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Lattice of transfer systems

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Abstract. We consider the lattice of all transfer systems on a given finite lattice L. We prove that it is semidistributive and as a corollary, we deduce a bijection between the transfer systems and the cliques of a graph whose vertices are the relations of L. As an application we find a lower bound for the number of transfer systems on a boolean lattice.

Résumé. On considère le treillis des systèmes de transferts sur un treillis fini L. On démontre que c'est un treillis semi-distributif et on obtient ainsi une bijection entre les systèmes de transferts et les cliques d'un graphe do not les sommets sont les relations de l'ensemble ordonné L. Comme application, nous donnons une borne inférieure pour le nombre de systèmes de transferts sur les treillis booléens.

Keywords: Transfer system, Boolean lattice, Semidistributive lattice, Clique complex

1 Introduction

A *transfer system* on a finite lattice $(L, \leqslant)$ is a combinatorial gadget which is the main object of study of an emerging field sometimes called *homotopical combinatorics* (we refer to [7] for recent survey on the topic). This field involves three a priori unrelated areas: *model structures* in the sense of Quillen, *G-equivariant topology* and classical *combinatorics of binary trees*. One of the main problem in this area is to classify, or to count the transfer systems on natural lattices, in particular on lattices of subgroups. There have been many interesting classification results for finite *total orders* ([1]), *diamond lattices* ([4]), lattice of subgroups of the quaternion group Q_8 and many others ([21]). There are also many natural lattices where we do not know how to count the transfer systems. The most striking example being the boolean lattices which are the lattices of subgroups of square-free elementary abelian groups.

In this long abstract (based on [18]), instead of trying to count the transfer systems, we study the abstract properties of the set of all transfer systems on a given lattice. It is known (and easy to see) that this set is a lattice. Our main result is the following theorem.

Theorem 1.1. *Let $(L, \leqslant)$ be a finite lattice. Then the lattice of transfer systems on L is* semidistributive, trim *and* congruence uniform.

These three notions of increasing technicalities are shared by other famous lattices such as the Tamari lattice (see e.g. [14]), the *cambrian lattices* [19]. More generally the lattice of torsion classes of the category of finite dimensional modules over a finite dimensional algebra (see [11]) is both semidistributive and congruence uniform. *Trimness* occurs for example for path algebras of *Dynkin type*.

In this article, we will focus on the *semidistributivity of the lattice* because it has the most interesting consequences for the problem of enumerating the transfer systems. We define the *elevating graph* of a lattice L as the graph whose vertices are the *non-trivial relations* (a, b) with $a < b$ of L. Two vertices (a, b) and (c, d) are conected by at *most one edge*. There is exactly *one edge* if and only if $(a, b) \boxslash (c, d)$ and $(c, d) \boxslash (a, b)$. (See Section 6.)

Then we have the following theorem:

Theorem 1.2. *Let $(L, \leqslant)$ be a finite lattice. There is a bijection between the set of transfer systems on L and the cliques of its elevating graph.*

This theorem can also be used as the base of a rather efficient algorithm for enumerating transfer systems.

The article is organized as follows. In the first three sections, we briefly give the three motivations. In the next section, we prove the semidistributivity of the lattice of transfer systems and we discuss the graph theoretical interpretation. The last section is devoted to the boolean lattices. We refer to Figure 2 for an example that illustrates the different concepts of this article.

2 Motivation 1: G-equivariant topology

In algebra and topology, there are various natural examples of multiplicative laws that are only associative up to higher homotopies, such as the loop space of a topological space, the Yoneda algebras etc. These algebras are called A_∞-algebras. The 'commutative' version of these algebras are called E_∞-algebras. As for all kinds of associativity laws, it is useful to see these algebras as algebras over special operads called A_∞ and E_∞ operads.

There is a very recent notion of $G - N_\infty$-operads associated with a finite group G. They are *equivariant generalization* of E_∞-operads and the algebras over these operads are equipped with a multiplication which is associative and commutative up to higher homotopies (an E_∞ structure) and multiplicative norm maps. This notion appears in the Hill–Hopkins–Ravenel [15] solution of the Kervaire invariant one problem and has received a lot of attention since then. One of the key results, is that the homotopy category of the $G - N_\infty$-operads is equivalent to the finite poset of G-transfer systems, viewed as a category. See [13, Theorem 3.6].

