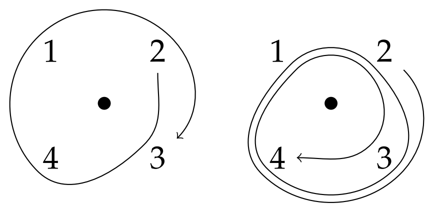


Figure 3: Two shard arcs. The arc on the left has TITO $[0,1][\underline{2,3}]$ and the arc on the right has TITO $[7,12,2,5]$.

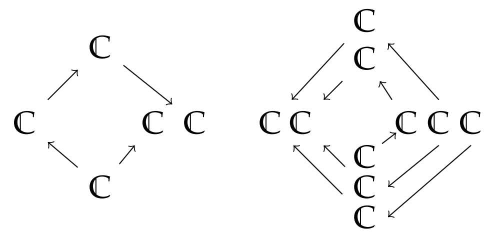


Figure 4: The string modules associated to the shard arcs in Figure 3.

of JIs of $\mathsf{Bic}(W)$. Given a biclosed set X, a **lower wall** of X is an element $\alpha \in X$ so that $X \setminus \{\alpha\}$ is also a biclosed set. If $X = \mathrm{inv}(\prec)$ for a TITO $\prec$ and $i < j$, then α_{ij} is a lower wall for X if and only if $j \prec i$ and j and i appear consecutively in the one-line notation of $\prec$. Each JI of $\mathsf{Bic}(W)$ has exactly one lower wall, but this is not enough to characterize them.

Lemma 3.1 ([2])**.** *A biclosed set X with exactly one lower wall is a JI if and only if it is of the form* $\mathrm{inv}(\prec)$*, where the window notation of $\prec$ does not contain two consecutive non-underlined windows.*

Both of the TITOs (2.2) and (2.3) have inversion sets with exactly one lower wall, but only (2.2) is a JI. In [2], the JIs of $\mathsf{Bic}(W)$ were described using arc diagrams. An arc diagram lives in $\mathbb{R}^2$ with the origin deleted. We label n points in the plane with $1, \ldots, n$ and call them *marked points*. We depict the marked points in a ring around the origin. An **arc** is a curve in the plane with its endpoints on marked points, and which otherwise does not pass through marked points or the origin. We consider arcs to be equivalent if they are connected by a homotopy fixing the endpoints which does not pass through marked points or the origin. We say the arc is **non-crossing** if it does not intersect itself. A **shard arc** is a non-crossing arc with distinct endpoints so that its angular coordinate is monotonic (i.e. it wraps around the origin without reversing direction). Two shard arcs are depicted in Figure 3.

We now summarize the main result of [2], which also appears in [3]. Given a JI $J = \mathrm{inv}(\prec)$, we build a shard arc $\sigma(J)$ as follows: let α_{ij} be the lower wall of J; without loss of generality assume $1 \leq i \leq n$. We imagine a bug driving a car in $\mathbb{R}^2$, starting at the marked point i. The bug drives clockwise around the origin, detouring around the marked points by turning either left or right. At the kth marked point it approaches, the bug turns left if and only if $i + k \prec i$. When the bug approaches the $(j-i)$th point, it drives straight to it and stops. The path driven is a shard arc denoted $\sigma(J)$.

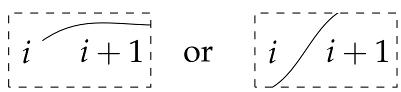
Theorem 3.2 ([2])**.** *The map $J \mapsto \sigma(J)$ is a bijection between JIs and shard arcs.*

We now introduce the map from $\mathsf{JIrr}(\mathsf{Bic}(W))$ to $\mathsf{Sph}(\Pi_Q)$. Let σ be a shard arc, and define a string module Π_σ as follows. Imagine our bug driving clockwise along the arc. Each time the bug is in a gap between i and $i+1$, it drops a copy of $\mathbb{C}$ into $(\Pi_\sigma)_i$. After the bug reaches the end of the arc, it then starts back at the beginning and drives around the arc another time. This time, when the bug turns to detour around i, it draws an arrow between the copy of $\mathbb{C}$ encountered previously and the copy of $\mathbb{C}$ it is about to encounter. The arrow points toward the previous $\mathbb{C}$ if the bug turns right, and the arrow points toward the next $\mathbb{C}$ if the bug turns left. Two examples of the resulting string modules are shown in Figure 4.

