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On z-superstable and critical configurations of chip-firing pairs

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Abstract. It is well known that there is a duality map between the superstable configurations and the critical configurations of a graph. This was extended to all M-matrices in (Guzmán and Klivans, 2015). We show a natural way to extend this to all (L, M)-chip firing pairs introduced in (Guzmán and Klivans, 2016). In addition, we study various properties of z-superstable configurations and critical configurations of (L, M)-chip firing pairs.

Résumé. Il est bien connu qu'il existe une dualité entre les configurations superstables et les configurations critiques d'un graphe. Dans (Guzmán et Klivans, 2015), ce résultat a été étendu à toutes les M-matrices. Nous démontrons une façon naturelle d'étendre cette dualité à toutes les paires (L, M) qui sont « chip-firing », introduites dans (Guzmán et Klivans, 2016). De plus, nous étudions diverses propriétés des configurations z-superstables et des configurations critiques des paires « chip-firing ».

Keywords: Chip-firing, Critical group, Avalanche finite, M-matrices, Chip-firing pairs

1 Introduction

Chip-firing is a game that takes place on a connected graph G, where chips are distributed across the vertices of G and moved to adjacent vertices based on a straightforward rule. This dynamical system has a profound theory that links to various fields in mathematics and physics [1, 9, 10, 14]. For further details, please see the recent textbooks [15] and [8].

Consider a simple graph G with $n + 1$ vertices, where one vertex is designated as the *sink*, and chips are placed on each non-sink vertex. The allocation of chips is described by an integer vector $\vec{c} \in \mathbb{Z}^n$, known as a *(chip) configuration*. A non-sink vertex with chips equal to or greater than its degree can fire, distributing one chip along each incident edge to adjacent vertices. A configuration $\vec{c}$ is termed *stable* if no non-sink vertex is able to fire.

For a connected graph G, any starting configuration will eventually reach stability via sequence of firings, with some chips moving to the sink vertex. Letting $v_1, ..., v_n$ represent the non-sink vertices of G, the chip-firing rules can be described using L_G, the $n \times n$ *reduced Laplacian* matrix of G.

The outcome of firing a vertex v_i on a chip configuration $\vec{c}$ is represented by $\vec{c} - L\vec{e_i}$, where $\vec{e_i}$ denotes the i-th standard basis vector. The matrix L_G establishes an equivalence relation among the vectors in $\mathbb{Z}^n$, with $\vec{c}$ and $\vec{d}$ being *firing equivalent* if $\vec{c} - \vec{d}$ lies within the image of L_G. This determines the *critical group* of G, as described by $\mathcal{K}(G) := \mathbb{Z}^n \,/\, \operatorname{Im} L_G$ [3].

A configuration $\vec{c}$ is considered *valid* (or *effective*) if for all $i = 1, \ldots, n$, the condition $c_i \geq 0$ holds. The goal is to identify notable valid configurations within each equivalence class $[\vec{c}] \in \mathcal{K}(G)$. It turns out that each $[\vec{c}]$ contains a distinct valid configuration that is *critical*, which means it is stable and can be derived by iteratively firing from a sufficiently large configuration $\vec{b}$. Stabilizing the sum of two critical configurations results in a critical configuration.

One can show that each $[\vec{c}]$ contains a unique valid configuration that is *superstable*, which means that it is stable under *set-firings*, the simultaneous firing of a set of vertices. A superstable configuration represents a solution to an energy minimization problem and aligns with the concept of a *G-parking function*.

For a connected graph G, there is a straightforward bijection linking the set of critical configurations and the set of superstable configurations, which are both correspondingly in bijection with the set of spanning trees of G. This simple map (taking a configuration, negating it coordinate-wise from a certain maximal configuration) is what is called the *duality map* between superstable configurations and critical configurations, and our focus is to extend this map to more general models.

