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The Ehrhart h*-polynomials of positroid polytopes

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Abstract. A positroid is a matroid realized by a matrix such that all maximal minors are non-negative. Positroid polytopes are matroid polytopes of positroids. In particular, they are lattice polytopes. The Ehrhart polynomial of a lattice polytope counts the number of integer points in the dilation of that polytope. The Ehrhart series is the generating function of the Ehrhart polynomial, which is a rational function with the numerator called the h^*-polynomial. We compute the h^*-polynomial of an arbitrary positroid polytope and an arbitrary half-open positroid polytope. Our result generalizes that of Katzman, Early, Kim, and Li for hypersimplices.

Keywords: positroids, Ehrhart theory

1 Introduction

A positroid is a matroid on an ordered set realized by a matrix such that all of its maximal minors are non-negative. Postnikov [22] showed that positroids are in bijection with several interesting classes of combinatorial objects, including Grassmann necklaces, decorated permutations, Ⅎ-diagrams, and equivalence classes of plabic graphs.

If $P \subseteq \mathbb{Z}^n$ is a d-dimensional lattice polytope, its *Ehrhart function/polynomial* is defined for every integer $t \geq 0$ by

$$E(P,t) := \#(t \cdot P) \cap \mathbb{Z}^n$$

where $t \cdot P$ is the dilation of P by a factor t, i.e., $t \cdot P = \{t \cdot v \mid v \in P\}$. It is well known from Ehrhart [10] that $E(P,t)$ is a polynomial function in t. The corresponding *Ehrhart series* is defined as $\sum_{t=0}^{\infty} E(P,t)z^t = \frac{h^*(P,z)}{(1-z)^{d+1}}$ where $h^*(P,z) = h_0 + h_1 z + \cdots + h_d z^d$ is a polynomial of degree at most d with non-negative coefficients [23], called the *Ehrhart h^*-polynomial* of P. Ehrhart theory naturally extends to half-open polytopes, which are polytopes with some facets removed. The Ehrhart h^*-polynomial of the whole polytope can then be obtained by inclusion-exclusion on the faces.

In this paper, we give explicit formulas for the h^*-polynomials of positroid polytopes and half-open positroid polytopes. Our work generalizes the work of Katzman [12], Early [9], Kim [13], and Li [17] on hypersimplices, as the hypersimplex $\Delta_{k,n}$ is the matroid polytope of the uniform matroid $U_{k,n}$, which is also a positroid. Apart from the special

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case of the uniform matroid, no prior combinatorial formula for the h^*-polynomial of an arbitrary positroid polytope was known.

Theorem 1.1. *Let $P_{\mathcal{J}}$ be any connected positroid polytope (see Definition 2.8), where $\mathcal{J}$ is the associated Grassmann necklace (see Definition 2.2). Let $D_{\mathcal{J}} \subset S_n$ be the subset of permutations that label the circuit triangulation of $P_{\mathcal{J}}$ (see Theorem 3.7). For any $w_0 \in D_{\mathcal{J}}$, let $(\mathcal{P}_{w_0,\mathcal{J}}, \prec)$ be the corresponding poset on $D_{\mathcal{J}}$ (see Definition 4.4). The cover statistic of $\mathcal{P}_{w_0,\mathcal{J}}$ gives the h^*-polynomial of $P_{\mathcal{J}}$, i.e.,*

$$h^*(P_{\mathcal{J}}, z) = \sum_{w \in D_{\mathcal{J}}} z^{\mathrm{cover}(w)}$$

where $\mathrm{cover}(w)$ *is the number of elements w covers in the poset $\mathcal{P}_{w_0,\mathcal{J}}$.*

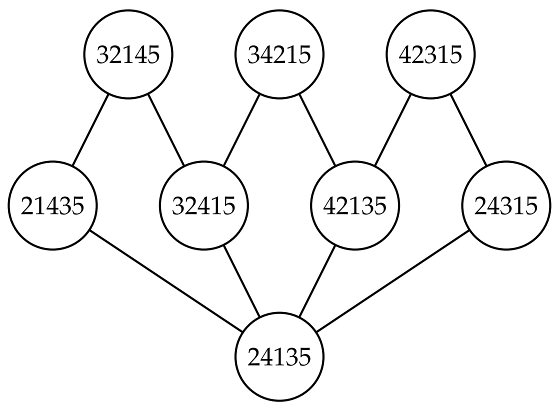

Figure 1: We show the graph of the circuit triangulation of the positroid polytope $P_{\mathcal{J}}$ associated to the positroid with Grassmann necklace $\mathcal{J} = (123, 235, 345, 145, 125)$, which coincides with the Hasse diagram of the poset $\mathcal{P}_{24135,\mathcal{J}}$. The h^*-polynomial of $P_{\mathcal{J}}$ is $1 + 4z + 3z^2$.