To explain this result, let us first recall that a poset $(P, \leqslant)$ can be viewed as a category whose objects are the elements of P, and there is a unique morphism from x to y if and only if $x \leqslant y$. Since the relation is reflexive, there is a unique morphism from x to x and this is id_x. The composition follows from the transitivity of the relation. For the notion of G-transfer systems we have the following definition:

Definition 2.1. Let G be a finite group. We denote by $\mathrm{Sub}(G)$ the poset of the subgroups of G. Then a *G-transfer system* is a partial ordering $\lhd$ on $\mathrm{Sub}(G)$ such that:

1. If $H \lhd K$, then $H \subseteq K$.
2. If $H \lhd K$ and $g \in G$, then $gHg^{-1} \lhd gKg^{-1}$.
3. If $H \lhd K$ and $M \leqslant K$, then $M \cap H \lhd M$.

The G-transfer systems are naturally *ordered by refinement*. This theorem on N_∞-operads is the main motivation behind the classification of the G-transfer systems. By classifying this relatively simple combinatorial object, we classify much more complicated objects.

The poset $\big(\mathrm{Sub}(G), \leqslant\big)$ is a lattice and it will be useful to define the notion of transfer system on a given lattice.

Definition 2.2. Let $(L, \leqslant)$ be a lattice. A *transfer system* $\lhd$ for L is a relation of partial ordering on L such that:

1. $i \lhd j$ implies $i \leqslant j$.
2. $i \lhd k$ and $j \leqslant k$ implies $(i \wedge j) \lhd j$.

Forgetting the second item in Definition 2.1 we see that a G-transfer system is in particular a *transfer system* on $\big(\mathrm{Sub}(G), \leqslant\big)$ in the sense of Definition 2.2. Moreover, when the group G is abelian or when all the subgroups of G are normal (e.g. Q_8) a G-transfer system in nothing but a transfer system.

Inclusion of relations naturally induces a poset structure on the set of transfer systems on L and we denote this poset by $\mathrm{Trs}(L)$. Since the intersection of two transfer systems is a transfer system, we have the following result.

Proposition 2.3. *The poset of transfer systems on $(L, \leqslant)$ is a finite lattice.*

Proof. This is [13, Proposition 3.7]. The join is the transitive closure of the union. □

Let G be a finite group. A poset $(X, \leqslant)$ is called a *G-poset* if there is a monotone G-action on it. That is X is a G-set and for every $x \leqslant y \in X$, and every $g \in G$ we have $g \cdot x \leqslant g \cdot y$. A lattice $(L, \leqslant)$ is a *G-lattice* if it is a G-poset and the action is compatible with the meets and the joins of the lattice. That is $g \cdot (x \wedge y) = g \cdot x \wedge g \cdot y$ and $g \cdot (x \vee y) = g \cdot x \vee g \cdot y$, for every $x, y \in L$ and $g \in G$.

Our first result is

Theorem 2.4. *Let G be a finite group and $L = \mathrm{Sub}(G)$ be its lattice of subgroup. Then*

1. *The lattice of transfer systems on L is a G-lattice.*

2. *The sublattice of G-fixed points is isomorphic to the lattice of G-transfer systems.*

Proof. See [18, Lemma 9.4]. $\square$

This result is very useful since it implies that Theorem 1.1 holds also for the lattice of G-transfer systems. Indeed semidistributivity and congruence uniformity are preserved by taking sublattices and trimness is preserved by taking G-fixed points.

3 Motivation 2: model structures on finite lattices

Let $\mathcal{C}$ be a category. A morphism f of $\mathcal{C}$ is said to *lift on the left* a morphism g of $\mathcal{C}$ if for every commutative square

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "b"] & Y \end{tikzcd}$$

there exists a lift $B \to X \in \mathcal{C}$ making the resulting diagrams commute. In other words, if $ga = bf$, there is $h : B \to X$ such that $hf = a$ and $gh = b$. In this case, we write $f \boxslash g$. If $\mathcal{S}$ is a class of morphisms in $\mathcal{C}$, we use the following notation

$$^{\boxslash}\mathcal{S} = \{f \in \mathrm{Mor}(\mathcal{C}) \mid f \boxslash g \ \forall g \in \mathcal{S}\}.$$

When $\mathcal{S} \subseteq {}^{\boxslash}\mathcal{T}$, we write $\mathcal{S} \boxslash \mathcal{T}$.