Recently, chip-firing has been extended to more general settings, where the reduced Laplacian of a graph is replaced by other matrices (see, for instance, [11, 12]). We use a matrix M to define a firing rule that mimics the graphical setting: firing v_i now takes a configuration $\vec{c}$ to $\vec{c} - M\vec{e_i}$, where $\vec{e_i}$ stands for the unit vector with 1 at the i-th coordinate. For a well-defined notion of chip-firing we require that M satisfies an *avalanche finite* property, so that repeated firings of any initial configuration eventually stabilize in an appropriate sense. The class of matrices with this property are known as *M-matrices*, and can be characterized in a number of ways (see Definition 2.1 below). In [12], Guzmán and Klivans have shown that the chip-firing theory defined by an M-matrix leads to good notions of critical and superstable configurations. They further generalized this model by introducing an invertible matrix L, calling (L, M) a *chip-firing pair*, and extending the definition of critical and superstable configurations to that model [13].

Our goal is to extend the duality between critical and superstable configurations to (L, M)-chip firing pairs. This will answer Question 5.2 of [6] in a much more general sense, since signed graphs are special cases of the chip-firing pair model. In Section 2 we summarize the Guzmán–Klivans theory of chip-firing pairs and also review the previously known duality between superstable and critical configurations of M-matrices. In Section 3 we provide our main result on extending the duality map to chip-firing pairs. In Section 4 we study some properties of the map discussed in the main result.

2 Prerequisites

In this section we review the definition of chip-firing pairs. After that we review the duality map between superstable configurations and critical configurations for M-matrices. Then we will go over the tools developed in [6] that we will use to deal with the z-superstable and critical configurations of chip-firing pairs.

2.1 Chip firing pairs

In [12], Guzmán and Klivans generalized the chip-firing on graphs to *M-matrices*. M-matrices are used in various fields such as economics or scientific computing [4, 7, 16, 18]. Guzmán and Klivans further generalized this by introducing an invertible integer matrix L in *chip-firing pairs* (L, M) introduced in [13].

Definition 2.1. *Suppose M is an $n \times n$ matrix such that $(M)_{ii} > 0$ for all i and $(M)_{ij} \leq 0$ for all $i \neq j$. Then M is called an (invertible)* M-matrix *if any of the following equivalent conditions hold:*

- *M is avalanche finite;*
- *The real part of the eigenvalues of M are positive;*
- *The entries of M^{-1} are non-negative;*
- *There exists a vector $\vec{x} \in \mathbb{R}^n$ with $\vec{x} \geq \vec{0}$ such that $M\vec{x}$ has all positive entries.*

The pair (L, M), an M-matrix M together with an invertible integer matrix L, is called a *chip-firing pair*. The relevant (chip) *configurations* $\vec{c} \in \mathbb{Z}^n$ are simply integer vectors with n entries, and chip-firing is dictated by the matrix L. In particular, (M, M) recovers the chip-firing on M-matrices and (L_G, L_G) when L_G is the (reduced) Laplacian of a graph recovers the classical chip-firing model on graphs.

Remark 2.2. *In this paper, we only focus on integral M-matrices (since the duality map for M-matrices given in [12] that we extend upon, is only given for integral matrices).*

Definition 2.3. *Suppose (L, M) is a chip-firing pair. A configuration $\vec{c}$ is* valid *if $\vec{c} \in S^+$, where*

$$S^+ = \{LM^{-1}\vec{x} : LM^{-1}\vec{x} \in \mathbb{Z}^n, \vec{x} \in \mathbb{R}^n_{\geq 0}\}.$$

Equivalently, a configuration $\vec{c}$ is valid if $ML^{-1}\vec{c} \in R^+$, where

$$R^+ = \{\vec{x} \in \mathbb{R}^n_{\geq 0} : LM^{-1}\vec{x} \in \mathbb{Z}^n\}.$$

In particular, for (M, M), being valid is the same as being a nonnegative integer vector.

Definition 2.4. *Suppose (L, M) is a chip-firing pair, and suppose that $\vec{c} \in S^+$ is a valid configuration. A site $i \in \{1, \ldots, n\}$ is* ready to fire *if*

$$\vec{c} - L\vec{e}_i \in S^+,$$

so that the vector obtained by subtracting the ith row of L from $\vec{c}$ is also valid.