1.1 Organization

In Section 2, we introduce positroids and related combinatorial objects. We reduce the problem to *connected positroids*; see Definition 2.8. In Section 3, we analyze the *circuit triangulation* of connected positroid polytopes. In Section 4, we give a family of shellings of connected positroid polytopes, which give formulas for the h^*-polynomial of an arbitrary connected positroid polytope, proving Theorem 1.1. In Section 5, we give a formula for the h^*-polynomial of any connected positroid polytope with *upper facets* removed. In Section 6, we apply our theorems to the special case of tree positroids and derive Corollary 6.8 in terms of the *circular extensions* of *partial cyclic orders*.

2 Positroids

In this section, we will start by defining *matroids* and *positroids*. We also define Postnikov's notion of *Grassmann necklace*, and explain how each one naturally labels a positroid. Then we introduce positroid polytopes and the notion of *connected* positroids, and we reduce the problem to connected positroids.

For a $k \times n$-matrix A of rank k and a k-element subset $I \subset [n]$, let A_I denote the $k \times k$-submatrix of A in the column set I, and let $\Delta_I(A) := \det(A_I)$ denote the corresponding *maximal minor* of A. The set of k-subsets $I \subset [n]$ such that $\Delta_I(A) \neq 0$ form the bases of a rank k matroid $M(A)$.

Definition 2.1. Suppose A is a $k \times n$ matrix of rank k with real entries such that all its maximal minors are nonnegative. The matroid $M(A)$ associated to A is called a *positroid*.

Definition 2.2. Let $k \leq n$ be a positive integer. A *Grassmann necklace* of type (k, n) is a sequence $(J_1, J_2, \ldots, J_n)$ of k-subsets $J_i \in \binom{[n]}{k}$ such that for any $i \in [n]$,

- if $i \in J_i$ then $J_{i+1} = J_i - \{i\} \cup \{j\}$ for some $j \in [n]$,
- if $i \notin J_i$ then $J_{i+1} = J_i$,

where $J_{n+1} = J_1$ by convention.

Towards the bijection between Grassmann necklaces and positroids, we need the notion of *i-order*. The *i-order* $<_i$ on the set $[n]$ is the total order

$$i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i i-2 <_i i-1.$$

Let $i \in [n]$. The *Gale order* on $\binom{[n]}{d}$ (with respect to $<_i$) is the partial order $\leq_i$ defined as follows: for any two d-subsets $S = \{s_1 <_i \cdots <_i s_d\} \subseteq [n]$ and $T = \{t_1 <_i \cdots <_i t_d\} \subseteq [n]$, we have $S \leq_i T$ if and only if $s_j \leq_i t_j$ for all $j \in [d]$.

Lemma 2.3 ([22, Lemma 16.3])**.** *For a matroid $M \subseteq \binom{[n]}{k}$ of rank k on the set $[n]$, let $\mathcal{J}_M = (J_1, \ldots, J_n)$ be the sequence of subsets in $[n]$ such that, for $i \in [n]$, J_i is the minimal basis of M with respect to the Gale order with respect to $<_i$ on $[n]$. The sequence $\mathcal{J}(M)$ is a Grassmann necklace of type (k, n).*

Theorem 2.4 ([22, 19])**.** *Let $\mathcal{J} = (J_1, \ldots, J_n)$ be a Grassmann necklace of type (k, n). Then the collection*

$$\mathcal{B}(\mathcal{J}) := \left\{ B \in \binom{[n]}{k} \mid B \geq_i J_i \text{ for all } i \in [n] \right\}$$

is the collection of bases of a rank k positroid $\mathcal{M}(\mathcal{J}) := ([n], \mathcal{B}(\mathcal{J}))$. Moreover, for any positroid M we have $\mathcal{M}(\mathcal{J}(M)) = M$.