Lemma 3.3. *The map $J \mapsto \Pi_{\sigma(J)}$ is a bijection* $\mathsf{JIrr}(\mathsf{Bic}(W)) \xrightarrow{\sim} \mathsf{Sph}(\Pi_Q)$.

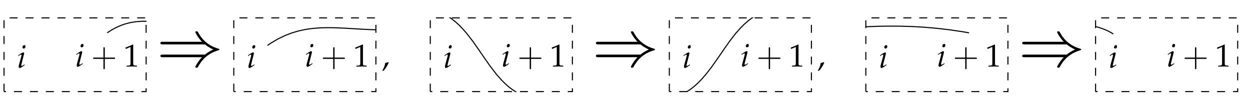
Proof. By Theorem 3.2, it is enough to verify that the map $\sigma \mapsto \Pi_\sigma$ is a bijection from shard arcs to $\mathsf{Sph}(\Pi_Q)$. First, we note that $\sigma \mapsto \Pi_\sigma$ is an injective map, since we can reverse the bug's algorithm to recover the shard arc σ from Π_σ. It remains to verify that Π_σ is in fact spherical, and that every spherical module arises in this way.

To prove these facts, we use *reflection functors*, as described in [10]. Write $\mathbf{NoSub}_i$ for the subcategory of modules that do not have S_i as a submodule, and $\mathbf{NoQuot}_i$ for the subcategory of modules that do not have S_i as a quotient. Reflection functors are inverse equivalences of categories $\Sigma_i : \mathbf{NoQuot}_i \rightleftarrows \mathbf{NoSub}_i : \Sigma_i^{-1}$. We omit their definition for brevity. A straightforward verification shows that Π_Σ has S_i as a quotient if and only if a subarc of Σ matches one of the following two patterns:



(∗)

A less straightforward computation shows that if σ is a shard arc, and Π_σ is in $\mathbf{NoQuot}_i$, then $\Sigma_i(\Pi_\sigma) = \Pi_{\sigma'}$ where σ' is another shard arc. More precisely, we can construct σ' by imagining putting a finger on each of the marked points i and $i+1$ in the plane and twisting 180° counter-clockwise. During this twist, we deform σ so that it avoids crossing any marked points. If one of i or $i+1$ is an endpoint of σ, then we also move that endpoint. After relabeling i and $i+1$ so they appear in their original order, the resulting arc is the new shard arc σ'. This process translates to several local moves (applied separately to each segment of σ which passes through the box):



By [10, Theorem 4.3], it follows that for any $\Pi_\sigma \in \mathbf{NoQuot}_i$ which is a real brick, we have that $\Pi_{\sigma'}$ is a real brick. Similar remarks apply to Σ_i^-, which acts by twisting 180° clockwise.

Given any shard arc σ, we can perform a 180° twist involving one of the endpoints of σ to make a shorter shard arc σ'. Without loss of generality, this twist is counter-clockwise. Then σ must avoid the patterns (∗) or the result of the twist would not be a shard arc. But then Π_σ is in $\mathbf{NoQuot}_i$, so it follows that $\Pi_\sigma = \Sigma_i^- \Sigma_i \Pi_\sigma = \Sigma_i^- \Pi_{\sigma'}$. By induction $\Pi_{\sigma'}$ is a real brick, hence so is Π_σ. This shows that the image of $\sigma \mapsto \Pi_\sigma$ consists of spherical modules.