Similarly, suppose $\vec{x} \in R^+$. Then a site $i \in \{1, \ldots, n\}$ is ready to fire *if*

$$\vec{x} - M\vec{e}_i \in R^+.$$

A configuration $\vec{c}$ (in S^+ or R^+) is stable *if no site is ready to fire.*

If i is ready to fire, we declare that $\vec{b} = \vec{c} - L\vec{e}_i \in S^+$ is derived from $\vec{c}$ through a *legal firing*. Repeating this process, a vector $\vec{a} \in S^+$ is said to be derived from $\vec{c}$ through a *sequence of legal firings*. For a configuration $\vec{c} \in S^+$ (or conversely, $\vec{d} \in R^+$), we define $\text{stab}_{S^+}(\vec{c})$ (and $\text{stab}_{R^+}(\vec{d})$) as the resulting configuration after executing a series of legal firings until no site remains eligible to fire. Adapting the proof presented in [15, Theorem 2.2.2], it can be established that both $\text{stab}_{S^+}(\vec{c})$ and $\text{stab}_{R^+}(\vec{d})$ are uniquely determined. When it is clear whether we are dealing with S^+ or R^+, we use $\text{stab}(\vec{x})$ to refer to the stabilization of the configuration $\vec{x}$.

The definition of critical and superstable configurations in this model are as follows.

Definition 2.5. *Given an (L, M) pair, a configuration $\vec{c} \in S^+$ is reachable if there exists some configuration $\vec{d} \in S^+$ satisfying:*

- $\vec{d} - L\vec{e}_i \in S^+$ *for all* $1 \leq i \leq n$
- $\vec{c} = \vec{d} - \sum_{j=1}^{k} L\vec{e}_j$ *and* $\vec{d} - \sum_{j=1}^{\ell} L\vec{e}_j \in S^+$ *for all* $\ell < k$.

Given an (L, M) pair, a configuration $\vec{c} \in S^+$ is critical if $\vec{c}$ is both stable and reachable.

Critical configurations are those that are both stable and reachable by chip-firing from a sufficiently large configuration. They are useful because they index the equivalence classes of the critical group.

Definition 2.6 ([12, Definition 4.3])**.** *A vector $f \in \mathbb{Z}^n$ with $f \geq 0$ is z-*superstable *if for every $z \in \mathbb{Z}^n$ with $z \geq 0$ and $z \neq 0$ there exists $1 \leq i \leq n$ such that $f_i - (Lz)_i < 0$.*

It turns out that in the equivalence class given by the matrix L in S^+, we can always find a unique representative that is critical and a unique representative that is z-superstable.

Theorem 2.7 ([13, Theorems 3.5, 4.3, 5.5])**.** *Suppose (L, M) is a chip-firing pair. Then there exists exactly one z-superstable configuration and one critical configuration in each equivalence class $[\vec{c}]_L$.*

Remark 2.8. *For chip-firing pairs, there is the notion of a χ-superstable configuration as well as a z-superstable configuration. From Theorem 2.7, it is the z-superstable configurations that have the same size as the critical configurations. For the remainder of the paper, we will focus only on z-superstable configurations, and we will call them the superstable configurations of the chip-firing pair, omitting the letter z.*

In the next subsection, we go over the duality that is known to exist when $L = M$.

2.2 Duality for M-matrices

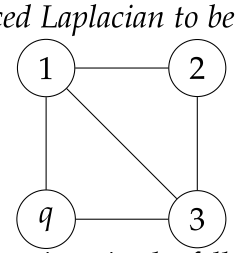
If we take a chip-firing pair (M, M), it recovers the chip-firing on M-matrices studied in [12]. Chip-firing on M-matrices generalizes many properties and results of the classical chip-firing on graphs, and one of them is the duality between superstable and critical configurations.

Given an M-matrix, it turns out that there is a critical configuration that is a coordinate-wise greater or equal to every other critical configuration. We call this configuration $\vec{c}_{\max}$, given by taking all diagonal entries of M minus one and forming a vector (in the classical case, this corresponds to having $\deg(v) - 1$ chips for each vertex v).