Example 2.5. Let $\mathcal{J}$ be the Grassmann necklace $(12, 23, 13, 14)$. The bases of the positroid associated to $\mathcal{J}$ is $\{12, 13, 14, 23, 24\}$.

Definition 2.6. Given a matroid $M = ([n], \mathcal{B})$, the (basis) *matroid polytope*

$$P_M := \text{convex}\{e_B \mid B \in \mathcal{B}\} \subset \mathbb{R}^n$$

of M is the convex hull of the indicator vectors of the bases of M, where $e_B = \sum_{i \in B} e_i$ and $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

The next proposition provides inequalities defining the matroid polytope of a positroid.

Proposition 2.7 ([1, 16]). *Let $\mathcal{J} = (J_1, J_2, \ldots, J_n)$ be a Grassmann necklace of type (k, n). For any $i \in [n]$, suppose the elements of J_i are $a_1^i <_i a_2^i <_i \cdots <_i a_k^i$. Then the matroid polytope $P_{\mathcal{I}}$ of the positroid associated to $\mathcal{J}$ can be described by the inequalities $x_i \geq 0$ for all $i \in [n]$ and*

$$x_1 + x_2 + \cdots + x_n = k$$
$$x_i + x_{i+1} + \cdots + x_{a_j^i - 1} \leq j - 1$$

for all $i \in [n]$ and $j \in [k]$, where all the subindices are taken modulo n.

To write the inequalities more concisely, we will use the following notation. Given $i, j \in [n]$, we define the *(cyclic) interval* $[i, j]$ to be the totally ordered set

$$[i,j] := \begin{cases} \{i <_i i+1 <_i \cdots <_i j\} & \text{if } i \leq j, \\ \{i <_i i+1 <_i \cdots <_i n <_i 1 <_i \cdots <_i j\} & \text{if } i > j, \end{cases}$$

and $x_{[i,j)} = x_i + \cdots + x_{j-1}$ with all indices modulo n.

We reduce the problem to connected positroids.

Definition 2.8. A matroid which cannot be written as the direct sum of two nonempty matroids is called *connected*.

Lemma 2.9 ([1, Lemma 7.3]). *Let M be a positroid on $[n]$ and write it as a direct sum of connected matroids $M = M_1 \oplus \cdots \oplus M_m$. Then each M_i is a positroid.*

Remark 2.10. The matroid polytope of a connected positroid on $[n]$ has dimension $n - 1$ [1, Theorem 8.2]. If a matroid M is equal to the direct sum of matroids $M = M_1 \oplus \cdots \oplus M_m$, then the matroid polytope P_M of M is equal to the direct product $P_M = P_{M_1} \times \cdots \times P_{M_m}$. The Ehrhart polynomial of P_M is equal to $E(P_M) = E(P_{M_1}) \cdots E(P_{M_m})$. Thus, to know the h^*-polynomial of all positroid polytopes, it suffices to give formulas for all connected positroid polytopes. From now on, we will focus on connected positroids.

3 Circuit triangulation of connected positroid polytopes

In this section, we analyze the triangulation of connected positroid polytopes in terms of (w)*-simplices*, defined by Parisi, Sherman-Bennett, Tessler, and Williams, and follow their conventions. We now introduce several definitions regarding descents/permutations that lead up to the characterization of (w)-simplices.

Definition 3.1. Let $w \in S_n$. A letter $i < n$ is a *left descent* of w if i occurs to the right of $i+1$ in w. In other words, $w^{-1}(i) > w^{-1}(i+1)$. We say that $i \in [n]$ is a *cyclic left descent* of w if either $i < n$ is a left descent of w or if $i = n$ and 1 occurs to the left of n in w, that is, $w^{-1}(1) < w^{-1}(n)$. We let $\mathrm{cDes}_L(w)$ denote the set of cyclic left descents of w and let $\mathrm{cdes}_L(w) = |\mathrm{cDes}_L(w)|$.