Definition 3.1. A *weak factorization system* on $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of subclasses of the morphisms of $\mathcal{C}$ such that:

1. Every morphism $f \in \mathcal{C}$ can be factored as $f = pi$ where $i \in \mathcal{L}$ and $p \in \mathcal{R}$.

2. $\mathcal{L} \boxslash \mathcal{R}$.

3. $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts ([18, Definition 2.1]).

Set theoretical issues aside, the class of weak factorization systems has a *natural structure of partial order*:

$$(\mathcal{L}, \mathcal{R}) \preceq (\mathcal{L}', \mathcal{R}') \text{ if } \mathcal{R} \subseteq \mathcal{R}' \text{ and } \mathcal{L}' \subseteq \mathcal{L}.$$

Very surprisingly, when $(L, \leqslant)$ is a finite lattice viewed as a category we have the following result [13, Theorem 4.13].

Theorem 3.2. *Let $(L, \leqslant)$ be a lattice. The map sending a transfer system $\mathcal{R}$ to $({}^{\boxslash}\mathcal{R}, \mathcal{R})$ is an isomorphism between the poset of transfer systems and the poset of weak factorization systems.*

4 Motivation 3: tree combinatorics

In this section, we consider the case where $(L, \leqslant)$ is the total order $1 \leqslant 2 \leqslant \cdots \leqslant n$ and we will see how the transfer systems on this poset are related to very classical combinatorial objects.

Let T be a *binary tree* with n inner vertices, we label its vertices by the integers $1, 2, \cdots, n$ in such a way that if x is a vertex in the left (right) subtree of y then the label if x is strictly smaller (larger) than the label of y. A binary tree T with such a labeling is called a *binary search tree*. We refer to Figure 1 for an illustration.

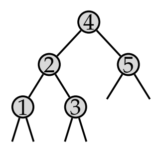


Figure 1: Binary search tree of size 5 labeled by the in-order algorithm.

If T is such a binary search tree, it induces a partial order $\lhd_T$ on $[n]$ by setting $i \lhd_T j$ if the vertex labelled by i is in the subtree with root the vertex labelled by j. Moreover, it is easy to classify the posets coming from such a binary tree. See for example [9, Proposition 2.21]. Surprisingly, this classification is much better explained in terms of weak factorization systems. If $\lhd$ is a poset of $[n]$, the relations of $\lhd$ can be splitted into two sets: the *increasing relations* ($i \lhd j$ such that $i \leqslant j$) and the *decreasing relations* ($i \lhd j$ such that $j \leqslant i$). The result of Châtel, Pilaud and Pons is easily rephrased as follows:

Proposition 4.1. *There is a binary tree T such that $\lhd = \lhd_T$ if and only if $(\mathrm{Inc}(\lhd), \mathrm{Dec}(\lhd)^{op})$ is a weak factorization system on $[n]$.*

It is also classical that for two binary trees, we have $T \leqslant T'$ in the Tamari lattice if and only if $\mathrm{Inc}(T') \subseteq \mathrm{Inc}(T)$ holds, or alternatively, if $\mathrm{Dec}(T) \subseteq \mathrm{Dec}(T')$ holds. Hence, we obtain an alternative proof of [1, Theorem 25].

Theorem 4.2. *Let L be the usual total ordering on $[n]$. The poset of weak factorization systems of L is isomorphic to the Tamari lattice on the binary trees with n inner vertices.*

Remark 4.3. The intervals of the Tamari lattice have received a lot of attention since their enumeration by Chapoton [8] and their relation with simple triangulations. Several families of intervals have been enumerated and several interesting bijections have been found ([12] and the references in these articles). With the point of view of transfer systems, there are natural families of intervals: for example the *model structures*, the *composition closed intervals* and the *compatible intervals*.