Theorem 2.9 ([12])**.** *Let M be an M-matrix. Let $\vec{c}_{\max}$ denote the vector where each entry is coming from the corresponding diagonal entry M_{ii} minus one. Then we have a bijection between superstable and critical configurations by the map $\vec{c} \to \vec{c}_{\max} - \vec{c}$.*

Example 2.10. *Consider the following graph that has the reduced Laplacian to be*

$$L_G = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} :$$



The superstable configurations and critical configurations are given in the following table:

Superstables	*Criticals*	*Superstables*	*Criticals*
(0,0,0)	(2,1,2)	(0,1,1)	(2,0,1)
(0,0,1)	(2,1,1)	(1,0,0)	(1,1,2)
(0,0,2)	(2,1,0)	(1,1,0)	(1,0,2)
(0,1,0)	(2,0,2)	(2,0,0)	(0,1,2)

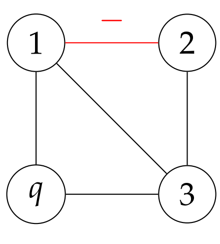
Notice that in the above example, we have a bijection between superstable configurations and critical configurations via the map $\vec{c} \to (2,1,2) - \vec{c}$.

Recall that our goal is to extend this to (L, M) chip-firing pairs. Next subsection will show that the map $\vec{c} \to \vec{c}_{\max} - \vec{c}$ does not work for chip-firing pairs.

2.3 The usual duality map does not work for chip-firing pairs

Recall that the usual duality map between the superstable and critical configurations for graphs (and also M-matrices) is given by the map $\vec{c} \to \vec{c}_{\max} - \vec{c}$ for some fixed $\vec{c}_{\max}$. As can be seen in the example below, this does not work for (L, M)-pairs in general. The examples from this point throughout will be using (L, M)-pairs coming from a signed graph. The systematic study of signed graphs and their Laplacian was initiated by Zaslavsky in [19] and also studied in [5, 17, 2].

Example 2.11. *We take L to be the (reduced) Laplacian of the following signed graph and M to be the (reduced) Laplacian of the underlying unsigned graph. The Laplacian of the signed graph is simply obtained from the Laplacian of the underlying graph, by changing the signs of entries corresponding to negative edges.*



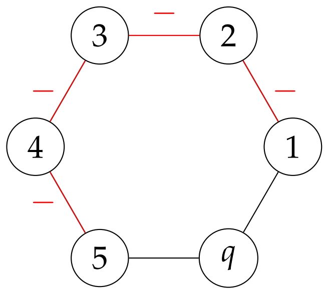
$$M = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix} \quad L = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

Configurations in S^+

Superstables	*Criticals*	*Superstables*	*Criticals*	*Superstables*	*Criticals*
$(0,0,0)$	$(6,4,2)$	$(2,2,0)$	$(8,6,2)$	$(5,4,0)$	$(8,6,1)$
$(1,1,0)$	$(7,5,2)$	$(3,3,0)$	$(9,7,2)$	$(6,5,0)$	$(9,7,1)$
$(4,3,2)$	$(8,6,0)$	$(3,2,0)$	$(6,4,1)$	$(6,4,0)$	$(6,5,2)$
$(5,4,2)$	$(9,7,0)$	$(4,3,0)$	$(7,5,1)$	$(7,5,0)$	$(7,6,2)$

Notice that in the table of superstable and critical configurations of the chip-firing pair, the coordinate-wise maximal critical configuration is $(9,7,2)$. However if we take the superstable configuration $(5,4,2)$, the vector we get by applying the traditional duality map $(9,7,2) - (5,4,2) = (4,3,0)$ is not a critical configuration.

Even worse, there are many cases where c_{max}, the critical configuration that has coordinate-wise maximal entries does not even exist.



Critical configurations
(9, 15, 17, 15, 9)
(12, 20, 23, 21, 13)
(13, 21, 23, 20, 12)
(7, 11, 12, 11, 7)
(10, 16, 18, 17, 11)
(11, 17, 18, 16, 10)

Example 2.12. *Consider the (L, M)-pair coming from the signed graph below. The underlying graph is C_6, the cycle on six vertices.*

As can be checked from the table of critical configurations above, there is no critical configuration that is the maximal in all coordinates.