Definition 3.2. Choose $0 \le k \le n-2$. We let D_n be the set of permutations $w \in S_n$ with $w_n = n$, and let $D_{k+1,n}$ to be the set of permutations $w \in D_n$ with $k+1$ cyclic left descents. Let (w) denote the cycle $(w_1, \cdots, w_n)$.

The definition of cyclic left descent only depends on the total order on $[n]$. That is, given any permutation of any totally ordered set that is not a singleton or empty set, the cyclic left descent of such a permutation can be defined analogously. This definition coincides with [15, Definition 6.2] in type A.

Definition 3.3. Let $w = w_1 \cdots w_n \in S_n$ and $i, j \in [n]$. Let $[i, j]$ denote the cyclic interval defined in Section 2. Let $w|_{[i,j]}$ be the restriction of w to the totally ordered set $[i, j]$, and let $\mathrm{cdes}_L(w|_{[i,j]})$ be the number of cyclic left descents of $w|_{[i,j]}$.

Example 3.4. Let $w = 32415$. Then $w|_{[1,3]} = 321$ and $w|_{[3,1]} = 3415$. Then $\mathrm{cDes}_L(w|_{[1,3]}) = \{1, 2\}$ and $\mathrm{cDes}_L(w|_{[3,1]}) = \{5, 1\}$.

Definition 3.5. For $w = w_1 w_2 \cdots w_n \in S_n$, let $w^{(a)}$ denote the cyclic rotation of w ending at a. We define

$$I_r(w) := \mathrm{cDes}_L(w^{(r)}).$$

Note that I_r only depends on the cycle (w), and $|I_1(w)| = \cdots = |I_n(w)|$.

We define the (w)-simplex $\Delta_{(w)}$ to be the convex hull of the points $e_{I_1(w)}, \ldots, e_{I_n(w)}$; this is an $(n-1)$-dimensional simplex. We call

$$I_{w_1} \to I_{w_2} \to \cdots \to I_{w_n} \to I_{w_1}$$

the *circuit* of $\Delta_{(w)}$.

Triangulations by (w)-simplices are often called *circuit triangulations* [14].

Example 3.6. The circuit of 32415 is $135 \to 235 \to 245 \to 124 \to 125 \to 135$. The vertices of $\Delta_{(32415)}$ are $11001, 10101, 01101, 01011, 11010$.

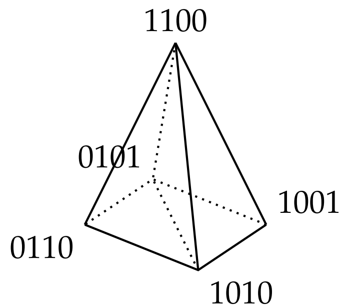

Figure 2: The positroid polytope associated to the Grassmann necklace $(12, 23, 13, 14)$, with bases $\{12, 13, 14, 23, 24\}$, consisting of the (1324)-simplex on the left and the (2134)-simplex on the right.

We now characterize the set of (w)-simplices in a positroid polytope.

Theorem 3.7. *Let $P_{\mathcal{J}}$ be any connected positroid polytope, where $\mathcal{J} = (J_1, \ldots, J_n)$ is the associated Grassmann necklace.*

For any $i \in [n]$, suppose the elements of J_i are $a^i_1 <_i \cdots <_i a^i_k$. Then the positroid polytope associated to π is triangulated by (w)-simplices for w in the set

$$D_{\mathcal{J}} := \{w \in D_{k+1,n} \mid \mathrm{cdes}_L(w|_{[i,a^i_j]}) \leq j-1 \text{ for all } i \in [n], j \in [k]\}$$
$$= \{w \in D_{k+1,n} \mid I_{w_i} \geq_j J_j \text{ for all } i, j \in [n]\},$$

where $I_{w_1} \to I_{w_2} \to \cdots \to I_{w_n} \to I_{w_1}$ is the circuit of w. The normalized volume of $P_{\mathcal{J}}$ is $\mathrm{vol}(P_{\mathcal{J}}) = |D_{\mathcal{J}}|$ the cardinality of $D_{\mathcal{J}}$.