The model structures and the composition closed intervals have been described in [3]. It turns out that the composition closed intervals are exactly the *exceptional intervals* in the sense of [20] and they are counted by the ternary trees. The compatible intervals have been counted in [16] are also counted by ternary trees. As this article is being written, there is no bijective proof of this fact.

5 Lattice of transfer systems

In this section we fix $(L, \leqslant)$ a finite lattice, and we illustrate in Figure 2 the various notions in the special case where L is a commutative square.

Semidistributive lattices are a generalization of the distributive lattices introduced by Jónsson [17] who was inspired by Whitman's solution to the word problem for free lattices involving the existence of a nice canonical form. We say that a lattice $(L, \wedge, \vee)$ is *semidistributive* if $\forall a, b, c \in L$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \text{ whenever } a \vee b = a \vee c,$$

and

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ whenever } a \wedge b = a \wedge c.$$

Theorem 5.1. *Let $(L, \leqslant)$ be a finite lattice. Then the lattice of transfer systems on L is a* semidistributive *lattice.*

Proof. Let $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3$ be three transfer systems on L such that $\mathcal{R}_1 \vee \mathcal{R}_2 = \mathcal{R}_1 \vee \mathcal{R}_3$. Note that $\mathcal{R}_1 \vee (\mathcal{R}_2 \wedge \mathcal{R}_3) \leqslant \mathcal{R}_1 \vee \mathcal{R}_2$ holds in any lattice. Conversely, we consider $(x, y) \in P = \mathcal{R}_1 \vee \mathcal{R}_2$. We assume that (x, y) is a cover relation in the poset P. Note that the $P = \mathcal{R}_1 \vee \mathcal{R}_2 = (\mathcal{R}_1 \cup \mathcal{R}_2)^{tc}$, the transitive closure of $\mathcal{R}_1 \cup \mathcal{R}_2$. Hence, the cover relation (x, y) is either in $\mathcal{R}_1$ or in $\mathcal{R}_2$. Since $P = \mathcal{R}_1 \vee \mathcal{R}_3 = (\mathcal{R}_1 \cup \mathcal{R}_3)^{tc}$, the relation (x, y) is also in $\mathcal{R}_1$ or in $\mathcal{R}_3$. It follows that (x, y) is either in $\mathcal{R}_1$ or in $\mathcal{R}_2 \cap \mathcal{R}_3$ and we have:

$$\mathcal{R}_1 \vee \mathcal{R}_2 \subseteq \big(\mathcal{R}_1 \cup (\mathcal{R}_2 \cap \mathcal{R}_3)\big)^{tc} = \mathcal{R}_1 \vee \big(\mathcal{R}_2 \wedge \mathcal{R}_3\big).$$

The other identity is obtained by playing with the duality between transfer systems on $(L, \leqslant)$ and 'co-transfer systems' on $(L, \leqslant)^{op}$. $\square$

In semidistributive lattices we have the existence of a so-called *canonical join decomposition* for their elements. Loosely speaking any element of a semidistributive lattice can be written in a unique minimal way as a join of *join-irreducible elements* (*i.e.* an element that covers a unique element). This is a lattice version of the fundamental theorem of arithmetic which says that any integer can be written in a unique way as product of prime numbers. Hence for any finite semidistributive lattice, there is a bijection between the elements of L and the subsets of the set of join-irreducible elements which appear as canonical join decomposition. To use this bijection it is therefore crucial to start by understanding the join-irreducible elements (the prime numbers of our lattice). For transfer systems, this boils down to the following.