2.4 Finding the superstable/critical configurations of chip firing pairs.

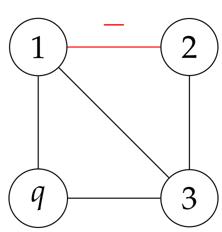
In this subsection, we discuss an alternative way to find the superstable and critical configurations of (L, M) chip-firing pairs, developed in [6].

Let $\text{sstab}(M)$ denote the set of superstable configurations of a M-matrix M and let $\text{crit}(M)$ denote the set of critical configurations. However, beware that we are not going to be using $\text{sstab}(L, M)$ to denote the set of superstable configurations of (L, M) and the same for $\text{crit}(L, M)$. It turns out that for configurations in S^+, it is important to look at their preimages in R^+. Given any vector $\vec{f}$, we use $\left\lfloor \vec{f} \right\rfloor$ to denote the vector obtained from f by taking the floor at every coordinate.

Theorem 2.13 ([6, Theorem 3.2])**.** *Given an (L, M) pair, a configuration $\vec{c} \in S^+$ is superstable/critical if and only if $\lfloor ML^{-1}\vec{c} \rfloor$ is a superstable/critical configuration of M.*

For example, we look at our running example coming from a signed graph.

Example 2.14. *Consider the signed graph studied in Example 2.11. The table lists all superstable and critical configurations in S^+, their preimage in R^+, and the floor of the preimage. We can notice that the floor of the preimages are the superstable and critical configurations of the underlying graph we saw in Example 2.10 (however, not all superstable/critical configurations of the underlying graph are used).*



LM^{-1}(sstab)	sstab	$\lfloor$sstab$\rfloor$	LM^{-1}(crit)	crit	$\lfloor$crit$\rfloor$
$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(6,4,2)$	$(2,0,2)$	$(2,0,2)$
$(1,1,0)$	$(0,1/2,0)$	$(0,0,0)$	$(7,5,2)$	$(2,1/2,2)$	$(2,0,2)$
$(4,3,2)$	$(2/3,1/3,2)$	$(0,0,2)$	$(8,6,0)$	$(8/3,4/3,0)$	$(2,1,0)$
$(5,4,2)$	$(2/3,5/6,2)$	$(0,0,2)$	$(9,7,0)$	$(8/3,11/6,0)$	$(2,1,0)$
$(2,2,0)$	$(0,1,0)$	$(0,1,0)$	$(8,6,2)$	$(2,1,2)$	$(2,1,2)$
$(3,3,0)$	$(0,3/2,0)$	$(0,1,0)$	$(9,7,2)$	$(2,3/2,2)$	$(2,1,2)$
$(3,2,0)$	$(4/3,1/6,0)$	$(1,0,0)$	$(6,4,1)$	$(7/3,1/6,1)$	$(2,0,1)$
$(4,3,0)$	$(4/3,2/3,0)$	$(1,0,0)$	$(7,5,1)$	$(7/3,2/3,1)$	$(2,0,1)$
$(5,4,0)$	$(4/3,7/6,0)$	$(1,1,0)$	$(8,6,1)$	$(7/3,7/6,1)$	$(2,1,1)$
$(6,5,0)$	$(4/3,5/3,0)$	$(1,1,0)$	$(9,7,1)$	$(7/3,5/3,1)$	$(2,1,1)$
$(6,4,0)$	$(8/3,1/3,0)$	$(2,0,0)$	$(6,5,2)$	$(2/3,4/3,2)$	$(0,1,2)$
$(7,5,0)$	$(8/3,5/6,0)$	$(2,0,0)$	$(7,6,2)$	$(2/3,11/6,2)$	$(0,1,2)$

Thanks to Theorem 2.13, it is much more convenient to deal with the preimages of configurations, especially when trying to check if it is superstable or critical. Given a superstable configuration in S^+, we will denote its preimage in R^+ as *superstable preimage* and for a critical configuration in S^+, we are going to denote its preimage in R^+ as *critical preimage*. We are also going to use sstab(L, M) to denote the set of superstable preimages of a (L, M) chip-firing pair and use crit(L, M) to denote the set of critical preimages.