Example 3.8. Consider the Grassmann necklace $\mathcal{J} = (12, 23, 13, 14)$. Then $D_{\mathcal{J}}$ consists of permutations in S_4 that end with 4 such that $\mathrm{cdes}_L(w) = 2$ and $\mathrm{cdes}_L(w|_{[3,4]}) \leq 1$, so $D_{\mathcal{J}} = \{1324, 2134\}$. The positroid polytope $P_{\mathcal{J}}$ is a pyramid, as in Figure 2. The (1324)-simplex has vertices $1100, 0101, 0110, 1010$, and the (2134)-simplex has vertices $1100, 0101, 1001, 1010$.

4 The h*-polynomial of connected positroid polytopes

In this section, we prove Theorem 1.1, which gives the h^*-polynomial of an arbitrary connected positroid polytope.

Definition 4.1. A *shelling* of a simplicial complex Γ is a linear order on its maximal faces $G_1, G_2, \ldots, G_s$ such that, for each $i \in [2, s]$, the set $G_i \cap (G_1 \cup \cdots \cup G_{i-1})$ is a union of facets of G_i.

Lemma 4.2 ([23, Corollary 2.6]). *Let Γ be a unimodular triangulation of a polytope P and let $G_1, \ldots, G_s$ be a shelling of Γ. Then the h^*-polynomial of P is equal to $h^*(P, z) = \sum_{i=1}^{s} z^{\alpha_i}$ where α_i is the number of facets of G_i in the intersection $G_i \cap (G_1 \cup \cdots \cup G_{i-1})$.*

Definition 4.3. Consider a connected positroid polytope $P_{\mathcal{J}}$ with Grassmann necklace $\mathcal{J}$. Let $\Gamma_{\mathcal{J}}$ be the graph whose vertices are $w \in D_{\mathcal{J}}$ and there is an edge between w and u if and only if $\Delta_{(w)}$ and $\Delta_{(u)}$ share a common facet. We call $\Gamma_{\mathcal{J}}$ the *graph of the circuit triangulation of $P_{\mathcal{J}}$*.

In the special case of the hypersimplex $\Delta_{k,n}$, the graph $\Gamma_{k,n}$ is defined in [14, Section 2.5]. For a generic connected positroid with Grassmann necklace $\mathcal{J}$ of type (k, n), the graph $\Gamma_{\mathcal{J}}$ is a connected subgraph of $\Gamma_{k,n}$.

Definition 4.4. Let $\Gamma = (V, E)$ be an undirected graph, and let $v_0 \in V$ be an arbitrary vertex of Γ. Define a partial order $(\mathcal{P}_{v_0,\Gamma}, \prec)$ on V with minimal element v_0 such that, for two distinct vertices $u, v \in V$, $u \prec v$ if and only if there exists a shortest path from v_0 to v passing through u.

In particular, the above definition applies to $\Gamma_{\mathcal{J}}$ and $w_0 \in D_{\mathcal{J}}$ for any Grassmann necklace $\mathcal{J}$. In this case, we will simplify our notation and denote $\mathcal{P}_{w_0,\Gamma_{\mathcal{J}}}$ by $\mathcal{P}_{w_0,\mathcal{J}}$. We can embed any connected positroid polytope into an *affine Coxeter arrangement*. The partial order $\mathcal{P}_{w_0,\mathcal{J}}$ is identified with the restriction of the weak order on a subset of the *affine symmetric group $\tilde{S}_n$*.

Benedetti–Knauer–Valencia-Porras proved the following for general type A alcoved polytopes in [5, Proposition 2.5]. In the spirit of [6, Theorem 2.1], Proposition 4.5 can be generalized to any other Coxeter group, studied by the author and Bullock in [7].

Proposition 4.5. *Consider a connected positroid polytope $P_{\mathcal{J}}$ with Grassmann necklace $\mathcal{J}$. Let $\Gamma_{\mathcal{J}}$ be the graph of the circuit triangulation of $P_{\mathcal{J}}$. For any $w_0 \in D_{\mathcal{J}}$, any linear extension of $\mathcal{P}_{w_0,\mathcal{J}}$ is a shelling of the circuit triangulation of $P_{\mathcal{J}}$.*

Theorem 1.1 follows naturally from Proposition 4.5.

5 The h*-polynomial of half-open connected positroid polytopes

In this section, we give a combinatorial formula for the h^*-polynomials of half-open connected positroid polytopes in terms of descents of permutations. We then compute the h^*-polynomial of a whole closed polytope by inclusion-exclusion on its half-open subpolytopes of smaller dimension. Our result generalizes [17] and [4].