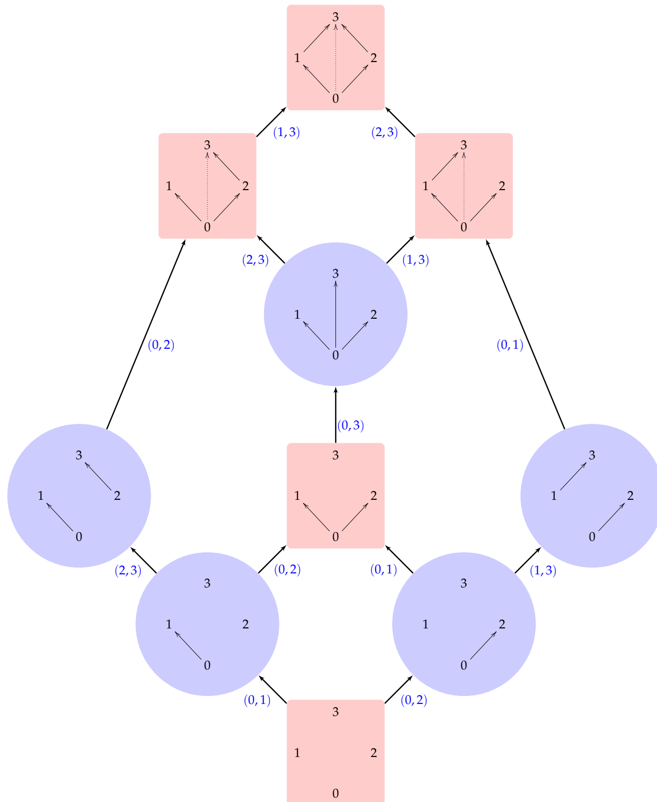


Figure 2: Lattice of transfer systems of the commutative square. The five transfer systems in circle blue are join-irreducible. The labelling of the edges is the join labelling.

We denote by $\mathrm{Rel}^*(L) = \{(a, b) \in L^2 \mid a \leqslant b \text{ and } a \neq b\}$ the set of non-trivial relations in L. For $(a, b) \in \mathrm{Rel}^*(L)$, we denote by $\mathrm{Tr}(a, b)$ the *smallest transfer system* containing the relation (a, b).

Proposition 5.2. *Let $(L, \leqslant)$ be a finite lattice. The map $(a, b) \mapsto \mathrm{Tr}(a, b)$ is a bijection between $\mathrm{Rel}^*(L)$ and the join-irreducible elements of the lattice of transfer systems on L.*

The next step is to explain how to find the *canonical join decomposition* of a given element. For this purpose we use the so-called *join-labelling* of the edges of the lattice (a consequence of semidistributivity). If x is an element of L, then its canonical decomposition is $j_1 \vee \cdots \vee j_n$ where $j_1, \cdots, j_n$ are the join-label of all arrows with target x.

Proposition 5.3. *Let $(L, \leqslant)$ be a finite lattice. Then the* join label *of a cover relation $\mathcal{R}' \lessdot \mathcal{R}$ is $\mathrm{Tr}(a, b)$ where (a, b) is the unique non trivial relation in $\mathcal{R} \cap {}^{\boxslash}\mathcal{R}'$.*

To simplify the pictures, we will simply label $\mathcal{R}' \lessdot \mathcal{R}$ by (a, b) instead of $\mathrm{Tr}(a, b)$.

6 Graph theoretical interpretation of transfer systems

For two relations (a,b) and (c,d) in $\mathrm{Rel}^*(L)$, we write $(a,b) \boxslash (c,d)$ if (a,b) lifts on the left (c,d). Concretely, we have:

$$(a,b) \boxslash (c,d) \text{ if and only if } \big(a \leqslant c \text{ and } b \leqslant d\big) \Longrightarrow b \leqslant c.$$

For every $S \subseteq \mathrm{Rel}^*(L)$ we say that S is an *elevating set* if for every $(a,b) \in S$ and $(c,d) \in S$ we have $(a,b) \boxslash (c,d)$ and $(c,d) \boxslash (a,b)$. The bijection discussed in the previous paragraph boils down to the following result.

Theorem 6.1. *Let $(L, \leqslant)$ be a finite lattice. There is a bijection between the set of weak factorization systems on L and the set of elevating subsets of* $\mathrm{Rel}^*(L)$.

The bijection is moreover explicit: if S is an elevating set, then the corresponding transfer system is the smallest transfer system containing S. Given a transfer system $\mathcal{R}$, the corresponding elevating set is the set consisting of the *join-labels* of all the arrows with target $\mathcal{R}$.