3 The Central Duality

In this section we establish the duality between the superstable configurations and critical configurations of (L, M)-pairs, that extends the canonical duality between superstable and critical configurations of M-matrices. We are mainly going to be dealing with the pre-images of the configurations in R^+.

3.1 The involution μ and the duality map.

We define an involution on the set of superstable configurations of M. Given any vector $\vec{f}$, recall that we used $\lfloor \vec{f} \rfloor$ to denote the vector obtained from f by taking the floor at every coordinate. We are going to let $\{\vec{f}\}$ denote $f - \lfloor \vec{f} \rfloor$.

Definition 3.1. *For any chip-firing pair (L, M), we define the map μ : sstab$(M) \to$ sstab(M):*

$$\mu(\vec{s}) = \begin{cases} \vec{s} & \text{if } \{LM^{-1}2\vec{s}\} = \{LM^{-1}\vec{c}_{\max}\}, \\ \text{sstab}(\vec{c}_{\max} - \vec{s}) & \text{otherwise.} \end{cases}$$

It is clear from definition that the above map is indeed an involution:

Proposition 3.2. *The map μ is an involution.*

In the special case where $L = M$ (this recovers the usual chip-firing on M-matrices, and in particular when M is the Laplacian of a graph, the classical chip-firing) the above involution is simply the identity map: since all superstable and critical preimages are integer vectors.

Lemma 3.3. *Let $\vec{a}, \vec{b}$ be integer vectors such that they are equivalent under M. Let $\vec{f}$ be a vector such that every entry f_i satisfies $0 \leq f_i < 1$. Then $\vec{a} + \vec{f} \in R^+$ if and only if $\vec{b} + \vec{f} \in R^+$.*

Example 3.4. *We use the same (L, M) pair coming from a signed graph studied in previous examples. From the configuration $\vec{a} = (1, 2, 0)$, we can fire vertex 2 to obtain the configuration $\vec{b} = (2, 0, 1)$, so these configurations are firing-equivalent under M. Take some non-negative rational vectors where the entries are bounded above by 1, say $\vec{f}_1 = (\frac{1}{3}, \frac{2}{3}, 0)$ and $\vec{f}_2 = (\frac{2}{3}, \frac{1}{3}, 0)$.*

Preimage	*Image under LM^{-1}*
$\vec{a} + \vec{f}_1 = (\frac{4}{3}, \frac{8}{3}, 0)$	$(8, 7, 0)$
$\vec{b} + \vec{f}_1 = (\frac{7}{3}, \frac{2}{3}, 1)$	$(7, 5, 1)$
$\vec{a} + \vec{f}_2 = (\frac{5}{3}, \frac{7}{3}, 0)$	$(8, \frac{27}{4}, 0)$
$\vec{b} + \vec{f}_2 = (\frac{8}{3}, \frac{1}{3}, 1)$	$(7, \frac{19}{4}, 1)$

Notice from the table that we have $\vec{a} + \vec{f}_1$ and $\vec{b} + \vec{f}_1$ are both in R^+ whereas $\vec{a} + \vec{f}_2$ and $\vec{b} + \vec{f}_2$ are both not in R^+, which is consistent with Lemma 3.3.

What the above lemma suggests is that we should group up the superstable/critical preimages having the same floor (which we call a *bucket*), and try to map the superstable configurations of M to critical configurations of M where the bucket structure is the same.

Theorem 3.5. *Let (L, M) be any chip-firing pair. The map $\chi : \mathrm{sstab}(L, M) \to \mathrm{crit}(L, M)$ given by $\vec{s} \mapsto \vec{c}_{\max} - \mu(\lfloor \vec{s} \rfloor) + \{\vec{s}\}$ is a bijection.*

Example 3.6. *Consider the signed graph from Example 2.14. Start with the superstable configuration $(5, 4, 0) \in S^+$ of the chip-firing pair. Then the corresponding superstable preimage is $\vec{s} = (\frac{4}{3}, \frac{7}{6}, 0)$.*

We can check that $\{LM^{-1}2\lfloor \vec{s} \rfloor\} = (0, \frac{1}{2}, 0) \neq (0, 0, 0) = \{LM^{-1}\vec{c}_{\max}\}$, so $\mu(\lfloor \vec{s} \rfloor) =$ $\mathrm{sstab}(\vec{c}_{\max} - \lfloor \vec{s} \rfloor)$. For this graph, we have $\vec{c}_{\max} = (2, 1, 2)$, so $\mu(\lfloor \vec{s} \rfloor) = (0, 0, 1)$ as $M^{-1}(\vec{c}_{\max} - \lfloor \vec{s} \rfloor - (0, 0, 1)) \in \mathbb{Z}$.