The facets of these (w)-simplices and positroid polytopes are all in the form $x_{[i,j)} = k$ for some $i, j \in [n]$ and $k \in \mathbb{Z}$. We will call a facet of a positroid polytope or (w)-simplex *upper* if it is of the form $x_{[i,j)} = k$ such that the polytope satisfies $x_{[i,j)} \leq k$.

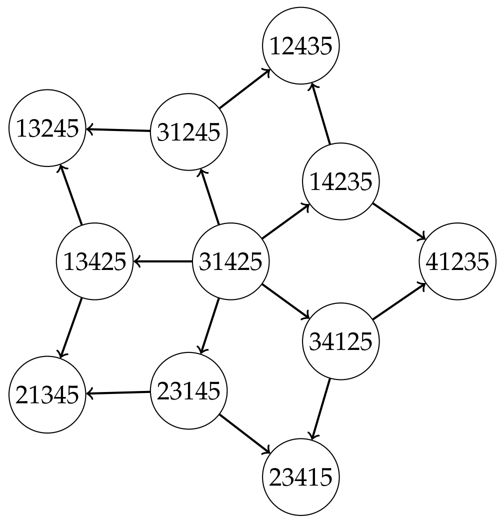

Figure 3: We draw the graph $\Gamma_{2,5}$ of the circuit triangulation of the hypersimplex $\Delta_{2,5}$. The vertices of $\Gamma_{2,5}$ are labeled by permutations in one-line notation. The arrows represent cover relations in the poset $\mathcal{P}_{31425,\mathcal{J}}$ for $\mathcal{J} = (12, 23, 34, 45, 51)$, pointing from a smaller element to a bigger element. We have $h^*(\Delta_{2,5}, z) = 1 + 5z + 5z^2$.

Theorem 5.1. *Let $P_{\mathcal{J}}$ be a connected positroid polytope, where $\mathcal{J}$ is the associated Grassmann necklace. Consider the half-open positroid polytope $\tilde{P}_{\mathcal{J}} \subset [0,1)^{n-1}$ which is the projection of $P_{\mathcal{J}}$ onto the first $(n-1)$ coordinates with all upper facets removed. Then the h^*-polynomial of $\tilde{P}_{\mathcal{J}}$ is equal to $h^*(\tilde{P}_{\mathcal{J}}, z) = \sum_{w \in D_{\mathcal{J}}} z^{\mathrm{des}(w)+1}$.*

To compute the h^*-polynomial of a polytope from the h^*-polynomials of its facets and the half-open polytope with those facets removed, we use the inclusion-exclusion principle. Let P be a polytope and let $F_1, \ldots, F_\ell$ be a collection of facets of P. Consider the restriction of the face poset of P to have coatoms $F_1, \ldots, F_\ell$. This poset $\mathcal{P}_{F_1,\ldots,F_\ell}$ describes all the faces of P in the intersections of $F_1, \ldots, F_\ell$. Let $\mu_{F_1,\ldots,F_\ell}$ be the Möbius function of this poset.

The next proposition follows from inclusion-exclusion on the face poset and additivity of Ehrhart polynomials.

Example 5.2. Consider the Grassmann necklace $\mathcal{J} = (12, 23, 13, 14)$. Then $D_{\mathcal{J}}$ consists of permutations in S_4 that end with 4 such that $\mathrm{cdes}_L(w|_{[3,4]}) \leq 1$ so $D_{\mathcal{J}} = \{1324, 2134\}$. Now $\mathrm{des}(132) = \mathrm{des}(213) = 1$, so $h^*(\tilde{P}_{\mathcal{J}}, z) = 2z^2$. To compute the h^*-polynomial of the whole positroid polytope P, which is a pyramid, we use inclusion-exclusion. The upper facets we removed are $F_1 : x_1 = 1, F_2 : x_2 = 1, F_3 : x_1 + x_2 + x_3 = 2$. We depict the poset $\mathcal{P}_{F_1,F_2,F_3}$ and the value of its Möbius function in Figure 4. Therefore, by inclusion-exclusion, the h^*-polynomial of $P_{\mathcal{J}}$ is $h^*(P_{\mathcal{J}}, z) = 2z^2 + 3(1-z) - 2(1-z)^2 = 1 + z$.