The set of elevating subsets (more generally, the set of all the canonical join-representations of the elements of a semidistributive lattice L) can be naturally viewed as a *simplicial complex*. It has been proved by Emily Barnard [5] that this simplicial complex is *flag*. This means that it is the *clique complex* of its 1-skeleton. In our setting we call it the *elevating graph* of the lattice L.

Definition 6.2. Let $(L, \leqslant)$ be a finite lattice. The elevating graph of L is the graph with vertices $\mathrm{Rel}^*(L)$ and there is an edge between (a,b) and (c,d) if and only if $(a,b) \boxslash (c,d)$ and $(c,d) \boxslash (a,b)$.

For the commutative square this graph is illustrated in Figure 3.

Recall that a clique is a complete (induced) subgraph. It follows that there are as *many transfer systems* on L as there are *cliques in the elevating graph* of L. For example, there are 10 cliques in the elevating graph of the commutative square: the empty graph, 5 vertices and 4 edges, so there are 10 transfer systems on this lattice.

We deduce from this result a *relatively efficient algorithm* for counting the weak factorization systems on a (not too big) lattice. It is known that counting cliques in a graph is a difficult task (it is $\sharp P$-complete) however the elevating graph is much smaller than the lattice of transfer systems. For example if $L = \mathcal{P}([4])$ is the boolean lattice with 16 elements, there are 5389480 transfer systems but the elevating graph of L has only 65 vertices and 1474 edges.

This approach is particularly good for lattices with few relations. For example, it is

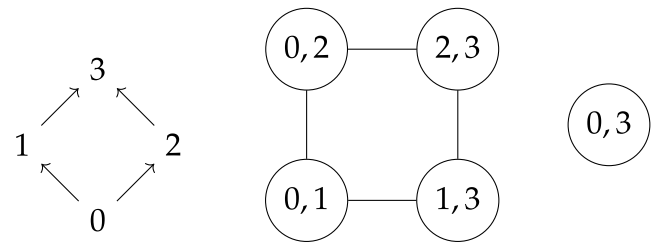
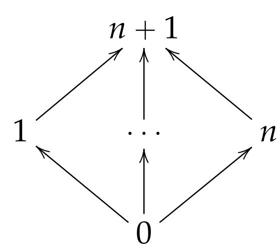


Figure 3: From left to right: the lattice L and its elevating graph. One can check that each clique of the graph corresponds to the set of labelings of the lower covers of a transfer system and conversely.

easy to describe the elevating graph of the diamond lattice



It follows that there are $2^{n+1} + n$ transfer systems for this lattice. This gives an alternative proof of [4, Theorem 5.4].

7 A lower bound in the case of the boolean lattice

The boolean lattices of subsets of $[n]$ for $n = 0, 1, 2$ are respectively total orders and a diamond lattice. The case $n = 3$ has been treated in [2] and is already quite complicated. With our approach counting transfer systems is equivalent to counting cliques in the *elevating graph*. We denote by B_n the boolean lattice of the subsets of $[n]$. The vertices of the elevating graph are the pairs of subsets (A, B) such that $A \subseteq B$ and $A \neq B$.

Lemma 7.1. *The elevating graph of B_n has $3^n - 2^n$ vertices and $\binom{3^n-2^n}{2} - (6^n - 5^n - 3^n + 2^n)$ edges.*

This sequence for the number of edges starts with $0, 0, 4, 99, 1474, 17715, 190414, \ldots$ and since $6^n - 5^n - 3^n + 2^n$ is small against $\binom{3^n-2^n}{2}$, when n is large enough the graph has almost all the possible edges. So we can expect a very large number of cliques, so a *very large number of transfer systems*. Since a subgraph of a clique is a clique, if a graph has a clique of size k, then it has at least 2^k cliques. Here the maximal size of a clique is denoted by $\mathrm{mcov}_\downarrow(L)$, and it is largest indegree of the Hasse diagram of the lattice of transfer systems.