Then

$$\chi(\vec{s}) = \vec{c}_{\max} - \mu(\lfloor \vec{s} \rfloor) + \{\vec{s}\} = (2, 1, 2) - (0, 0, 1) + \left(\frac{1}{3}, \frac{1}{6}, 0\right) = \left(\frac{7}{3}, \frac{7}{6}, 1\right)$$

gives a critical preimage in R^+. Its image in S^+ is $(8,6,1)$, so we have mapped the superstable configuration $(5,4,0)$ to the critical configuration $(8,6,1)$.

It can be verified that χ recovers all of the critical preimages (and hence the critical configurations) in the table from Example 2.14 starting from all superstable preimages.

Remark 3.7. *The duality χ recovers the classical duality for chip-firing on graphs when L is the (reduced) Laplacian of a signed graph and M is the (reduced) Laplacian of the underlying unsigned graph. It also recovers the duality for M-matrices when $L = M$.*

Example 3.8. *Take the M-matrix coming from the unsigned graph in Example 2.10. Now let us study the (M, M) chip-firing pair. We have that $(0,0,1)$ and $(1,1,0)$ are superstable configurations of M. Since the preimages of all superstable and critical configurations are obviously integral (MM^{-1} is the identity matrix), they are fixed points in our involution μ.*

Then

$$\chi(0,0,1) = \vec{c}_{\max} - \mu(\lfloor(0,0,1)\rfloor) + \{(0,0,1)\} = (2,1,2) - (0,0,1) + (0,0,0) = (2,1,1)$$

$$\chi(1,1,0) = \vec{c}_{\max} - \mu(\lfloor(1,1,0)\rfloor) + \{(1,1,0)\} = (2,1,2) - (1,1,0) + (0,0,0) = (1,0,2)$$

This aligns with the usual duality between superstable and critical configurations on graphs.

3.2 Counting the number of fixed points of μ.

Let F_0^M stand for the set of superstable preimages of an (L, M) chip-firing pair such that the all entires are integers. Then this turns out to be a subgroup of the critical group of M. Then from this we can compute the number of fixed points of μ:

Theorem 3.9. *The number of fixed points of the involution map μ is equal to either 0 or $|F_0^M|d$, where d is the number of elements of $\mathcal{K}(M)/F_0^M$ with order at most 2.*

Example 3.10. *Let's revisit the running example of a (L, M)-pair coming from a signed graph.*

sstab(M)	*Its image (under LM^{-1})*	*Is it a fixed point of μ?*
$(0,0,0)$	$(0,0,0)$	*Yes*
$(0,0,1)$	$(1,\frac{3}{4},1)$	*No*
$(0,0,2)$	$(2,\frac{3}{2},2)$	*Yes*
$(0,1,0)$	$(2,2,0)$	*Yes*
$(0,1,1)$	$(3,\frac{11}{4},1)$	*No*
$(1,0,0)$	$(2,\frac{5}{4},1)$	*No*
$(1,1,0)$	$(4,\frac{13}{4},1)$	*No*
$(2,0,0)$	$(4,\frac{5}{2},2)$	*Yes*
$\vec{c}_{\max} = (2,1,2)$	$(8,6,4)$	

We have $\mathcal{K}(M) = \mathbb{Z}_8$ and $|F_0^M| = 2$. Therefore, $\mathcal{K}(M)/F_0^M \cong \mathbb{Z}_4$, which has 2 elements of order at most 2. Using Theorem 3.9 with $d = 2$ and $|F_0^M| = 2$, we get 4 fixed points in μ.

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