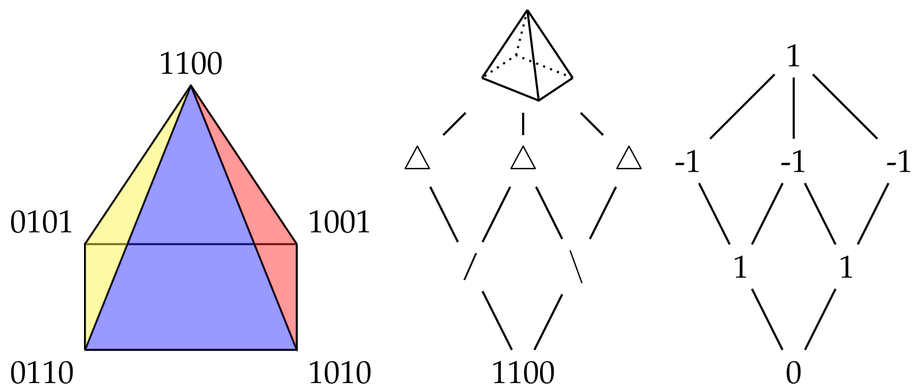

Figure 4: The positroid polytope associated to the Grassmann necklace $\mathcal{J} = (12, 23, 13, 14)$ is a pyramid. The red facet corresponds to $F_1 : x_1 = 1$; the yellow facet corresponds to $F_2 : x_2 = 1$; the blue facet corresponds to $F_3 : x_1 + x_2 + x_3 = 2$. These are all the upper facets of this positroid polytope. The poset $\mathcal{P}_{F_1,F_2,F_3}$ and the value of its Möbius function $\mu_{F_1,F_2,F_3}(-, P_{\mathcal{J}})$.

6 Tree positroids

When the plabic graph of a positroid is acyclic, we call it a *tree positroid*. In this section, we apply Theorem 1.1 to the special case of a tree positroid. To each plabic graph, one can associate a *plabic tiling* [20]. Tree positroids are positroids whose plabic tilings are *bicolored subdivision*. We start with several defintions, following the conventions of [21].

Definition 6.1. Let $\mathbf{P}_n$ be a convex n-gon with vertices labeled from 1 to n in clockwise order. A *bicolored subdivision* τ is a partition of $\mathbf{P}_n$ into black and white polygons such that two polygons sharing an edge have different colors. We say that τ has *type* (k, n) if any triangulation of the black polygons consists of exactly k black triangles.

Remark 6.2. Given a Grassmann necklace $\mathcal{J} = (J_1, \ldots, J_n)$, we can define a graph $G_{\mathcal{J}}$ on $[n]$ such that $\{i, j\}$ is an edge if and only if $|J_i \setminus J_j| = |J_j \setminus J_i| = 1$. The positroid $P_{\mathcal{J}}$ associated with the Grassmann necklace $\mathcal{J}$ is a tree positroid if and only if the graph $G_{\mathcal{J}}$ is a subdivision of a convex n-gon into polygons by diagonals.

The tree positroid polytopes are triangulated by (w)-simplices where w extends a *partial cyclic order* [21].

Definition 6.3. A *(partial) cyclic order* on a finite set X is a ternary relation $C \subset \binom{X}{3}$ such that for all $a, b, c, d \in X$:

$$(a,b,c) \in C \implies (c,a,b) \in C \qquad \text{(cyclicity)}$$
$$(a,b,c) \in C \implies (c,b,a) \notin C \qquad \text{(asymmetry)}$$
$$(a,b,c) \in C \text{ and } (a,c,d) \in C \implies (a,b,d) \in C \qquad \text{(transitivity)}$$

A cyclic order C is *total* if for all $a,b,c \in X$, either $(a,b,c) \in C$ or $(a,c,b) \in C$. A total cyclic order C' is a *circular extension* of C if $C \subseteq C'$. The set of total cyclic orders on $[n]$ is in bijection with the set D_n (see Definition 3.2). We denote the set of all circular extensions of C by $\mathrm{Ext}(C)$.