Proposition 7.2. *Let $(L, \leqslant)$ be a finite lattice. Then $2^{\mathrm{mcov}_{\downarrow}(L)} \leqslant |\operatorname{Trs}(L)|$.*

For $0 \leqslant k \leqslant 2n-1$, we consider the relation $\mathcal{R}_k$ on B_n defined by

$$X \leqslant_{\mathcal{R}_k} Y \text{ if and only if } X = Y \text{ or } X \subseteq Y \text{ and } |X| + |Y| \leqslant k.$$

Lemma 7.3. *Let B_n be the boolean lattice of the subsets of $[n]$.*

1. *The relation $\mathcal{R}_k$ is a transfer system for the boolean lattice B_n.*
2. *The number of arrows with target $\mathcal{R}_k$ in the lattice of transfer systems is*

$$\sum_{j=0, j \neq \frac{k}{2}}^{n} \binom{n}{j} \binom{n-j}{k-2j}.$$

Removing $\mathcal{R}_0$, these numbers naturally form a triangle, see Figure 4. Note that the

1
2 1 *2*
3 3 *7* 3 3
4 6 *16* 13 *16* 6 4
5 10 30 35 *51* 35 30 10 5
6 15 50 75 *126* 121 *126* 75 50 15 6
7 21 77 140 266 322 *393* 322 266 140 77 21 7

Figure 4: Triangle of the number of lower cover relations of the $\mathcal{R}_k$ for $k = 1, \ldots, 2n-1$.

maximal numbers in each row (in red) appear in the expansion of $(1 + x + x^2)^n$. When n is odd, this is the coefficient of x^n and when n is even, it is the coefficient of x^{n-1} or x^{n+1}.

Proposition 7.4. *Let $n \in \mathbb{N}$ and $L = B_n$ the boolean lattice of the subsets of $[n]$. Then*

1. *If n is odd, $\mathrm{mcov}_{\downarrow}(L) \geqslant \sum_{j=0}^{\frac{n-1}{2}} \binom{n}{j}\binom{n-j}{n-2j}$.*
2. *If n is even, $\mathrm{mcov}_{\downarrow}(L) \geqslant \sum_{j=1}^{\frac{n}{2}} \binom{n}{j}\binom{n-j}{n+1-2j}$.*

Question 7.5. *Let $L = B_n$ be the boolean lattice of the subsets of $[n]$. Is $\mathrm{mcov}_{\downarrow}(\operatorname{Trs}(L))$ given by Proposition 7.4 ?*

Using the computer and the software SageMath, we were are able to check that Question 7.5 has a positive answer for $n \leqslant 6$.

n	1	2	3	4	5	6	7
$\mathrm{mcov}_{\downarrow}$	1	2	7	16	51	126	$\geqslant 393$
$2^{\mathrm{mcov}_{\downarrow}} \approx$	2	4	128	65536	$2.2 \cdot 10^{15}$	$8.5 \cdot 10^{37}$	$\geqslant 2 \cdot 10^{118}$
$\lvert \mathrm{Trs}(B_n) \rvert$	2	10	450	5389480	?	?	

8 Perspectives

1. Look for a bijective proof for the number of *compatible intervals* of the Tamari lattice. The compatible intervals are easily described using the interval-posets of [10]. There is a simple bijection between interval-posets and *blossoming trees* (see [12]) which seems well-suited for this problem.

2. By a famous theorem of Day, a lattice is congruence unform if and only if it is obtained from a singleton lattice by a sequence of '*interval doublings*'. Can we obtain a nice description of such a sequence for lattices of transfer systems?

3. With our approach we obtained an easy lower bound for the number of transfer systems for the boolean lattices. Can we find more information about the elevating graph of the boolean lattice and can we hope for a closed formula ?

4. The number $\mathrm{mcov}_{\downarrow}(L)$ is crucial in our approach. When L is a total order, the lattice of transfer systems is isomorphic to the Tamari lattice which is a *regular* lattice. However, the lattice of transfer system is not regular in general and this number seems rather difficult to compute. Can we compute $\mathrm{mcov}_{\downarrow}(L)$ for natural families of lattices?

5. The transfer systems on finite lattices and the torsion classes of finite dimensional algebras share the same lattice properties. We refer to [18, Section 11] for more details. It remains to understand if this is only a combinatorial coincidence or if there is more behind it.

6. The topology of the simplicial complex consisting of the canonical join representations of the Tamari lattice has been studied by Barnard in [6]. Can we find interesting topological properties the canonical complex of the lattice of transfer systems?

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