Now let M be any real brick. Then $M = \Sigma_i M'$ or $M = \Sigma_i^- M'$ for some real brick $M' \in \mathbf{NoQuot}_i$ (resp. $M' \in \mathbf{NoSub}_i$) of smaller dimension [10, Theorem 5.1]. Without loss of generality, $M = \Sigma_i M'$. By induction M' is of the form Π_σ for some shard arc σ. Then $\Sigma_i \Pi_\sigma = \Pi_{\sigma'}$ for some shard arc σ'. We conclude that $\sigma \mapsto \Pi_\sigma$ is surjective. $\square$

Remark 3.4. Under homological mirror symmetry, the operation of twisting an arc 180° corresponds to applying a *Dehn twist* around a Lagrangian sphere.

4 Proof of Theorems 1 and 2

In this section we will sketch the proof of Theorems 1 and 2. We refer the reader to [3, 4] for more details. We will assume familiarity with the "Fundamental Theorem of Semidistributive Lattices" [14], and will freely use notation and terms from there. Our first goal is to show that there is a quotient of $\mathsf{Tors}(\Pi_Q)$ given by contracting exactly the imaginary bricks. This reduces to the following lemma, which implies that imaginary bricks do not directly force real bricks.

Lemma 4.1. *If M is a real brick and $D \subseteq M$ is a submodule which is an imaginary brick, then there is a brick $D' \not\cong D$ in the torsion class $\overline{\{D\}}$ so that $D' \subseteq M$. The dual statement also holds.*

We will also make use of the following result, which follows quickly from the fact that $\mathsf{Bic}(W)$ is an inverse limit of finite semidistributive lattices (in turn due to arguments in [6] and semidistributivity in rank 3).

Lemma 4.2. $\mathsf{Bic}(W)$ *is a completely semidistributive well-separated κ-lattice.*

Hence the full power of [14] applies to $\mathsf{Bic}(W)$. In particular, Lemma 4.1 implies there is a lattice quotient of $\mathsf{Tors}(\Pi_Q)$ contracting exactly the imaginary bricks. Then Theorem 1.2 identifies that quotient with $\mathsf{Bic}(W)$ using its factorization system; hence Theorem 1.2 implies Theorem 1.1.

For $J_1, J_2 \in \mathsf{JIrr}(\mathsf{Bic}(W))$, we write $J_1 \to J_2$ if $J_1 \not\leq \kappa(J_2)$. Then Theorem 1.2 reduces to showing that $J_1 \to J_2$ if and only if $\mathrm{Hom}_{\Pi_Q}(\Pi_{\sigma(J_1)}, \Pi_{\sigma(J_2)}) \neq 0$. To prove this, we use *folding*. Let $\widehat{W} = S_\infty$ be the group of permutations of $\mathbb{Z}$ fixing all but finitely many elements; this is a Coxeter group of type A_∞. Let $\widehat{\Pi}$ be its preprojective algebra. Each $J \in \mathsf{JIrr}(\mathsf{Bic}(W))$ has an unfolding $\widehat{J} \in \mathsf{JIrr}(\mathsf{Bic}(\widehat{W}))$, and each spherical module $\Pi_{\sigma(J)}$ has an unfolding $\widehat{\Pi}_{\sigma(J)}$, which is a real brick for $\widehat{\Pi}$. There is a "translation by n" operator Tr_n on both $\mathsf{Bic}(\widehat{W})$ and the $\widehat{\Pi}$-modules. Unfolding has the property that $J_1 \to J_2$ if and only if $\widehat{J}_1 \to \mathsf{Tr}_n^k \widehat{J}_2$ for some k, and $\mathrm{Hom}_{\Pi_Q}(\Pi_{\sigma(J_1)}, \Pi_{\sigma(J_2)}) \neq 0$ if and only if $\mathrm{Hom}_{\widehat{\Pi}}(\widehat{\Pi}_{\sigma(J_1)}, \mathsf{Tr}_n^k \widehat{\Pi}_{\sigma(J_2)}) \neq 0$ for some k. Furthermore, $\widehat{J}_1 \to \mathsf{Tr}_n^k \widehat{J}_2$ if and only if $\mathrm{Hom}_{\widehat{\Pi}}(\widehat{\Pi}_{\sigma(J_1)}, \mathsf{Tr}_n^k \widehat{\Pi}_{\sigma(J_2)}) \neq 0$, by Theorem 1.2 in type A (a consequence of [13]). It follows that $J_1 \to J_2$ if and only if $\mathrm{Hom}_{\Pi_Q}(\Pi_{\sigma(J_1)}, \Pi_{\sigma(J_2)}) \neq 0$, completing the proof.