Definition 6.4. Let $x_1, \dots, x_m$ be a sequence of m distinct elements of $[n]$ (for $3 \leq m \leq n$). We let $C = C_{(x_1, x_2 \dots, x_m)}$ denote the partial cyclic order on $[n]$ in which for each triple $1 \leq i < j < \ell \leq m$ we have $(x_i, x_j, x_\ell) \in C$ (which implies by cyclicity that also (x_j, x_ℓ, x_i) and (x_ℓ, x_i, x_j) lie in C). We call this partial cyclic order a *chain*.

Definition 6.5. Let τ be a bicolored subdivision of $\mathbf{P}_n$ with q polygons $P_1, \dots, P_q$ which are black or white. If P_a is white (respectively, black), we let $v_1, \dots, v_r$ denote its list of vertices read in clockwise (respectively, counterclockwise) order. We then associate the chain $C_a = C_{(v_1, \dots, v_r)}$ to P_a. Finally we define the *τ-order* to be the partial cyclic order which is the union of the partial cyclic orders associated to the black and white polygons:

$$C_\tau := C_1 \cup \cdots \cup C_q.$$

Remark 6.6. Not all cyclic orders have a circular extension [18], that is, $\mathrm{Ext}(C)$ could be empty. Moreover, the problem of determining whether a cyclic order has a circular extension is NP-complete [18], which implies that it is NP-complete to test if two partial cyclic order C, C' has a common circular extension $\mathrm{Ext}(C \cup C')$, and it is in general hard to tell if two positroid polytopes intersect or not.

Proposition 6.7 ([21, Corollary 4.8])**.** *Let σ be a bicolored subdivision of type (k,n). Let $\mathcal{J}_\sigma$ be the Grassmann necklace associated with the tree positroid of σ. Then the positroid polytope associated with $\mathcal{J}_\sigma$ is a union of (w)-simplices for (w) that cyclically extends C_σ:*

$$P_{\mathcal{J}_\sigma} = \bigcup_{(w) \in \mathrm{Ext}(C_\sigma)} \Delta_{(w)}.$$

Corollary 6.8. *Let σ be a bicolored subdivision of type (k,n). Let $\mathcal{J}_\sigma$ be the Grassmann necklace associated with the tree positroid of σ. Then we have $\mathrm{Ext}(C_\sigma) = D_{\mathcal{J}_\sigma}$.*

7 Future work

The *polypositroid* is a *polymatroid*, or equivalently, *generalized permutohedron*, that is also alcoved [16]. They are parametrized by *Coxeter necklaces* and *membranes*, as analogies to *Grassmann necklaces* and *plabic graphs*. How much does the positroid results generalize to polypositroids?

Early gave a formula for the h^*-polynomial of any dilated hypersimplices in terms of the *decorated ordered set partitions* [9]. Is there a bijection between his formula and our shelling formula? Bullock and the author conjectured a bijection for $\Delta_{2,n}$ [7].

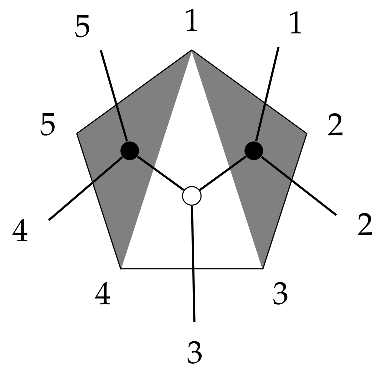

Figure 5: The tree plabic graph and bicolored subdivision associated to the Grassmann necklace $\mathcal{J} = (124, 234, 134, 145, 125)$. We have that C_τ consists of the chains $(3, 2, 1), (1, 3, 4)$ and $(5, 4, 1)$.

Elias–Kim–Supina [11] and Clarke–Kölbl [8] studied the equivariant Ehrhart H^* of the hypersimplices and showed that they are related to the S_n-representation on the decorated ordered set partitions. Ardila–Supina–Vindas-Meléndez and Ardila–Schindler–Vindas-Meléndez [3, 2] studied the equivariant Ehrhart theory of the regular permutohedron. Is there a relation between the equivariant Ehrhart H^* of the permutohedron and decorated ordered set partitions?

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