As a corollary, we deduce the following, which resolves [10, Conjecture 6.11][4].

Lemma 4.3. *If $M_1, M_2 \in \mathsf{Sph}(\Pi_Q)$ and $\underline{\dim}(M_1) = \underline{\dim}(M_2)$, then $\mathrm{Hom}(M_1, M_2) \neq 0$.*

5 Proof of Theorem 3

In this section, we assume the reader is familiar with cluster algebras and their additive categorifications. Let Q be a quiver which is an orientation of the cycle graph and A_Q the principal coefficients cluster algebra associated to Q. We wish to show that there is a lattice quotient L of $\mathsf{Bic}(W)$ so that the Hasse diagram of L contains the ordered exchange graph of A_Q. The key is that there is a known lattice quotient L of $\mathsf{Tors}(\Pi_Q)$ which has the property we desire. If Q is acyclic, then we set $\Lambda := \mathbb{C}[Q]$ to be the path algebra of Q. If Q is the oriented cycle, then we set Λ to be the quotient of $\mathbb{C}[Q]$ which is a cluster-tilted algebra of type D. This is the quotient by the ideal generated by paths of length $n-1$. In both cases, Λ is a quotient algebra of Π_Q, so there is a quotient map $\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathsf{Tors}(\Lambda)$ [11]. When Λ is a path algebra of an acyclic quiver or a cluster-tilted algebra, then it is known (see, e.g., [9, Section 4.5]) that the Hasse diagram of $\mathsf{Tors}(\Lambda)$ contains the ordered exchange graph of A_Q in the connected component of the bottom element.

Our goal is now to prove that there is a common quotient of $\mathsf{Tors}(\Lambda)$ and $\mathsf{Bic}(W)$. We use the theory of quotient lattices from [14], and define L to be the quotient of $\mathsf{Tors}(\Lambda)$ given by contracting all imaginary bricks of Λ. By Lemma 4.1 and [14, Theorem 6.3], no other bricks are contracted in this quotient. Then $\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathsf{L}$ factors through the quotient $\mathsf{Tors}(\Pi_Q) \twoheadrightarrow \mathsf{Bic}(W)$, since all bricks contracted in the second quotient are also contracted in the first.

We now wish to show that the quotient $\mathsf{Tors}(\Lambda) \twoheadrightarrow \mathsf{L}$ maps the ordered exchange graph isomorphically onto its image. Since each edge in the exchange graph has a brick label (cf. [14, Remark 3.12]), the only way this could fail is if the brick label of some edge of the exchange graph is contracted in the quotient. In other words, we wish to show that the bricks labeling the edges of the exchange graph are all real bricks. Now, if M is a brick labeling an edge of the exchange graph, then $\underline{\dim}(M)$ is a c-vector of the cluster algebra [9, $\mathsf{f\text{-}tors} \to \mathsf{int\text{-}t\text{-}str} \to \mathsf{2\text{-}smc} \to \mathsf{c\text{-}mat}$]. It is known that the c-vectors of acyclic cluster algebras [9, Theorem 3.23] and of the oriented cycle cluster algebra (e.g., [5]) are real roots. Hence all brick labels must be real, and $\mathsf{Tors}(\Lambda) \twoheadrightarrow \mathsf{L}$ restricts to an isomorphism on the ordered exchange graph.

[4] We give a different proof of this conjecture (for all types) in [4].

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